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CONFIGURATION OF MUTUALLY PERSPECTIVE TRIANGLES

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A configuration of mutually perspective triangles [1] has been considered, and a Partially Balanced Incomplete Block Desing (PBIBD) corresponding to the configuration has been obtained.

1. CONFIGURATION OF MUTUALLY PERSPECTIVE TRIANGLES

The configuration consists of ${}^nc_2$ points and ${}^nc_3$ lines, with $(n - 2)$ lines passing through each of the points and three points lying on each of the lines.

The configuration is denoted by $\{({}^nc_2)_{n-2}, ({}^nc_3)_3\}$. We may denote the vertices of the configuration by (ij), the same as (ji), and its sides by ijk, so that the vertices ij, jk, ki, lie on the side ijk, with

$$(i, j, k = 1, 2, \ldots, n;\ i \neq j \neq k)\,.$$

The existence of the configuration is a consequence of the following theorem on mutually perspective triangles.

Theorem. *If $(n - 3)$ $(n \geqq 5)$ mutually perspective triangles have the same axis (i.e., all pairs of the triangles have the same axis) of perspectivity, then the ${}^{n-3}c_2$ centres of perspectivities form a configuration of $(n - 6)$ mutually perspective triangles.*

For $n = 5, 6, 7, 8$ the centres of perspectivities are, respectively: a point, three collinear points, vertices of a quadrilateral and vertices of a Desargues configuration.

For $n = 5$ the theorem reduces to the well known Desargues theorem on perspective triangles.

Proof of the theorem. Let ij $(i = 4, 5, \ldots, n;\ j = 1, 2, 3 \ldots)$ be the $3(n - 3)$ vertices of the $(n - 3)$ triangles.

Therefore, every pair of triangles formed of ik, il, im and tk, tl, tm are perspective from the vertex (it) and the side klm.

The centres of perspectivities it $(i, t = 4, 5, \ldots, n)$ are ${}^{n-3}c_2$ points, and the axis of perspectivity is 123, hence the Arguesian points are 12, 23, 31.

Similarly, these ${}^{n-3}c_2$ centres of perspectivities form a configuration of $(n - 6)$ mutually perspective triangles whose centres of perspectivities are ${}^{n-6}c_2$ points it $(i, t = 7, 8, \ldots, n)$ and whose axis is 456.

The dual of the configuration is the configuration $\{({}^{n}c_3)_3, ({}^{n}c_2)_{n-2}\}$.

Now, for $n = k - 1$, k we have the configurations

$$\{({}^{k-1}c_2)_{k-3}, ({}^{k-1}c_3)_3\} \quad \text{and} \quad \{({}^{k}c_2)_{k-2}, ({}^{k}c_3)_3\}\,,$$

respectively. The configuration for $n = k$ can be split into two configurations, namely

(i) $$\{{}^{(k-1)}k - 2, ({}^{k-1}c_2)_2\}$$

i.e., $a(k - 1)$-gon, with its diagonals joined; and

(ii) $$\{({}^{k-1}c_2)_{k-3}, ({}^{k-1}c_3)_3\}$$

such that the latter configuration is inscribed to the $(k - 1)$-gon.

Consequently, the configuration for $n = k$ can be split into (i) a $(k - 1)$-gon with its diagonals joined,

(ii) $$(i - 2)\text{-gons} \quad (i = k, \ldots, 7)$$

with their diagonals joined, and

(iii) a self-inscribed decagon.

For $k = 6$, we get $(15_4, 20_3)$ configuration, which can be split into a self-inscribed decagon and a pentagon with diagonals joined, such that the vertices of the self-inscribed decagon lie on the sides of the pentagon.

1a) SELF-INSCRIBED DECAGONS

For $n = 5$ we get the Desargues configuration, which can be read as a self-inscribed decagon. Therefore, from n elements, we get ${}^{n}c_5$ self-inscribed decagons.

Since the same desargues configuration can be read as 24 selfinscribed decagons [2], we get altogether $24 \,.\, {}^{n}c_5$ selfinscribed decagons.

1b) PAIRS OF MUTUALLY INSCRIBED PENTAGONS

Since the same Desargues configuration reveals 6 pairs of mutually inscribed pentagons [2], we get totally $6 \,.\, {}^{n}c_5$ pairs of mutually inscribed pentagons from the configuration.

1c) PAIRS OF QUADRILATERALS INSCRIBED INTO QUADRANGLES

The same Desargues configuration reveals five pairs of quadrilaterals and quadrangles, the former inscribed to the latter. Therefore, the total number of pairs of quadrilaterals and quadrangles is $5 \, . \, {}^n c_5$.

The group G of symmetries of the configuration of mutually perspective triangles which leaves both the set P of ${}^n c_2$ points and the set L of ${}^n c_3$ lines invariant is the group of permutations of the ${}^n c_2$ points which preserves the set L of lines.

The stabilizers of

(i) a point, say 12, is the permutation group of $2!\,(n-2)!$ permutation operations, as product of $(n-2)!$ permutations of $3, 4, \dots, n$ and two of $1, 2$;

(ii) a line, say 123, is the permutation group of $3!\,(n-3)!$ permutation operations as product of $(n-3)!$ permutations of $4, \dots, n$ and $3!$ of $1, 2, 3$.

Hence G has order

$${}^n c_2 \, . \, 2!\,(n-2)! = {}^n c_3 \, . \, 3!\,(n-3)! = n!\,.$$

If we also allow reciprocity which interchanges P and L we obtain the group G' of order $2n!$.

2. A PBIBD FROM THE CONFIGURATION

Considering the lines and points of the configuration as blocks and objects, respectively, we get a PBIB Design, with the parameters

$$b = {}^n c_3\,, \quad v = {}^n c_2\,, \quad r = n-2\,, \quad k = 3\,,$$

$$\lambda_1 = 1\,, \quad \lambda_2 = 0\,, \quad n_1 = 2(n-2)\,, \quad n_2 = {}^{n-2} c_2$$

and

$$P^1_{ij} = \begin{pmatrix} n-2, & n-3 \\ n-3, & {}^{n-3} c_2 \end{pmatrix}, \quad P^2_{ij} = \begin{pmatrix} 4, & 2(n-4) \\ 2(n-4), & {}^{n-4} c_2 \end{pmatrix}$$

This arrangement can be made in $n!$ ways.

The above design can be split into the following designs:

(1) BIBD's with the parameters

$$b = {}^{i-2} c_2\,, \quad v = i-2\,, \quad r = i-3\,, \quad k = 2\,, \quad \lambda = 1$$

with $i = 7, 8, \dots, k, (k+1)$

and

(2) a PBIBD with the parameters

$$b = v = 10\,, \quad k = r = 3\,; \quad \lambda_1 = 1\,, \quad n_1 = 6\,, \quad \lambda_2 = 0\,, \quad n_2 = 3$$