Časopis pro pěstování matematiky, roč. 101 (1976), Praha

A REMARK ON THE DIFFERENTIAL EQUATIONS ON THE SPHERE

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(Received October 15, 1975)

1. Let S^n be the unit-sphere in $\mathscr{R}^{n+1}$. A function $f : S^n \to \mathscr{R}$ is called *linear* if $f(m) = \langle m, a\rangle$, m being the position vector of S^n and a a constant vector. Let g be the metric tensor of S^n and ∇ the covariant differentiation with respect to it. Introduce the following differential operators for functions on S^n:

$$\Delta f = g^{ij}\nabla_i\nabla_j f, \tag{1.1}$$

$$\mathscr{L}f = \Delta f + nf, \tag{1.2}$$

$$\mathscr{M}f = \frac{\det(\nabla_i\nabla_j f)}{\det(g_{ij})} + f\Delta f + f^2; \tag{1.3}$$

Δ is, of course, the Laplacian, $\mathscr{M}$ is the so called Weingarten operator. The following assertion is known: *The only solutions* $f : S^n \to \mathscr{R}$ *of* $\mathscr{L}f = 0$ *or* $\mathscr{M}f = 0$ *resp. are linear.* For the proofs, see, p. ex., [1] and [2]. U. Simon [2] proves the linearity of solutions of a class of more general operators. In what follows, I propose, for $n = 2$, to present another class of operators with the desired property taking in regard the boundary conditions as well. Namely, I am going to prove the following theorems.

Theorem 1. *Let* $D \subset S^2$ *be a domain,* ∂D *its boundary and* $f : \bar{D} \to \mathscr{R}$ *a function. If*

$$\mathscr{L}f = 0 \quad \text{in} \quad D, \tag{1.4}$$

$$\mathscr{M}f = 0 \quad \text{on} \quad \partial D, \tag{1.5}$$

f is linear.

Theorem 2. *Let* $D \subset S^2$ *be a domain,* ∂D *its boundary and* $f : \bar{D} \to \mathscr{R}$ *a function. Let* $F : \mathscr{R} \to \mathscr{R}$ *be a function satisfying, for each* $t \in \mathscr{R}$,

$$F(t) > F'(t) \, . \, (t - F'(t)) \quad \text{or} \quad F(t) = 0 \quad \text{resp.} \tag{1.6}$$

If

$$\mathscr{M}f = F(\mathscr{L}f) \quad \text{in} \quad D, \tag{1.7}$$

$$(1.8) \qquad (\mathscr{L}f)^2 - 4F(\mathscr{L}f) = 0 \quad \text{on} \quad \partial D\,,$$

f is linear.

For the omitted details of the proofs, see [3].

2. On S^2, consider a domain G which may be covered by a system of tangent orthonormal frames $\sigma = \{m, v_1, v_2, v_3\}$. We then have

$$(2.1) \qquad dm = \omega^1 v_1 + \omega^2 v_2\,, \quad dv_1 = \omega_1^2 v_2 + \omega^1 v_3\,, \quad dv_2 = -\omega_1^2 v_1 + \omega^2 v_3\,,$$
$$dv_3 = --\omega^1 v_1 - \omega^2 v_2$$

with the usual integrability conditions. For a function $f : G \to \mathscr{R}$ introduce the covariant derivatives $f_i, f_{ij}, P, \ldots, S, T_1, \ldots, T_5$ with respect to σ by means of formulae (2.2), (2.4), (2.6) and (2.8):

$$(2.2) \qquad df = f_1\omega^1 + f_2\omega^2\,;$$

$$(2.3) \qquad (df_1 - f_2\omega_1^2) \wedge \omega^1 + (df_2 + f_1\omega_1^2) \wedge \omega^2 = 0\,;$$

$$(2.4) \qquad df_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2\,, \quad df_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2\,;$$

$$(2.5) \qquad \{df_{11} - 2f_{12}\omega_1^2\} \wedge \omega^1 + \{df_{12} + (f_{11} - f_{22})\,\omega_1^2\} \wedge \omega^2 = f_2\omega^1 \wedge \omega^2\,,$$
$$\{df_{12} + (f_{11} - f_{22})\,\omega_1^2\} \wedge \omega^1 + \{df_{22} + 2f_{12}\omega_1^2\} \wedge \omega^2 = -f_1\omega^1 \wedge \omega^2\,;$$

$$(2.6) \qquad df_{11} - 2f_{12}\omega_1^2 = P\omega^1 + Q\omega^2\,,$$
$$df_{12} + (f_{11} - f_{22})\,\omega_1^2 = (Q + f_2)\,\omega^1 + (R + f_1)\,\omega^2\,,$$
$$df_{22} + 2f_{12}\omega_1^2 = R\omega^1 + S\omega^2\,;$$

$$(2.7) \qquad \{dP - (3Q + 2f_2)\,\omega_1^2\} \wedge \omega^1 + \{dQ + (P - 2R - 2f_1)\,\omega_1^2\} \wedge \omega^2 =$$
$$= 2f_{12}\omega^1 \wedge \omega^2\,,$$
$$\{dQ + (P - 2R - 2f_1)\,\omega_1^2\} \wedge \omega^1 + \{dR + (2Q - S + 2f_2)\,\omega_1^2\} \wedge \omega^2 =$$
$$= 2(f_{22} - f_{11})\,\omega^1 \wedge \omega^2\,,$$
$$\{dR + (2Q - S + 2f_2)\,\omega_1^2\} \wedge \omega^1 + \{dS + (3R + 2f_1)\,\omega_1^2\} \wedge \omega^2 =$$
$$= -2f_{12}\omega^1 \wedge \omega^2\,;$$

$$(2.8) \qquad dP - (3Q + 2f_2)\,\omega_1^2 = T_1\omega^1 + T_2\omega^2\,,$$
$$dQ + (P - 2R - 2f_1)\,\omega_1^2 = (T_2 + 2f_{12})\,\omega^1 + (T_3 + 2f_{11})\,\omega^2\,,$$
$$dR + (2Q - S + 2f_2)\,\omega_1^2 = (T_3 + 2f_{22})\,\omega^1 + (T_4 + 2f_{12})\,\omega^2\,,$$
$$dS + (3R + 2f_1)\,\omega_1^2 = T_4\omega^1 + T_5\omega^2\,.$$

It is easy to see that, in our notation,

$$\mathscr{L}f = f_{11} + f_{22} + 2f\,, \quad \mathscr{M}f = f_{11}f_{22} - f_{12}^2 + f(f_{11} + f_{22} + f)\,. \tag{2.9}$$

From this

$$(\mathscr{L}f)^2 - 4\mathscr{M}f = (f_{11} - f_{22})^2 + 4f_{12}^2 \geqq 0\,, \tag{2.10}$$

and we have

$$\begin{aligned} \mathrm{d}\mathscr{L}f &= (P + R + 2f_1)\,\omega^1 + (Q + S + 2f_2)\,\omega^2\,, \\ \mathrm{d}\mathscr{M}f &= \{(f_{22} + f)\,P - 2f_{12}Q + (f_{11} + f)\,R + f_1\mathscr{L}f - 2f_2f_{12}\}\,\omega^1 + \\ &\quad + \{(f_{22} + f)\,Q - 2f_{12}R + (f_{11} + f)\,S + f_2\mathscr{L}f - 2f_1f_{12}\}\,\omega^2\,. \end{aligned} \tag{2.11}$$

On G, consider the 1-form

$$\begin{aligned} \tau = &\{(f_{11} - f_{22})\,(Q + f_2) + f_{12}(R - P)\}\,\omega^1 + \\ &+ \{(f_{11} - f_{22})\,(R + f_1) + f_{12}(S - Q)\}\,\omega^2\,. \end{aligned} \tag{2.12}$$

It may be shown that τ does not depend on the choice of the frames σ. We have

$$\begin{gathered} \mathrm{d}\tau = -2\{\Phi + \tfrac{1}{2}(\mathscr{L}f)^2 - 2\mathscr{M}f\}\,\omega^1 \wedge \omega^2 \\ \text{with} \quad \Phi = (Q + f_2)\,(Q - S) + (R + f_1)\,(R - P)\,; \end{gathered} \tag{2.13}$$

our main tool in proving Theorems 1 and 2 will be the Stokes formula $\int_{\partial D} \tau = \int_D \mathrm{d}\tau$.

First of all, let us prove that the suppositions of our Theorems imply $\Phi \geqq 0$ in D. Suppose (1.4). Then, see (2.11),

$$P + R + 2f_1 = 0\,, \quad Q + S + 2f_2 = 0\,, \tag{2.14}$$

and we have

$$\Phi = 2(Q + f_2)^2 + 2(R + f_1)^2 \geqq 0\,. \tag{2.15}$$

Next, let

$$\mathscr{M}f = 0 \quad \text{in} \quad D\,. \tag{2.16}$$

Then (2.11_2) implies

$$\begin{aligned} (f_{22} + f)\,(P - R) + \mathscr{L}f\,.\,(R + f_1) - 2f_{12}(Q + f_2) &= 0\,, \\ (f_{11} + f)\,(S - Q) + \mathscr{L}f\,.\,(Q + f_2) - 2f_{12}(R + f_1) &= 0\,. \end{aligned} \tag{2.17}$$

ſet $m \in D$ be a fixed point; the frames σ may be always chosen in such a way that $L_{12}(m) = 0$. If $\mathscr{L}f(m) \neq 0$, we have

$$\Phi(m) = (\mathscr{L}f)^{-1}\,\{(f_{11} + f)\,(Q - S)^2 + (f_{22} + f)\,(R - P)^2\}\,|_m\,.$$

Now, quite generally,

$$(f_{11}+f)\mathscr{L}f = \mathscr{M}f + f_{12}^2 + (f_{11}+f)^2\,, \quad (f_{22}+f)\mathscr{L}f = \mathscr{M}f + f_{12}^2 + (f_{22}+f)^2\,,$$

i.e, $\Phi(m) \geqq 0$. In the case $\mathscr{L}f(m) = 0$, there are two possibilities: a) $\mathscr{L}f = 0$ in a neighborhood of m, b) there is a sequence $\{m_i\}$, $m_i \to m$, such that $\mathscr{L}f(m_i) \neq 0$ for each m_i. The preceding results prove $\Phi(m) \geqq 0$ in these cases, too. Finally, consider the general supposition of Theorem 2. From (1.7) and (2.11), we get

$$\begin{aligned}(2.18)\qquad & (f_{22}+f)P - 2f_{12}Q + (f_{11}+f)R + f_1\mathscr{L}f - 2f_2f_{12} - \\ & - F'(P + R + 2f_1) = 0\,, \\ & (f_{22}+f)Q - 2f_{12}R + (f_{11}+f)S + f_2\mathscr{L}f - 2f_1f_{12} - \\ & - F'(Q + S + 2f_2) = 0\,,\end{aligned}$$

i.e.,

$$\begin{aligned}(2.19)\qquad & (f_{22}+f-F')(P-R) + (\mathscr{L}f - 2F')(R+f_1) - 2f_{12}(Q+f_2) = 0\,, \\ & (f_{11}+f-F')(S-Q) + (\mathscr{L}f - 2F')(Q+f_2) - 2f_{12}(R+f_1) = 0\,.\end{aligned}$$

Suppose $\mathscr{L}f - 2F'(\mathscr{L}f) = 0$, i.e., $\mathscr{M}f = F(\mathscr{L}f) = \frac{1}{4}(\mathscr{L}f)^2 + c$, $c = \text{const}$. The condition (1.6_1) implies $\frac{1}{4}t^2 + c > \frac{1}{2}t(t - \frac{1}{2}t)$, i.e, $c > 0$. On the other hand, (2.10) implies $-4c = (\mathscr{L}f)^2 - 4\mathscr{M}f \geqq 0$, which is a contradiction. Thus $\mathscr{L}f - 2F'(\mathscr{L}f) \neq 0$ in D. Let $m \in D$ be again a point, and suppose $f_{12}(m) = 0$. Then

$$\Phi(m) = (\mathscr{L}f - 2F')^{-1}\{(f_{11}+f-F')(Q-S)^2 + (f_{22}+f-F')(R-P)^2\}\,|_m\,.$$

It is easy to verify

$$\begin{aligned}&(f_{11}+f-F')(\mathscr{L}f - 2F') = F'^2 - F'\,.\,\mathscr{L}f + \mathscr{M}f + (f_{11}+f-F')^2\,, \\ &(f_{22}-f-F')(\mathscr{L}f - 2F') = F'^2 - F'\,.\,\mathscr{L}f + \mathscr{M}f + (f_{22}+f-F')^2\,;\end{aligned}$$

because of (1.7) and (1.6),

$$(f_{11}+f-F')(\mathscr{L}f - 2F') > 0\,, \quad (f_{22}+f-F')(\mathscr{L}f - 2F') > 0\,,$$

and $\Phi(m) \geqq 0$ follows.

By means of (2.10), we get

$$(2.20)\qquad f_{11} - f_{22} = f_{12} = 0 \quad \text{on} \quad \partial D\,.$$

in all cases. Thus $\tau = 0$ on ∂D, and we get

$$(2.21)\qquad f_{11} - f_{22} = f_{12} = 0 \quad \text{in} \quad D$$