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MARTINGALE CONVERGENCE TO THE POISSON DISTRIBUTION

G. K. EAGLESON, Cambridge

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1. INTRODUCTION

Consider a double array of random variables (r.vs), whose rows are martingale difference sequences, i.e. for each $n = 1, 2, \ldots$, we have r.vs $X_{n1}, \ldots, X_{nk_n}$ on a probability space $(\Omega, \mathscr{F}, P)$ with sub-σ-fields $\mathscr{F}_{n0} \subset \mathscr{F}_{n1} \subset \ldots \subset \mathscr{F}_{nk_n}$ of $\mathscr{F}$ such that X_{nj} is $\mathscr{F}_{nj}$ - measurable and $E(X_{nj} \mid \mathscr{F}_{n,j-1}) = 0$ almost surely (a.s.) for $j = 1, 2, \ldots, k_n$, where $k_n \to \infty$ as $n \to \infty$. Such arrays are called *martingale arrays.* Let

$$S_n = X_{n1} + \ldots + X_{nk_n}, \quad \sigma_{nj}^2 = E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}),$$

$$V_{nk}^2 = \sum_{j=1}^{k} \sigma_{nj}^2, \quad b_n = \max_{j \leq k_n} \sigma_{nj}^2.$$

A martingale array is called a *martingale elementary system* if it satisfies

a) $EX_{nj}^2 < \infty$, for all j and n;

b) there exists a finite constant C such that

$$\lim_{n \to \infty} P(V_{nk_n}^2 > C) = 0;$$

and

c) $b_n \to^P 0$ as $n \to \infty$.

If the r.vs $X_{n1}, \ldots, X_{nk_n}$ are independent for each n, then $V_{nk_n}^2$ is a.s. constant and equal to ES_n^2. The conditions (a), (b) and (c) then constitute the requirements for the triangular array to form an elementary system (see GNENDENKO [5], p. 316).

A sufficient condition (BROWN and EAGLESON [3]), for the row sums of a martingale elementary system to converge in law to the Poisson distribution with zero mean and parameter λ (written $\mathscr{P}(\lambda)$) is that

$$\text{(1)} \qquad \text{for all } \varepsilon > 0, \quad \sum_j E(X_{nj}^2 I(|X_{nj} - 1| > \varepsilon) \mid \mathscr{F}_{n,j-1}) \to^P 0, \quad \text{as} \quad n \to \infty,$$

and

$$\text{(2)} \qquad \sum_j E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) \to^P \lambda, \quad \text{as} \quad n \to \infty.$$

The conditions (1) and (2) involve both truncation and conditioning. In this paper, we will investigate the possibility of removing these difficulties by proving a martingale analogue of Alda's ([1]) condition for convergence of the row sums of an elementary system of independent r.vs to the Poisson distribution.

2. RESULTS

Theorem 1. *Let $\{X_{nj}, \mathscr{F}_{n,j}\}$ be a martingale array. The two conditions*

$$\sum_j X_{nj}(X_{nj} - 1) \to^P \lambda \quad as \quad n \to \infty\,, \tag{3}$$

and

$$\sum_j X_{nj}^2(X_{nj} - 1)^2 \to^{L^1} 0 \quad as \quad n \to \infty\,, \tag{4}$$

together imply (1) *and* (2).

Proof. First note that

$$E(\sum_j X_{nj}^2 I(|X_{nj} - 1| > \varepsilon)) \leqq \varepsilon^{-2} E(\sum_j X_{nj}^2(X_{nj} - 1)^2) \to 0\,,$$

so that (1) follows immediately from (4).

For each n, the sequence of r.vs,

$$U_{nj} = X_{nj}(X_{nj} - 1) - E(X_{nj}(X_{nj} - 1) \mid \mathscr{F}_{n,j-1}) =$$
$$= X_{nj}(X_{nj} - 1) - E(X_{nj}^2 \mid \mathscr{F}_{n,j-1})\,, \quad j = 1, \ldots, k_n\,,$$

is a sequence of martingale differences. Thus

$$E(\sum_j U_{nj})^2 = \sum_j E(U_{nj}^2) \leqq \sum_j E(X_{nj}^2(X_{nj} - 1)^2) \to 0\,.$$

So (2) follows from (3).

Corollary 1. *If $\{X_{nj}, \mathscr{F}_{nj}\}$ is a martingale array, for which*

$$\sum_j X_{nj}(X_{nj} - 1) \to^P \lambda \quad \text{as} \quad n \to \infty\,, \tag{3}$$

and

$$\lim_{n\to\infty} \sum_j EX_{nj}^2 = \lim_{n\to\infty} \sum_j EX_{nj}^3 = \lim_{n\to\infty} \sum_j EX_{nj}^4 = \lambda\,, \tag{5}$$

then (1) and (2) hold.

Corollary 2. *If $\{X_{nj}, \mathscr{F}_{nj}\}$ is a martingale elementary system satisfying either* (3) *and* (4) *or* (3) *and* (5), *then $S_n \to^{\mathscr{D}} \mathscr{P}(\lambda)$ as $n \to \infty$.*

Unfortunately, the moment condition (4) or some other, equally strong, moment condition seems to be indispensible in proving that (3) implies the conditions (1) and (2). In the reverse direction, most of these moment conditions may be removed at the cost of more delicate computations.

Theorem 2. *Let $\{X_{nj}, \mathcal{F}_{nj}\}$ be a martingale array. The three conditions:*

$$\text{for all } \varepsilon > 0, \quad \sum_j E(X_{nj}^2 I(|X_{nj} - 1| > \varepsilon) \mid \mathcal{F}_{n,j-1}) \to^p 0 \quad \text{as} \quad n \to \infty, \tag{1}$$

$$\sum_j E(X_{nj}^2 \mid \mathcal{F}_{n,j-1}) \to^p \lambda \quad \text{as} \quad n \to \infty, \tag{2}$$

and

$$\lim_{n\to\infty} \sum X_{nj}^2 = \lambda, \tag{6}$$

together imply

$$\max_{j \leqq k_n} |X_{nj}(X_{nj} - 1)| \to^p 0 \quad as \quad n \to \infty \tag{7}$$

and

$$\sum_j X_{nj}(X_{nj} - 1) \to^p \lambda \quad as \quad n \to \infty. \tag{3}$$

Proof. The proof is divided into a number of lemmas.

Lemma 1. *Under the conditions of the Theorem ((1), (2) and (6)), for all ε, $0 < \varepsilon < \frac{1}{2}$,*

$$\lim_{n\to\infty} \sum_j E(X_{nj}^2 I(|X_{nj} - 1| > \varepsilon)) = 0, \tag{8}$$

$$\lim_{n\to\infty} \sum_j E(X_{nj}^2 I(|X_{nj}| \leqq \varepsilon)) = 0. \tag{9}$$

Proof. Clearly

$$0 \leqq \sum_j E(X_{nj}^2 I(|X_{nj}| \leqq \varepsilon) \mid \mathcal{F}_{n,j-1}) \leqq$$

$$\leqq \sum_j E(X_{nj}^2 I(|X_{nj} - 1| > \varepsilon) \mid \mathcal{F}_{n,j-1}) \leqq \sum_j E(X_{nj}^2 \mid \mathcal{F}_{n,j-1}).$$

The hypotheses of the Theorem imply that the last term $\to^p \lambda$, and the two middle terms $\to^p 0$, as $n \to \infty$. As (6) holds, the result follows from Pratt's Theorem [8].

Set $M_n = \max_{j \leqq k_n} |X_{nj}(X_{nj} - 1)|$.

Lemma 2. *Under the conditions of the Theorem, $M_n \to^p 0$ as $n \to \infty$.*

Proof. For fixed $\varepsilon > 0$ ($\varepsilon \leqq \frac{1}{2}$), if $|x(x - 1)| > \varepsilon$, then there are neighbourhoods N_0 and N_1 of 0 and 1 respectively such that $x \notin N_0 \cup N_1$. Let

$$\eta = \tfrac{1}{2} \min(\text{length of } N_0, \text{ length of } N_1).$$

Then for all $x \notin N_0 \cup N_1$,

$$|x(x-1)| \leqq (1+\eta^{-1})\,x^2 .$$

Now

$$P(M_n > \varepsilon) \leqq \sum_j P(|X_{nj}(X_{nj}-1)| > \varepsilon) \leqq$$

$$\leqq \varepsilon^{-1}(1+\eta^{-1}) \sum_j E(X_{nj}^2\, I(|X_{nj}(X_{nj}-1)| > \varepsilon)) \leqq$$

$$\leqq \varepsilon^{-1}(1+\eta^{-1}) \sum_j E(X_{nj}^2\, I(|X_{nj}-1| > \eta)),$$

which tends to zero as $n \to \infty$ by Lemma 1.

Lemma 3 *Under the conditions of the Theorem, for all* $\varepsilon > 0$

$$\sum_j X_{nj}(X_{nj}-1)\, I(|X_{nj}| > \varepsilon) \to^P 0 \text{ as } n \to \infty.$$

Proof.

$$\Big|\sum_j X_{nj}(X_{nj}-1)\, I(|X_{nj}| > \varepsilon)\Big| \leqq$$

$$\leqq M_n \sum_j I(|X_{nj}| > \varepsilon) \leqq M_n \varepsilon^{-2} \sum_j X_{nj}^2\, I(|X_{nj}| > \varepsilon).$$

But $M_n \to^P 0$ and $\varepsilon^{-2} \sum_j X_{nj}^2\, I(|X_{nj}| > \varepsilon)$ is bounded in probability since its expectation is bounded by $\varepsilon^{-2} \sum_j E(X_{nj}^2) \leqq A$, where A is a constant independent of n.

Lemma 4. *Under the conditions of the Theorem, for all* $\varepsilon > 0$

$$\sum_j X_{nj}(X_{nj}-1)\, I(|X_{nj}| \leqq \varepsilon) - \sum_j E(X_{nj}(X_{nj}-1)\, I(|X_{nj}| \leqq \varepsilon) \mid \mathscr{F}_{n,j-1}) \to^{L^2} 0$$

$$as \quad n \to \infty .$$

Proof. For each n, the sequence of r.vs

$$W_{nj} = X_{nj}(X_{nj}-1)\, I(|X_{nj}| \leqq \varepsilon) - E(X_{nj}(X_{nj}-1)\, I(|X_{nj}| \leqq \varepsilon) \mid \mathscr{F}_{n,j-1})$$

is a sequence of martingale differences. So

$$E\Big(\sum_j W_{nj}\Big)^2 = \sum_j E(W_{nj}^2) \leqq \sum_j E(X_{nj}^2 (X_{nj}-1)^2\, I(|X_{nj}| \leqq \varepsilon)).$$

But $(X_{nj}-1)^2\, I(|X_{nj}| \leqq \varepsilon) \leqq (1+\varepsilon)^2$ and $\lim_{n\to\infty} \sum_j E(X_{nj}^2\, I(|X_{nj}| \leqq \varepsilon) = 0$ by Lemma 1. The proof is completed.

Finally, note that for all n and j $E(X_{nj}(X_{nj}-1) \mid \mathscr{F}_{n,j-1}) = E(X_{nj}^2 \mid \mathscr{F}_{n,j-1})$, so that combining the results of the above Lemmas, the proof of the Theorem follows if we establish the following Lemma:

Lemma 6. *For all* $\varepsilon > 0$,

$$\sum_j E(X_{nj}(X_{nj} - 1)\, I(|X_{nj}| > \varepsilon) \mid \mathscr{F}_{n,j-1}) \to^p 0 \quad \text{as} \quad n \to \infty\,.$$

Proof. Fix $\delta > 0$. Then

$$\Big|\sum_j E(X_{nj}(X_{nj} - 1)\, I(|X_{nj}| > \varepsilon \ \text{ and } \ |X_{nj} - 1| > \delta) \mid \mathscr{F}_{n,j-1})\Big| \leqq$$

$$\leqq (1 + \varepsilon^{-1}) \sum_j E(X^2_{nj}\, I(|X_{nj} - 1)| > \delta) \mid \mathscr{F}_{n,j-1}) \to^p 0$$

by hypothesis.

Also

$$\sum_j E(X_{nj}(X_{nj} - 1)\, I(|X_{nj}| > \varepsilon \ \text{ and } \ |X_{nj} - 1| \leqq \delta) \mid \mathscr{F}_{n,j-1}) \leqq$$

$$\leqq \delta\varepsilon^{-1} \sum_j E(X^2_{nj} \mid \mathscr{F}_{n,j-1}) \to^p \lambda\delta\varepsilon^{-1}\,,$$

by hypothesis. As δ may be chosen arbitrarily small, the result follows.

Corollary 3. *If* $\{X_{nj}, F_{nj}\}$ *is a martingale array satisfying the moment conditions,*

(5) $$\lim_{n\to\infty} \sum_j EX^2_{nj} = \lim_{n\to\infty} \sum_j EX^3_{nj} = \lim_{n\to\infty} \sum_j EX^4_{nj} = \lambda\,,$$

then the conditions (1) *and* (2) *together are equivalent to* (3).

3. REMARKS

1. Analogous results for convergence to normality were obtained by Scott [11]. Scott proved the martingale version of Raikov's Theorem [9] by showing that if $\{X_{nj}, \mathscr{F}_{nj}\}$ is a martingale array then the conditions

(10) $$\sum_j E(X^2_{nj} \mid \mathscr{F}_{n,j-1}) \to^p 1 \quad \text{as} \quad n \to \infty\,,$$

and

(11) $$\text{for all} \quad \varepsilon > 0\,, \quad \sum_j E(X^2_{nj}\, I(|X_{nj}| > \varepsilon) \mid \mathscr{F}_{n,j-1}) \to^p 0 \quad \text{as} \quad n \to \infty$$

are equivalent to

(12) $$\sum_j X^2_{nj} \to^p 1 \quad \text{as} \quad n \to \infty\,,$$

and

(13) $$\max_{j \leqq k_n} X^2_{nj} \to^p 0 \quad \text{as} \quad n \to \infty\,,$$

provided the following moment condition holds

$$\lim_{n\to\infty} \sum_j EX_{nj}^2 = 1 . \tag{14}$$

Although Scott's proof in [11] is in terms of a single sequence of martingale differences, the result is also true for triangular arrays. In fact, looking at Scott's proof, one sees that it is possible to prove that (10) and (11) together imply (12) and (13), and that (12) and (13) together with (14) imply (10) and (11). Thus it would seem that the martingale version of Lindeberg's conditions, (10) and (11), are stronger than the Raikov-type conditions (12) and (13). That this is in fact so has been shown recently by McLeish [6] who proved a martingale central limit theorem under rather weak moment restrictions.

As the situation for convergence to the Poisson distribution is similar (the conditions (1) and (2) seem to be stronger than (3)) one wonders whether a theorem about convergence to the Poisson could be proved without using the conditions (1) and (2).

2. At first sight, the use of the r.v.

$$\sum_j X_{nj}(X_{nj} - 1) \tag{15}$$

seems a little arbitrary. However, it should be remembered that in a discrete distribution it is often more natural to use factorial moments, so that (15) might well play the role of a sample variance. Further, if $Y(t)$, $t \geqq 0$, is a Poisson process with parameter λ, and $Z(t) = Y(t) - \lambda t$, then it is easy to show that if $0 = t_0 < t_1 < \ldots$ $\ldots < t_n = 1$,

$$\sum_{i=0}^{n} (Z(t_i) - Z(t_{i-1}))(Z(t_i) - Z(t_{i-1}) - 1) \to^{P} \lambda$$

as the partition becomes finer i.e., as $\max_i |t_i - t_{i-1}| \to 0$.

3. Though the moment condition (4) may seem excessive, even in the case of independent $\{X_{nj}\}$, (4) together with (6) is necessary and sufficient for the convergence in law of the row sums to a Poisson r.v. T_0 and for

$$\lim ES_n^4 = ET_0^4 .$$

(see Brown and Eagleson [2]).

4. There have been a number of results giving sufficient conditions for the convergence of row sums of arrays of dependent r.vs to the Poisson (see for example, Freedman [4], Mihailov [7] and Sevast'yanov [10]), but all of these papers deal with the special case of 0–1 r.v's.