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ON THE DENSITY OF THE DILATIONS AND TRANSLATES OF FUNCTION IN L_1

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A well known theorem of WIENER [2] asserts that linear combinations of the translates $k(\xi + t)$ of a fixed function $k(\xi)$ in L_1 are dense in L_1, provided that the Fourier transform of k never vanishes on the real axis. Suppose that in addition to the translates of k we also allow dilations:

$$k(\delta^{-1}[\xi + t] \qquad (\delta > 0) \tag{1}$$

and then ask, for which k's is the span of this set of functions dense in L_1? Clearly, the span will be dense if k is the characteristic function of an interval – this just amounts to the fact that the class of step functions vanishing outside of a finite interval is dense in L_1. More generally we have the following result.

Theorem. *The necessary and sufficient condition for the set of functions* (1) *to have a dense span in L_1 is that $\int k(\xi)\, d\xi \neq 0$.*

Proof. The necessity follows from the observation that if $\int k(\xi)\, d\xi = 0$, then the integral of any linear combination of the functions (1) will also be zero:

$$\int \sum_{j=1}^{n} a_j\, k(\delta_j^{-1}[\xi + t_j])\, dx = \left(\sum_{j=1}^{n} a_j \delta_j\right) \int k(\xi)\, d\xi = 0\,.$$

Consequently, it would be impossible to approximate any function f in L_1 with a non-vanishing integral by such combinations of the functions (1).

To prove the sufficiency we employ a well-known criterion, based on the Hahn-Banach theorem, for denseness of the span of a set of elements $\mathscr{S}$ in a normed linear space X (cf. [1], p. 65); namely that the only bounded linear functional on X vanishing for each of the elements of $\mathscr{S}$ be the identically zero functional. By the Riesz representation theorem, all linear functionals $I(f)$ on L_1 are known to have the form $I(f) = \int f(\xi)\, g(\xi)\, d\xi$ where g is a bounded measurable function. It will, therefore, be sufficient to show that if the relations

$$\int k\left(\frac{x-\xi}{\delta}\right) g(\xi)\, d\xi = 0$$

hold for all x and all $\delta > 0$, then $g(\xi)$ must be zero almost everywhere. In turn, this will be an immediate consequence of the following result regarding approximations of the identity which is of some interest in itself.

Lemma. *Let $k \in L_1$ and $g \in L_\infty$, then*

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int k\left(\frac{x-\xi}{\delta}\right) g(\xi)\, d\xi = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int k\left(\frac{\xi}{\delta}\right) g(x-\xi)\, d\xi = g(x) \int k(\xi)\, d\xi \tag{2}$$

holds on the Lebesgue set of g.

Proof. We recall the definition of the Lebesgue set of g as the set of all points x at which the relation

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{|\xi| < \delta} |g(x-\xi) - g(x)|\, d\xi = 0$$

holds.

We will first prove (2) under the additional assumption that k is a bounded measurable function vanishing outside of a finite interval, say $k(\xi) = 0$ for $|\xi| \geqq a$. Letting $\|k\|_\infty$ denote the essential supremum of k, a straightforward computation then yields

$$\left|\frac{1}{\delta} \int k(\xi/\delta)\, g(x-\xi)\, d\xi - g(x) \int k(\xi)\, d\xi\right| = \frac{1}{\delta} \left|\int_{|\xi| \leqq a\delta} k(\xi/\delta)\, [g(x-\xi) - g(x)]\, d\xi\right| \leqq$$

$$\leqq (a\|k\|_\infty) \left(\frac{1}{a\delta} \int_{|\xi| \leqq a\delta} |g(x-\xi) - g(x)|\, d\xi\right);$$

from which (2) follows immediately in this case.

Suppose now that $k \in L_1$. To reduce this situation to the one just considered, we construct a sequence of bounded measurable functions $k^{(n)}$, $n = 1, 2, \ldots$, vanishing outside of $[-n, +n]$, and which converge to k in the L_1 norm:

$$\|k^{(n)} - k\|_1 \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3}$$

Setting

$$\gamma_n = \int k^{(n)}(\xi)\, d\xi \quad \text{and} \quad \gamma = \int k(\xi)\, d\xi\,,$$

this implies that

$$\gamma_n \to \gamma \quad \text{as} \quad n \to \infty\,. \tag{4}$$