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HARMONIC MAPPINGS OF SURFACES

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We are going to study the harmonic and slightly less than harmonic mappings $f : M \to N$ in the case $\dim M = \dim N = 2$. For further details, see [1]–[9].

1. Let M, N be Riemannian manifolds, $\dim M = \dim N = 2$, $f : M \to N$ a mapping; everything be of class C^∞. Let us suppose that M and N are oriented and $f : M \to N$ is orientation preserving. Let M be covered by a system of domains such that in each of them we are able to choose a field of orthonormal frames $\{v_1, v_2\}$, let $\{\omega^1, \omega^2\}$ be the dual bases. The Euclidean connection of M is then given by

$$(1.1)\qquad \nabla m = \omega^1 v_1 + \omega^2 v_2\,, \quad \nabla v_1 = \omega_1^2 v_2\,, \quad \nabla v_2 = -\omega_1^2 v_1\,;$$

$$d\omega^1 = -\omega^2 \wedge \omega_1^2\,, \quad d\omega^2 = \omega^1 \wedge \omega_1^2\,, \quad d\omega_1^2 = -K\omega^1 \wedge \omega^2\,,$$

K being the curvature of M. Analoguously, the connection on N be given by

$$(1.2)\qquad \nabla^* n = \Omega^1 v_1^* + \Omega^2 v_2^*\,; \quad \nabla^* v_1^* = \Omega_1^2 v_2^*\,, \quad \nabla^* v_2^* = -\Omega_1^2 v_1^*\,;$$

$$d\Omega^1 = -\Omega^2 \wedge \Omega_1^2\,, \quad d\Omega^2 = \Omega^1 \wedge \Omega_1^2\,, \quad d\Omega_1^2 = -K^*\Omega^1 \wedge \Omega^2\,.$$

On M, we get the induced forms

$$(1.3)\qquad \tau^1 := f^*\Omega^1\,, \quad \tau^2 := f^*\Omega^2\,, \quad \tau_1^2 := f^*\Omega_1^2$$

satisfying

$$(1.4)\qquad d\tau^1 = -\tau^2 \wedge \tau_1^2\,, \quad d\tau^2 = \tau^1 \wedge \tau_1^2\,, \quad d\tau_1^2 = -K^*\tau^1 \wedge \tau^2\,.$$

Let us write

$$(1.5)\qquad \tau^1 = a_1\omega^1 + a_2\omega^2\,, \quad \tau^2 = a_3\omega^1 + a_4\omega^2\,.$$

Then

$$(1.6)\qquad \tau^1 \wedge \tau^2 = \mu\omega^1 \wedge \omega^2\,, \quad \text{where} \quad \mu = a_1a_4 - a_2a_3 \geqq 0\,.$$

By means of successive exterior differentiations of (1.5), we get the existence of functions $b_1, \ldots, b_6, c_1, \ldots, c_8$ such that

$$(1.7)\qquad (\mathrm{d}a_1 - a_2\omega_1^2 - a_3\tau_1^2) \wedge \omega^1 + (\mathrm{d}a_2 + a_1\omega_1^2 - a_4\tau_1^2) \wedge \omega^2 = 0$$
$$(\mathrm{d}a_3 - a_4\omega_1^2 + a_1\tau_1^2) \wedge \omega^1 + (\mathrm{d}a_4 + a_3\omega_1^2 + a_2\tau_1^2) \wedge \omega^2 = 0\,;$$

$$(1.8)\qquad \begin{aligned} \mathrm{d}a_1 - a_2\omega_1^2 - a_3\tau_1^2 &= b_1\omega^1 + b_2\omega^2\,,\\ \mathrm{d}a_3 - a_4\omega_1^2 + a_1\tau_1^2 &= b_4\omega^1 + b_5\omega^2\,,\\ \mathrm{d}a_2 + a_1\omega_1^2 - a_4\tau_1^2 &= b_2\omega^1 + b_3\omega^2\,,\\ \mathrm{d}a_4 + a_3\omega_1^2 + a_2\tau_1^2 &= b_5\omega^1 + b_6\omega^2\,; \end{aligned}$$

$$(1.9)\qquad (\mathrm{d}b_1 - 2b_2\omega_1^2 - b_4\tau_1^2) \wedge \omega^1 + \{\mathrm{d}b_2 + (b_1 - b_3)\,\omega_1^2 - b_5\tau_1^2\} \wedge \omega^2 =$$
$$= (a_2K + a_3\mu K^*)\,\omega^1 \wedge \omega^2\,,$$
$$\{\mathrm{d}b_2 + (b_1 - b_3)\,\omega_1^2 - b_5\tau_1^2\} \wedge \omega^1 + (\mathrm{d}b_3 + 2b_2\omega_1^2 - b_6\tau_1^2) \wedge \omega^2 =$$
$$= (-a_1K + a_4\mu K^*)\,\omega^1 \wedge \omega^2\,,$$
$$(\mathrm{d}b_4 - 2b_5\omega_1^2 + b_1\tau_1^2) \wedge \omega^1 + \{\mathrm{d}b_5 + (b_4 - b_6)\,\omega_1^2 + b_2\tau_1^2\} \wedge \omega^2 =$$
$$= (a_4K - a_1\mu K^*)\,\omega^1 \wedge \omega^2\,,$$
$$\{\mathrm{d}b_5 + (b_4 - b_6)\,\omega_1^2 + b_2\tau_1^2\} \wedge \omega^1 + (\mathrm{d}b_6 + 2b_5\omega_1^2 + b_3\tau_1^2) \wedge \omega^2 =$$
$$= (-a_3K - a_2\mu K^*)\,\omega^1 \wedge \omega^2\,;$$

$$(1.10)\qquad \begin{aligned} \mathrm{d}b_1 - 2b_2\omega_1^2 - b_4\tau_1^2 &= c_1\omega^1 + c_2\omega^2\,,\\ \mathrm{d}b_2 + (b_1 - b_3)\,\omega_1^2 - b_5\tau_1^2 &= (c_2 + a_2K + a_3\mu K^*)\,\omega^1 +\\ &\quad + (c_3 + a_1K - a_4\mu K^*)\,\omega^2\,,\\ \mathrm{d}b_3 + 2b_2\omega_1^2 - b_6\tau_1^2 &= c_3\omega^1 + c_4\omega^2\,,\\ \mathrm{d}b_4 - 2b_5\omega_1^2 + b_1\tau_1^2 &= c_5\omega^1 + c_6\omega^2\,,\\ \mathrm{d}b_5 + (b_4 - b_6)\,\omega_1^2 + b_2\tau_1^2 &= (c_6 + a_4K - a_1\mu K^*)\,\omega^1 +\\ &\quad + (c_7 + a_3K + a_2\mu K^*)\,\omega^2\,,\\ \mathrm{d}b_6 + 2b_5\omega_1^2 + b_3\tau_1^2 &= c_7\omega^1 + c_8\omega^2\,. \end{aligned}$$

Of course, we have

$$(1.11)\qquad \mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2\,,$$
$$\mathrm{d}s_*^2 = (\tau^1)^2 + (\tau^2)^2 =$$
$$= (a_1^2 + a_3^2)\,(\omega^1)^2 + 2(a_1a_2 + a_3a_4)\,\omega^1\omega^2 + (a_2^2 + a_4^2)\,(\omega^2)^2\,.$$

The fundamental invariants of f are

(1.12) $$I_1 = a_1^2 + a_2^2 + a_3^2 + a_4^2 , \quad I_2 = (a_1 - a_4)^2 + (a_2 + a_3)^2 .$$

The mapping f is called *constant* if $I_1 = 0$; f is said to be *conformal* if $I_2 = 0$; the geometrical signification is obvious. To each point $m \in M$, we get the induced quadratic mapping

(1.13) $$f_{**} : T_m(M) \to T_{f(m)}(N) ,$$

$$f_{**}(xv_1 + yv_2) = (b_1x^2 + 2b_2xy + b_3y^2)\, v_1^* + (b_4x^2 + 2b_5xy + b_6y^2)\, v_2^* ;$$

see [2]. Further, we get the mapping

(1.14) $$t : M \to T(N) , \quad t(m) \in T_{f(m)}(N) ; \quad t = (b_1 + b_3)v_1^* + (b_2 + b_4)\, v_2^* ;$$

t is the so-called *tension field*. The expressions

(1.15) $$J_1 = (b_1 + b_3)^2 + (b_4 + b_6)^2 , \quad J_2 = b_1^2 + 2b_2^2 + b_3^2 + b_4^2 + 2b_5^2 + b_6^2$$

are invariants of f as well; f is said to be *harmonic* if $J_1 = 0$, and it is *totally geodesic* if $J_2 = 0$.

2. Let us produce several integral formulas.

First of all, consider the 1-form

(2.1) $$\varphi_1 = \{(a_1 - a_4)(b_2 + b_4) - (a_2 + a_3)(b_1 - b_5)\}\, \omega^1 + \\ + \{(a_1 - a_4)(b_3 + b_5) - (a_2 + a_3)(b_2 - b_6)\}\, \omega^2 ;$$

φ_1 is invariant. Then

(2.2) $$\int_{\partial M} \varphi_1 = \int_M \{2L_1 - I_2(K + \mu K^*)\}\, \omega^1 \wedge \omega^2 ,$$

$$\text{where} \quad L_1 = \begin{vmatrix} b_1 - b_5 & b_2 - b_6 \\ b_2 + b_4 & b_3 + b_5 \end{vmatrix} .$$

For the invariant form

(2.3) $$\varphi_2 = \{(a_1 + a_4)(b_2 - b_4) - (a_2 - a_3)(b_1 + b_5)\}\, \omega^1 + \\ + \{(a_1 + a_4)(b_3 - b_5) - (a_2 - a_3)(b_2 + b_6)\}\, \omega^2 ,$$

we get

(2.4) $$\int_{\partial M} \varphi_2 = \int_M \{2L_2 + I_3(\mu K^* - K)\}\, \omega^1 \wedge \omega^2 ,$$

$$\text{where} \quad L_2 = \begin{vmatrix} b_1 + b_5 & b_2 + b_6 \\ b_2 - b_4 & b_3 - b_5 \end{vmatrix}$$

and

$$(2.5) \qquad I_3 = (a_1 + a_4)^2 + (a_2 - a_3)^2 .$$

Further,

$$(2.6) \qquad \tfrac{1}{2}\, dI_1 = (a_1 b_1 + a_2 b_2 + a_3 b_4 + a_4 b_5)\, \omega^1 + $$
$$+ (a_1 b_2 + a_2 b_3 + a_3 b_5 + a_4 b_6)\, \omega^2 ,$$

i,e,,

$$(2.7) \quad \frac{1}{2}\int_{\partial M} * \, dI_1 = \int_M \{a_1(c_1 + c_3) + a_2(c_2 + c_4) + a_3(c_5 + c_7) + a_4(c_6 + c_8) +$$
$$+ J_2 + I_1 K - 2\mu^2 K^*\}\, \omega^1 \wedge \omega^2 .$$

Analoguously,

$$(2.8) \qquad \tfrac{1}{2}\, dI_2 = \{(a_1 - a_4)(b_1 - b_5) + (a_2 + a_3)(b_2 + b_4)\}\, \omega^1 +$$
$$+ \{(a_1 - a_4)(b_2 - b_6) + (a_2 + a_3)(b_3 + b_5)\}\, \omega^2$$

and

(2.9)

$$\frac{1}{2}\int_{\partial M} * \, dI_2 = \int_M \{(a_1 - a_4)(c_1 + c_3 - c_6 - c_8) + (a_2 + a_3)(c_2 + c_4 + c_5 + c_7) +$$
$$+ (b_1 - b_5)^2 + (b_2 + b_4)^2 + (b_2 - b_6)^2 + (b_3 + b_5)^2 + I_2(K + \mu K^*)\}\, \omega^1 \wedge \omega^2 .$$

Next,

$$(2.10) \qquad \tfrac{1}{2}\, dI_3 = \{(a_1 + a_4)(b_1 + b_5) + (a_2 - a_3)(b_2 - b_4)\}\, \omega^1 +$$
$$+ \{(a_1 + a_4)(b_2 + b_6) + (a_2 - a_3)(b_3 - b_5)\}\, \omega^2$$

and

$$(2.11) \qquad \frac{1}{2}\int_{\partial M} * \, dI_3 = \int_M \{(a_1 + a_4)(c_1 + c_3 + c_6 + c_8) +$$
$$+ (a_2 - a_3)(c_2 + c_4 - c_5 - c_7) + (b_1 + b_5)^2 + (b_2 - b_4)^2 +$$
$$+ (b_2 + b_6)^2 + (b_3 - b_5)^2 + I_3(K - \mu K^*)\}\, \omega^1 \wedge \omega^2 .$$

Finally, consider the invariant 1-form

$$(2.12) \qquad \varphi_3 = \{(b_1 + b_3)(c_5 + c_7) - (b_4 + b_6)(c_1 + c_3)\}\, \omega^1 +$$
$$+ \{(b_1 + b_3)(c_6 + c_8) - (b_4 + b_6)(c_2 + c_4)\}\, \omega^2 .$$

From (1.10), we get

$$(2.13) \qquad \mathrm{d}(b_1 + b_3) - (b_4 + b_6)\,\tau_1^2 = (c_1 + c_3)\,\omega^1 + (c_2 + c_4)\,\omega^2\,,$$
$$\mathrm{d}(b_4 + b_6) + (b_1 + b_3)\,\tau_1^2 = (c_5 + c_7)\,\omega^1 + (c_6 + c_8)\,\omega^2\,.$$

The exterior differentiation yields

$$(2.14) \qquad \{\mathrm{d}(c_1 + c_3) - (c_2 + c_4)\,\omega_1^2 - (c_5 + c_7)\,\tau_1^2\} \wedge \omega^1 +$$
$$+ \{\mathrm{d}(c_2 + c_4) + (c_1 + c_3)\,\omega_1^2 - (c_6 + c_8)\,\tau_1^2\} \wedge \omega^2 = (b_4 + b_6)\,\mu K^* \omega^1 \wedge \omega^2\,,$$
$$\{\mathrm{d}(c_5 + c_7) - (c_6 + c_8)\,\omega_1^2 + (c_1 + c_3)\,\tau_1^2\} \wedge \omega^1 +$$
$$+ \{\mathrm{d}(c_6 + c_8) + (c_5 + c_7)\,\omega_1^2 + (c_2 + c_4)\,\tau_1^2\} \wedge \omega^2 = -(b_1 + b_3)\,\mu K^* \omega^1 \wedge \omega^2$$

and the existence of functions $e_1, \ldots, e_6$ such that

$$(2.15) \qquad \mathrm{d}(c_1 + c_3) - (c_2 + c_4)\,\omega_1^2 - (c_5 + c_7)\,\tau_1^2 = e_1\omega^1 + (e_2 - b_4\mu K^*)\,\omega^2\,,$$
$$\mathrm{d}(c_2 + c_4) + (c_1 + c_3)\,\omega_1^2 - (c_6 + c_8)\,\tau_1^2 = (e_2 + b_6\mu K^*)\,\omega^1 + e_3\omega^2\,,$$
$$\mathrm{d}(c_5 + c_7) - (c_6 + c_8)\,\omega_1^2 + (c_1 + c_3)\,\tau_1^2 = e_4\omega^1 + (e_5 + b_1\mu K^*)\,\omega^2\,,$$
$$\mathrm{d}(c_6 + c_8) + (c_5 + c_7)\,\omega_1^2 + (c_2 + c_4)\,\tau_1^2 = (e_5 - b_3\mu K^*)\,\omega^1 + e_6\omega^2\,.$$

By means of (2.13) and (2.15), we get the integral formula

$$(2.16) \qquad \int_{\partial M} \varphi_3 = \int_M (2L_3 - J_1\mu K^*)\,\omega^1 \wedge \omega^2\,, \quad \text{where} \quad L_3 = \begin{vmatrix} c_1 + c_3 & c_2 + c_4 \\ c_5 + c_7 & c_6 + c_8 \end{vmatrix}.$$

3. Let us explain the geometrical interpretation of the invariants L_i. Introduce the invariant operator

$$(3.1) \qquad * : T_m(M) \to T_m(M)\,, \quad m \in M\,; \quad *(xv_1 + yv_2) = -yv_1 + xv_2\,;$$

satisfying $\omega(*v) + *\omega(v) = 0$ for $v \in T_m(M)$, $\omega \in T_m^*(M)$. To f_{**} (1.13), consider the associated bilinear mapping

$$(3.2) \qquad \mathscr{L} : T_m(M) \times T_m(M) \to T_{f(m)}(N)\,,$$
$$\mathscr{L}(x^1v_1 + x^2v_2, y^1v_1 + y^2v_2) = (b_1x^1y^1 + b_2x^1y^2 + b_2x^2y^1 + b_3x^2y^2)\,v_1^* +$$
$$+ (b_4x^1y^1 + b_5x^1y^2 + b_5x^2y^1 + b_6x^2y^2)\,v_2^*\,.$$

Finally, consider the operator

$$(3.3) \qquad * : T_n(N) \to T_n(N)\,, \quad n \in N\,; \quad *(\xi v_1^* + \eta v_2^*) = -\eta v_1^* + \xi v_2^*\,.$$

Lemma 1. *Let $v \in T_m(M)$ be an arbitrary unit vector. Then*

$$(3.4) \qquad L_1 = \langle \mathscr{L}(v, v) + *\mathscr{L}(v, *v),\ \mathscr{L}(*v, *v) - *\mathscr{L}(v, *v) \rangle ,$$
$$L_2 = \langle \mathscr{L}(v, v) - *\mathscr{L}(v, *v),\ \mathscr{L}(*v, *v) + *\mathscr{L}(v, *v) \rangle .$$

Proof. Because of the invariance of L_1, L_2 and $\mathscr{L}$, we may choose the frames such that $v = v_1$, i.e., $*v = v_2$ at $m \in M$. Then

$$\mathscr{L}(v_1, v_1) = b_1 v_1^* + b_4 v_2^* ,$$
$$\mathscr{L}(v_1, v_2) = b_2 v_1^* + b_5 v_2^* , \quad \mathscr{L}(v_2, v_2) = b_3 v_1^* + b_6 v_2^* ,$$

and our Lemma follows. QED.

Lemma 2. *Let t (1.14) be the tension field, $v \in T_m(M)$ an arbitrary unit vector and*

$$(3.5) \qquad V := f_* v , \quad W := f_*(*v) .$$

Then

$$(3.6) \qquad L_3 = \langle *\nabla_V^* t,\ \nabla_W^* t \rangle .$$

Proof. We have

$$\nabla^* t = (b_4 + b_6)(\tau_1^2 - \Omega_1^2) v_1^* + (b_1 + b_3)(\Omega_1^2 - \tau_1^2) v_2^* +$$
$$+ \{(c_1 + c_3)\omega^1 + (c_2 + c_4)\omega^2\} v_1^* + \{(c_5 + c_7)\omega^1 + (c_6 + c_8)\omega^2\} v_2^* .$$

Notice that, for each form $\Omega \in T^*(N)$ and each vector $v \in T(M)$, we have $f^* \Omega(v) = \Omega(f_* v)$. Set $v = v_1$ at $m \in M$. Then

$$\nabla_V^* t = (c_1 + c_3) v_1^* + (c_5 + c_7) v_2^* , \quad \nabla_W^* t = (c_2 + c_4) v_1^* + (c_6 + c_8) v_2^* ,$$

and the Lemma follows. QED.

Lemma 3. *Let t (1.14) be the tension field and V, W be defined by (3.5). Let $\varepsilon = \pm 1$. Then*

$$(3.7) \qquad \nabla_V^* t + \varepsilon * \nabla_W^* t = 0$$

for each $v \in T(M)$ if and only if

$$(3.8) \qquad c_1 + c_3 - \varepsilon(c_6 + c_8) = c_2 + c_4 + \varepsilon(c_5 + c_7) = 0 .$$

Proof. Let $v = x v_1 + y v_2$. Then

$$\nabla_V^* t = \{(c_1 + c_3) x + (c_2 + c_4) y\} v_1^* + \{(c_5 + c_7) x + (c_6 + c_8) y\} v_2^* ,$$
$$\nabla_W^* t = \{(c_2 + c_4) x - (c_1 + c_3) y\} v_1^* + \{(c_6 + c_8) x - (c_5 + c_7) y\} v_2^*$$

and

$$\nabla^*_V t + \varepsilon * \nabla^*_W t = \{c_1 + c_3 - \varepsilon(c_6 + c_8)\}(xv_1^* - \varepsilon yv_2^*) +$$
$$+ \{c_2 + c_4 + \varepsilon(c_5 + c_7)\}(yv_1^* + \varepsilon xv_2^*).$$

The Lemma follows easily. QED.

4. Our main task is to obtain several typical geometric consequences of our integral formulas (2.2, 4, 7, 9, 11, 16). In all theorems, M and N are Riemannian manifolds, $\dim M = \dim N = 2$, $f : M \to N$ is an orientation preserving mapping, ∂M the boundary of M. All other notions have been explained above.

First of all, let us state the following

Lemma 4. *The condition*

(4.1) f is harmonic

and/or the condition

(4.2) for each $m \in M$ there is $\dim f_{**}(T_m(M)) \leqq 1$ and there exists a vector $0 \neq v \in T_m(M)$ such that $f_{**}(v) = 0$

implies

(4.3) $$L_1 \leqq 0, \quad L_2 \leqq 0.$$

Proof. The condition (4.1) is equivalent to $b_1 + b_3 = b_4 + b_6 = 0$. Hence

$$L_1 = -(b_1 - b_5)^2 - (b_2 + b_4)^2 \leqq 0, \quad L_2 = -(b_1 + b_2)^2 - (b_2 - b_4)^2 \leqq 0.$$

From (4.2), we get – see (1.13) – the existence of functions $\varrho, \sigma, B_1, B_2, B_3$ such that

$$b_1 = \varrho B_1, \quad b_2 = \varrho B_2, \quad b_3 = \varrho B_3, \quad b_4 = \sigma B_1, \quad b_5 = \sigma B_2, \quad b_6 = \sigma B_3$$

and

$$f_{**}(xv_1 + yv_2) = (B_1 x^2 + 2B_2 xy + B_3 y^2)(\varrho v_1^* + \sigma v_2^*).$$

Further,

$$L_1 = L_2 = (\varrho^2 + \sigma^2)(B_1 B_3 - B_2^2),$$

and our Lemma follows. QED.

Theorem 1. *Suppose:* (i) $L_1 \leqq 0$ *on* M, (ii) $K > 0$ *on* M, (iii) $K^* \geqq 0$ *on* $f(M) \subset N$, (iv) $I_2 = 0$ *on* ∂M. *Then f is conformal.*

Proof is a direct consequence of (2.2). QED.

Theorem 2. *Suppose:* (i) *M is compact,* (ii) $L_2 \leqq 0$ *on* M, (iii) $K > 0$ *on* M,

(iv) $K^* \leqq 0$ *on* $f(M) \subset N$. *Then* f *is a constant mapping. We may suppose* (i′) $J_2 = 0$ *on* ∂M *instead of* (i).

Proof. From (2.4), $a_1 + a_4 = a_2 - a_3 = 0$. Hence $\mu = -a_1^2 - a_2^2$; from $\mu \geqq 0$, we get $a_1 = a_2 = 0$. QED.

Theorem 3. *Suppose*: (i) f *is harmonic*, (ii) $K \geqq 0$ *on* M, (iii) $K^* \leqq 0$ *on* $f(M) \subset N$, (iv) $J_2 = 0$ *on* ∂M. *Then* f *is totally geodesic. Replacing* (ii) *by* (ii′) $K > 0$ *on* M, f *has to be a constant mapping.*

Proof is a consequence of (2.7). QED.

Theorem 4. *Suppose*: (i) *for each* $v \in T(M)$, *we have* $\nabla_V^* t + *\nabla_W^* t = 0$, t *being the tension field and* $V := f_* v$, $W := f_*(*v)$, (ii) $K > 0$ *on* M, (iii) $K^* \geqq 0$ *on* $f(M) \subset N$, (iv) $I_2 = 0$ *on* ∂M. *Then* f *is conformal.*

Proof follows from Lemma 3 and (2.9). QED.

Theorem 5. *Suppose*: (i) *for each* $v \in T(M)$, *we have* $\nabla_V^* t + *\nabla_W^* t = 0$, t *being the tension field and* $V := f_* v$, $W := f_*(*v)$, (ii) $K \geqq 0$ *on* M, (iii) $K^* \geqq 0$ *on* $f(M) \subset N$, (iv) $I_2 = 0$ *on* ∂M. *Then* f *is harmonic and we have*

$$I_2(K + \mu K^*) = 0 \tag{4.4}$$

at each point $m \in M$.

Proof. From Lemma 3 and (2.9),

$$b_1 - b_5 = b_2 + b_4 = b_2 - b_6 = b_3 + b_5 = 0$$

and f is to be harmonic. From (1.10),

$$\begin{aligned}
&c_1 - c_6 - a_4 K + a_1 \mu K^* = 0\,, && c_2 - c_7 + a_2 K + a_3 \mu K^* = 0\,,\\
&c_2 - c_7 - a_3 K - a_2 \mu K^* = 0\,, && c_3 - c_8 + a_1 K - a_4 \mu K^* = 0\,,\\
&c_2 + c_5 + a_2 K + a_3 \mu K^* = 0\,, && c_3 + c_6 + a_4 K - a_1 \mu K^* = 0\,,\\
&c_3 + c_6 + a_1 K - a_4 \mu K^* = 0\,, && c_4 + c_7 + a_3 K + a_2 \mu K^* = 0\,.
\end{aligned}$$

By the elimination of $c_1, \dots, c_8$ from these equations and from (3.8) for $\varepsilon = 1$, we get

$$(a_1 - a_4)(K + \mu K^*) = (a_2 + a_3)(K + \mu K^*) = 0\,,$$

i.e., (4.4). QED.

Theorem 6. *Suppose*: (i) M *is compact*, (ii) *for each* $v \in T(M)$, *we have* $\nabla_V^* t = *\nabla_W^* t$, t *being the tension field and* $V := f_*(v)$, $W := f_*(*v)$, (iii) $K > 0$ *on* M, *(iv) $K^* \leqq 0$ on $f(M) \subset N$. Then f is a constant mapping. Instead of (i), it is sufficient to suppose (i') $J_2 = 0$ on ∂M.*

Proof. From Lemma 3 and (2.11), we get $I_3 = 0$, i.e., $\mu = -a_1^2 - a_2^2$. From $\mu \geqq 0$, we get $a_1 = a_2 = 0$. QED.

Theorem 7. *Suppose: (i) M is compact, (ii) for each $v \in T(M)$, $\nabla_V^* t = *\nabla_W^* t$, t being the tension field and $V := f_*(v)$, $W := f_*(*v)$, (iii) $K \geqq 0$ on M, (iv) $K^* \leqq 0$ on $f(M) \subset N$. Then f is harmonic and we have*

$$I_3(K - \mu K^*) = 0 \tag{4.5}$$

at each point $m \in M$. Instead of (i), it is sufficient to suppose (i') $J_2 = 0$ on ∂M.

Proof. From Lemma 3 and (2.11),

$$b_1 + b_5 = b_2 - b_4 = b_2 + b_6 = b_3 - b_5 = 0,$$

and f is harmonic. From (1.10),

$$c_1 + c_6 + a_4 K - a_1 \mu K^* = 0, \quad c_2 + c_7 + a_2 K + a_3 \mu K^* = 0,$$

$$c_2 + c_7 + a_3 K + a_2 \mu K^* = 0, \quad c_3 + c_8 + a_1 K - a_4 \mu K^* = 0,$$

$$c_2 - c_5 + a_2 K + a_3 \mu K^* = 0, \quad c_3 - c_5 - a_4 K + a_1 \mu K^* = 0,$$

$$c_3 - c_6 + a_1 K - a_4 \mu K^* = 0, \quad c_4 - c_7 - a_3 K - a_2 \mu K^* = 0.$$

The elimination of $c_1, \dots, c_8$ from these equations and from (3.8) for $\varepsilon = -1$ implies

$$(a_1 + a_4)(K - \mu K^*) = 0, \quad (a_2 - a_3)(K - \mu K^*) = 0,$$

i.e., (4.5). QED.

Theorem 8. *Suppose: (i) $L_3 \geqq 0$ on M, (ii) $K^* < 0$ on $f(M) \subset N$, (iii) $f_*(T_m(M)) = T_{f(m)}(N)$ for each $m \in M$, (iv) $J_1 = 0$ on ∂M. Then f is a harmonic mapping.*

Proof. From (2.16), $J_1 = 0$ on M. QED.

Theorem 9. *Suppose: (i) M is compact, (ii) $J_1 = \text{const.} \neq 0$ on M, (iii) $K^* > 0$ or $K^* < 0$ on $f(M) \subset N$. Then $\dim f_*(T_m(M)) \leqq 1$ for each $m \in M$.*

Proof. From $J_1 = $ const. and (1.10), we get

$$(b_1 + b_3)(c_1 + c_3) + (b_4 + b_6)(c_5 + c_7) = 0,$$

$$(b_1 + b_3)(c_2 + c_4) + (b_4 + b_6)(c_6 + c_8) = 0$$