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ON THE MINIMUM DEGREE AND EDGE-CONNECTIVITY OF A GRAPH

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Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set of G and the edge set of G, respectively. If $v \in V(G)$, then we denote by $\deg_G v$ the degree of v in G. Moreover, we denote by $\delta(G)$ and $\Delta(G)$ the minimum degree of G and the maximum degree of G, respectively. If U is a nonempty subset of $V(G)$, then we denote by $\langle U \rangle_G$ the graph G' such that $V(G') = U$ and

$$E(G') = \{e \in E(G);\ e \text{ is incident with no vertex in } V(G) - U\}\ .$$

Let G be a nontrivial connected graph. We say that a set $S \subseteq E(G)$ is a cut set of G, if the graph $G - S$ is disconnected. A cut set S with $|S| = n$ is referred to as an n-cut set. We denote by $\varkappa_1(G)$ the minimum integer k such that there is a k-cut set of G; the integer $\varkappa_1(G)$ is called the edge-connectivity of G. (The terms not defined here can be found in M. BEHZAD and G. CHARTRAND [1].)

It is obvious that for any nontrivial connected graph G, $\varkappa_1(G) \leqq \delta(G)$. A sufficient condition for $\varkappa_1(G) = \delta(G)$ is due to D. R. LICK [3]; note that Lick's result is an analogue of R. HALIN's theorem on the vertex-connectivity [2]. In the present note it will be shown that an analysis of a nontrivial connected graph from the point of view of its edge-connectivity can lead to an upper bound for the minimum degree. In fact, we obtain an upper bound for a more general characteristic: if G is a graph and U is a nonempty subset of $V(G)$, then we denote

$$\deg_G U = \min \{\deg_G u;\ u \in U\}\ .$$

Obviously, $\delta(G) = \deg_G V(G)$.

Let G be a nontrivial connected graph, and let U be a nonempty subset of $V(G)$. We denote $f_G(U) = \Delta(G_0)$, where G_0 is the spanning subgraph of $\langle U \rangle_G$ with the property that $e \in E(G_0)$ if and only if $e \in E(\langle U \rangle_G)$ and $\varkappa_1(G - e) = \varkappa_1(G)$. We denote by $h_G(U)$ the minimum integer i such that there is an i-cut set R_0 of G with the property that for at least one component F_0 of the graph $G - R_0$ it holds that $V(F_0) \subseteq U$. Obviously, $\varkappa_1(G) \leqq h_G(U)$. It is clear that if U_1 and U_2 are subsets of $V(G)$ such that $\emptyset \neq U_1 \subseteq U_2$, then $f_G(U_1) \leqq f_G(U_2)$ and $h_G(U_2) \leqq h_G(U_1)$. Denote $f_G = f_G(V(G))$. Clearly, $h_G(V(G)) = \varkappa_1(G)$.

The following theorem is the main result of this note:

Theorem. *Let G be a nontrivial connected graph, and let U be a nonempty subset of $V(G)$. Then*

$$h_G(U) \leqq \deg_G U \leqq \max(f_G(U), h_G(U)).$$

Proof. It is obvious that for each $u \in U$, the set of edges incident with u in G is a cut set of G. Therefore, $h_G(U) \leqq \deg_G U$.

We shall prove the inequality $\deg_G U \leqq \max(f_G(U), h_G(U))$. Clearly, there is a nonempty subset U_0 of U such that $h_G(U_0) = h_G(U)$ and that for each nonempty subset U' of U, $|U'| < |U_0|$ implies $h_G(U') > h_G(U)$. Obviously, $U_0 \neq V(G)$. We denote by F the graph $\langle U_0 \rangle_G$. It is obvious that there is an $h_G(U)$-cut set R of G such that F is a component of $G - R$. It is easy to see that $E(F) \cap R = \emptyset$. Denote $n = |U_0|$. Obviously,

$$\text{(1)} \qquad \deg_G U \leqq \deg_G U_0 \leqq \Delta(F) + \left[\frac{h_G(U)}{n}\right] \leqq n - 1 + \frac{h_G(U)}{n}.$$

(Note that if x is a real number, then $[x]$ denotes the maximum integer j such that $j \leqq x$.)

Let $n \leqq h_G(U)$. If $h_G(U) < \deg_G U$, then it follows from (1) that $h_G(U).(n - 1) < n(n - 1)$, and thus $h_G(U) < n$, which is a contradiction. Hence $\deg_G U \leqq h_G(U) \leqq \max(f_G(U), h_G(U))$.

Let $h_G(U) < n$. From (1) it follows that $\deg_G U \leqq \Delta(F)$. We distinguish two cases:

(I) For each $e \in E(F)$, $\varkappa_1(G - e) = \varkappa_1(G)$. Then $f_G(U) \geqq \Delta(F)$. Therefore, $\deg_G U \leqq \Delta(F) \leqq f_G(U) \leqq \max(f_G(U), h_G(U))$.

(II) There exists $e \in E(F)$ such that $\varkappa_1(G - e) \neq \varkappa_1(G)$. Then there exists a $\varkappa_1(G)$-cut set S of G such that $e \in S$. Obviously, the graph $G - S$ has precisely two components, say G_1 and G_2, and $E(G_1) \cap S = \emptyset = E(G_2) \cap S$. We denote by H the graph $G - U_0$. It is easy to see that $E(H) \cap R = \emptyset$. Next, we denote $V_{11} = U_0 \cap V(G_1)$, $V_{12} = U_0 \cap V(G_2)$, $V_{21} = V(H) \cap V(G_1)$, and $V_{22} = V(H) \cap V(G_2)$. Finally, we denote by

$$E_1, \ldots, E_5, \text{ and } E_6$$

the set of all $e \in R \cup S$ with the property that e is incident

$$\text{with } V_{11} \text{ and } V_{21},$$

$$\text{with } V_{12} \text{ and } V_{22},$$

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and

$$\text{with } V_{12} \text{ and } V_{21},$$

respectively.

It is clear the sets $E_1, \ldots, E_5, E_6$ are mutually disjoint, $R = E_1 \cup E_2 \cup E_5 \cup E_6$ and $S = E_3 \cup E_4 \cup E_5 \cup E_6$. Since $E(F) \cap S \neq \emptyset$, we have $V_{11} \neq \emptyset \neq V_{12}$. Since $V(H) \neq \emptyset$, we have that either $V_{21} \neq \emptyset$ or $V_{22} \neq \emptyset$. Without loss of generality we assume that $V_{21} \neq \emptyset$. We distinguish two subcases:

(1) $V_{22} = \emptyset$. Then $E_2 = E_4 = E_5 = \emptyset$. Therefore $S = E_3 \cup E_6$. This implies that $h_G(V_{12}) \leqq \varkappa_1(G) \leqq h_G(U)$, which is a contradiction.

(2) $V_{22} \neq \emptyset$. Then both $E_1 \cup E_4 \cup E_6$ and $E_2 \cup E_4 \cup E_5$ are cut sets of G. Therefore, $|E_1 \cup E_4 \cup E_6| \geqq \varkappa_1(G)$ and $|E_2 \cup E_4 \cup E_5| \geqq \varkappa_1(G)$. Clearly, $E_1 \cup \cup E_3 \cup E_5$ and $E_2 \cup E_3 \cup E_6$ are also cut sets of G. Since $h_G(V_{11}) > h_G(U)$ and $h_G(V_{12}) > h_G(U)$, we have $|E_1 \cup E_3 \cup E_5| > h_G(U)$ and $|E_2 \cup E_3 \cup E_6| > h_G(U)$. Thus $2|R| + 2|S| = 2\, h_G(U) + 2\, \varkappa_1(G) < |E_1 \cup E_3 \cup E_5| + |E_2 \cup E_3 \cup E_6| + |E_1 \cup \cup E_4 \cup E_6| + |E_2 \cup E_4 \cup E_5| \leqq 2|R| + 2|S|$, which is a contradiction.

Hence the proof is complete.

Proofs of the following corollaries are omitted:

Corollary 1. *Let G be a nontrivial connected graph. Then $\varkappa_1(G) \leqq \delta(G) \leqq \leqq \max(f_G, \varkappa_1(G))$.*

Corollary 2. *Let G be a nontrivial connected graph, and let U be a nonempty subset of $V(G)$. If $f_G(U) \leqq h_G(U)$, then $\deg_G U = h_G(U)$.*

Corollary 3. *Let G be a nontrivial connected graph, and let n be a positive integer such that $n \geqq \varkappa_1(G)$. Then there exists a vertex u of G such that $\deg_G u = n$ if and only if there exists a nonempty subset U of $V(G)$ such that $f_G(U) \leqq h_G(U) = n$.*

Corollary 4. *Let G be a nontrivial connected graph. Then $\delta(G) = \varkappa_1(G)$ if and only if there exists a nonempty subset U of $V(G)$ such that $f_G(U) \leqq h_G(U) = \varkappa_1(G)$.*