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ON THE EXISTENCE OF PERIODIC BOUNDARY CONDITIONS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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In this note, we are concerned with the following non-linear second order differential equation

(1) $$x'' + Kx = f(t, x, x'),$$

(2) $$x(0) = x(\omega), \quad x'(0) = x'(\omega).$$

It shall be shown that if certain conditions are imposed on the function f, then there exists a unique solution of (1) satisfying the boundary conditions (2). In so doing we shall use a result reported by ĎURIKOVIČ in [1].

If in addition to our assumptions given below, the function f is ω-periodic function of t, i.e.

$$f(t + \omega, x, x') = f(t, x, x')$$

then the result of our paper gives an ω-periodic solution (see [3]).

In the sequel it is assumed that:

(A_1) $f(t, x, x')$ is a real-valued, continuous, bounded function with the domain $E = [0, T] \times R^2$, $T > 0$,

(A_2) $$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqq K_2\{|x_1 - x_2| + 1/\sqrt{|K|}\,|y_1 - y_2|\}, \quad K_2 > 0,$$

$$\frac{2K_2}{|K|} < 1.$$

For $\omega \in (0, \pi/\sqrt{K})$, $K > 0$, let $G(t, s)$ be the Green's function

$$(3)\qquad G(t,s)=\begin{cases}\dfrac{1}{2\sqrt{K}}\cdot\dfrac{\cos\sqrt{K}\left(\dfrac{\omega}{2}+s-t\right)}{\sin\sqrt{K}\,\dfrac{\omega}{2}} & \text{for}\quad 0\leqq s\leqq t\leqq\omega\,,\\[2ex] \dfrac{1}{2\sqrt{K}}\cdot\dfrac{\cos\sqrt{K}\left(\dfrac{\omega}{2}+t-s\right)}{\sin\sqrt{K}\,\dfrac{\omega}{2}} & \text{for}\quad 0\leqq t\leqq s\leqq\omega\,,\end{cases}$$

and, $\omega\in(0,+\infty)$, $K<0$, let $G(t,s)$ be defined as

$$(4)\qquad G(t,s)=\begin{cases}\dfrac{\exp\left[-\sqrt{|K|}\,(t-s)\right].\exp\left[\sqrt{|K|}\,w\right]+\exp\left[\sqrt{|K|}\,(t-s)\right]}{2\sqrt{|K|}\left[1-\exp\sqrt{|K|}\,\omega\right]} & \text{for}\quad s\leqq t\,,\\[2ex] \dfrac{\exp\left[-\sqrt{|K|}\,(s-t)\right].\exp\left[\sqrt{|K|}\,\omega\right]+\exp\left[\sqrt{|K|}\,(s-t)\right]}{2\sqrt{|K|}\left[1-\exp\sqrt{|K|}\,\omega\right]} & \text{for}\quad s\geqq t\end{cases}$$

and $K>0$,

$$(5)\qquad G_t(t,s)=\begin{cases}\dfrac{1}{2}\,\dfrac{\sin\sqrt{K}\left(\dfrac{\omega}{2}+s-t\right)}{\sin\sqrt{K}\,\dfrac{\omega}{2}} & \text{for}\quad 0\leqq s\leqq t\leqq\omega\,,\\[2ex] -\dfrac{1}{2}\,\dfrac{\sin\sqrt{K}\left(\dfrac{\omega}{2}+t-s\right)}{\sin\sqrt{K}\,\dfrac{\omega}{2}} & \text{for}\quad 0\leqq t\leqq s\leqq\omega\,,\end{cases}$$

$K<0$,

$$(6)\qquad G_t(t,s)=\begin{cases}\dfrac{\exp\left[\sqrt{|K|}\,(t-s)\right]-\exp\left[-\sqrt{|K|}\,(t-s)\right].\exp\left[\sqrt{|K|}\,\omega\right]}{2\left[1-\exp\sqrt{|K|}\,\omega\right]} & \text{for}\quad s\leqq t\,,\\[2ex] \dfrac{-\exp\left[\sqrt{|K|}(s-t)\right]+\exp\left[-\sqrt{|K|}(s-t)\right].\exp\left[\sqrt{|K|}\,\omega\right]}{2\left[1-\exp\sqrt{|K|}\,\omega\right]} & \text{for}\quad s\geqq t\,.\end{cases}$$

It is easy to see that

$$\int_0^\omega |G(t,s)|\,\mathrm{d}s\leqq\frac{1}{|K|}\,,$$

$$\int_0^\omega |G_t(t,s)|\,\mathrm{d}s\leqq\frac{1}{\sqrt{|K|}}\,.$$

Then, the equation (1) together with boundary conditions (2) is equivalent to solving the integrodifferential equation

$$(7) \qquad x(t) = \int_0^\omega G(t, s) f(s, x(s), x'(s)) \, ds .$$

With respect to (7), the sequence of Piccard's successive approximations $\{x_n(t)\}_1^\infty$ is defined by the equation

$$(8) \qquad x_n(t) = \int_0^\omega G(t, s) f(s, x_{n-1}(s), x'_{n-1}(s)) \, ds$$

and the sequence of derivatives is determined by

$$(9) \qquad x'_n(t) = \int_0^\omega G_t(t, s) f(s, x_{n-1}(s), x'_{n-1}(s)) \, ds .$$

Theorem 1. *Let $f(t, x, x')$ satisfy the assumptions $(A_1)-(A_2)$. Then there exists one and only one solution $x(t)$ of the problem (1) satisfying boundary conditions (2). Moreover Piccard's sequence of successive appriximations which is defined by (8) for any function $x_0(t)$ specified below converges uniformly to $x(t)$.*

Proof. We consider the space S of all continuous function with continuous first derivative on $[0, T]$, $[0, \omega] \subset [0, T]$. Let us define the distance

$$(10) \qquad \max_{t\in[0,T]} \frac{|x_1(t) - x_2(t)| + \frac{1}{\sqrt{|K|}} |x'_1(t) - x'_2(t)|}{|K|^p} , \quad p \in R$$

for arbitrary pair of elements $x_1(t), x_2(t) \in S$. It is obvious that the set S, on which the distance is defined by equality (10) is a complete metric space X (see [1]). Let us define an operator U on X as

$$(11) \qquad U\, x(t) = \int_0^\omega G(t, s) f(s, x(s), x'(s)) \, ds .$$

The operator U maps the space X into itself. Moreover

$$\frac{d}{dt} U\, x(t) = \int_0^\omega G_t(t, s) f(s, x(s), x'(s)) \, ds .$$

To complete the proof, we have to show that all the hypothesis of Luxemburg's Theorem 1, [1] are satisfied. Proof of the property 1°. Let $x_1(t), x_2(t)$ be two arbitrary functions from the space X, then

$$|U\, x_1(t) - U\, x_2(t)| \leqq \int_0^\omega |G(t, s)| \, |f(s, x_1(s), x'_1(s)) - f(s, x_2(s), x'_2(s))| \, ds \leqq$$

$$\leqq K_2 \int_0^{\omega} |G(t, s)| \frac{|x_1(s) - x_2(s)| + \frac{1}{\sqrt{|K|}} |x_1'(s) - x_2'(s)|}{|K|^p} |K|^p \, \mathrm{d}s \leqq$$

$$\leqq \frac{K_2}{|K|} d(x_1, x_2) \, |K|^p ,$$

and

$$\frac{1}{\sqrt{|K|}} \left| \frac{\mathrm{d}}{\mathrm{d}t} U \, x_1(t) - \frac{\mathrm{d}}{\mathrm{d}t} U \, x_2(t) \right| \leqq$$

$$\leqq \frac{K_2}{\sqrt{|K|}} \int_0^{\omega} |G_t(t, s)| \frac{|x_1(s) - x_2(s)| + \frac{1}{\sqrt{|K|}} |x_1'(s) - x_2'(s)|}{|K|^p} \leqq$$

$$\leqq \frac{K_2}{|K|} d(x_1(t), x_2(t)) \, . \, |K|^p \, .$$

From the given inequalities, it follows

$$d(U \, x_1(t), U \, x_2(t)) \leqq \frac{2K_2}{|K|} d(x_1(t), x_2(t)) \, .$$

Thus, the first condition of Theorem 1 is proved. It follows from the définition of the metric (10) that the arbitrary two elements of S have a finite distance. So conditions 2° and 3° of Theorem 1, [1] is obvious. Thereby, we have proved the existence and the uniqueness of the solution of the integro-differential equation (3), and the uniform convergence of successive approximation (8) to this solution for any function $x_0(t) \in S$.

In the following two theorems, we shall assume $K > 0$. It is easy to see that $G(t, s)$ is nonnegative in $0 \leqq s \leqq t \leqq \omega$ and in $0 \leqq t \leqq s \leqq \omega$. Since $\omega \in (0, \pi/\sqrt{K})$, and hence

$$\sin \sqrt{K} \frac{\omega}{2} \geqq \frac{2}{\pi} \sqrt{K} \frac{\omega}{2}$$

which implies

$$G(t, s) \leqq \frac{\pi}{2K\omega}$$

and

$$|G_t(t, s)| \leqq \frac{\pi}{2\sqrt{(K)}\,\omega} \, .$$

Now, let us assume that U is the operator defined by (11) and US is the set of all images S under the mapping U. If we denote the complete metric space which was obtained by the completion of the metric space $[US, d_2]$ in the sence of the distance

$$d_2(z_1(t), z_2(t)) = \underset{[0,T]}{\text{Max}} \left(|z_1(t) - z_2(t)| + \frac{1}{\sqrt{K}} |z'_1(t) - z'_1(t)| \right)$$

by $[S^*, d_2]$, then the following theorems hold.

Theorem 2. *Let $f(t, x, y)$ be a function defined and continuous on E, and let it fulfil the following conditions*

$$(A_3) \qquad |f(t, x, y)| \leqq \frac{K}{2\pi} t^p, \quad p \geqq 0, \quad (t, x, y) \in E,$$

$$(A_4) \qquad |f(t, x_1, y_1) - f(t, x_2, y_2)| < \frac{K}{2\pi t^r} \left(|x_1 - x_2|^q + \left[\frac{1}{\sqrt{K}} |y_1 - y_2| \right]^q \right),$$

for $(t, x_i, y_i) \in E$, $i = 1, 2$ where $q \geqq 1$, $0 < r < 1$, $r = p(q - 1)$ and

$$\frac{1}{(1 - r)} \left(\frac{1}{p + 1} \right)^{q-1} < 1 .$$

Then there exists one and only one solution $x(t) \in S^*$ of problem (1), (2). And moreover the sequence of Piccord's approximations defined by (8) for any $x_0(t) \in S$ in $[0, T]$ converges uniformly on $[0, T]$ to this unique solution.

Proof. The proof will be given similarly as that of Theorem 1. The set S^* is a subset of the set S. On the set we can define the distance

$$(12) \qquad d(z_1(t), z_2(t)) = \max_{t \in [0,T]} \frac{|z_1(t) - z_2(t)| + \frac{1}{K} |z'_1(t) - z'_2(t)|}{\omega^p} .$$

Again the set S^*, on which the distance is defined by equality (12) is a complete metric space X (see [1]). The operator U by (11) maps X into itself. To complete the proof we have to show all conditions of Luxemburg's theorem are satisfied. The proof of condition 1°. Let $z_1(t), z_2(t)$ be two arbitrary elements of X. Then from (11) and (A_3), we obtain

$$|z_1(t) - z_2(t)| \leqq \frac{K}{\pi} \int_0^\omega |G(t, s)| \, s^p \, ds < \frac{1}{2(p + 1)} \omega^p$$

and

$$\frac{1}{\sqrt{K}} |z'_1(t) - z'_2(t)| \leqq \frac{K}{\pi \sqrt{K}} \int_0^\omega |G_t(t, s)| \, s^p \, ds < \frac{1}{2(p + 1)} \omega^p .$$

Moreover by (A_4) and (12), we obtain

$$|U z_1(t) - U z_2(t)| \leqq \frac{K}{\pi}\int_0^{\omega} \frac{|z_1(s) - z_2(s)|^q + \left[\frac{1}{\sqrt{K}}|z_1'(s) - z_2'(s)|\right]^q}{s^r} G(t, s)\,\mathrm{d}s \leqq$$

$$\leqq \frac{K}{\pi}\,\omega^p\left(\frac{\omega^p}{p+1}\right)^{q-1}\cdot\frac{\pi}{2K\omega}\int_0^{\omega}\frac{|z_1(s) - z_2(s)| + \left|\frac{1}{\sqrt{K}}\right||z_1'(s) - z_2'(s)|}{\omega^p}\,\frac{\mathrm{d}s}{s^r} \leqq$$

$$\leqq \frac{\omega^p}{2}\cdot\frac{1}{(p+1)^{q-1}}\cdot\frac{d(z_1(t), z_2(t))}{1-r}$$

and similarly

$$\frac{1}{\sqrt{K}}\left|\frac{\mathrm{d}}{\mathrm{d}t}U z_1(t) - \frac{\mathrm{d}}{\mathrm{d}t}U z_2(t)\right| \leqq \frac{\omega^p}{2}\cdot\frac{1}{(p+1)^{q-1}}\cdot\frac{d(z_1(t), z_2(t))}{1-r}.$$

From the last two inequalities it follows that

$$d(U z_1(t), U z_2(t)) \leqq \frac{1}{(p+1)^{q-1}}\cdot\frac{1}{1-r}\,d(z_1(t), z_2(t)).$$

Hence Condition 1° is proven. Condition 2° follows directly from (12) because we have $d(x_n, x_{n+1}) \leqq (p+1)^{-1} < \infty$, for $n = 1, 2, \ldots$. From (12) we also obtain Condition (3°) too.

Remark. The assumption (A_3) of Theorem (2) guarantees the boundedness of the function $f(t, x, y)$ in E. In the following theorem we shall show that this assumption is not necessary.

Theorem 3. *Let the function $f(t, x, y)$ be continuous in E and let it satisfy the following conditions*

(A_5) $$|f(t, x, y)| \leqq \frac{K}{\pi}t^{-p}, \quad 0 < p < 1 \quad \text{for all} \quad (t, x, y) \in E.$$

(A_6) $$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqq \frac{K}{\pi}t^{p(q-1)}\left\{|z_1 - x_2|^q + \left[\frac{1}{\sqrt{K}}|y_1 - y_2|\right]^q\right\}$$

and

$$\left(\frac{1}{1-p}\right)^{q-1}\cdot\frac{1}{p(q-1)+1} < 1.$$

Then, there exists one and only one solution of problem (1) *satisfying boundary conditions* (2), *and moreover the sequence of Piccard's approximations defined by* (8) *for arbitrary function $x_0(t) \in S$ in $t \in [0, T]$, converges uniformly on $[0, T]$ to this unique solution.*

Proof. The operator U is defined by relation (11) as in proceeding theorems.