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A CONSTRUCTION OF TOLERANCES ON MODULAR LATTICES

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It is well-known that there exists a one-to-one correspondence between congruences and ideals in rings and Ω-groups (see [4]) and between congruences and normal subgroups in groups. This correspondence exists also between congruences and ideals in Boolean algebras (see [1] or [5]), however, an analogous correspondence does not exist for distributive lattices in the general case, as is shown in [5]. It is only proved in [3] (Theorem 2.2) that each ideal of a lattice L is a kernel of at least one congruence relation if and only if L is distributive. The aim of this paper is to give a relationship between ideals and compatible tolerances for modular lattices.

1.

By a *tolerance relation*, or briefly a *tolerance*, on a set A we mean a reflexive and symmetric binary relation on A. Thus each equivalence relation on A is a tolerance relation on A.

Let $\mathfrak{A} = (A, F)$ be an algebra with the support A and a set F of fundamental operations. Further, let T be a tolerance relation on the support A. The relation T is called a *compatible tolerance relation* on $\mathfrak{A}$ (or briefly a *compatible tolerance* on $\mathfrak{A}$) if for each n-ary $f \in F$, $n \geqq 1$, and for arbitrary $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ such that $a_i\, T\, b_i$ $(i = 1, \ldots, n)$ we have also $f(a_1, \ldots, a_n)\, T f(b_1, \ldots, b_n)$.

Especially, each congruence on $\mathfrak{A}$ is a compatible tolerance on $\mathfrak{A}$. The concept of compatible tolerance has been introduced for algebraic structures by B. ZELINKA in [6] and studied for lattices in [7] and [8].

Definition 1. Let $\mathfrak{A} = (A, F)$ be an algebra, $S = \{A_\gamma, \gamma \in \Gamma\}$ a system of subsets $A_\gamma \subseteq A$. S is called a *covering* of $\mathfrak{A}$ if $\bigcup_{\gamma \in \Gamma} A_\gamma = A$. The covering S is called *compatible* on $\mathfrak{A}$, if for each n-ary $f \in F$ and arbitrary $\gamma_1, \ldots, \gamma_n$ there exists $\gamma_0 \in \Gamma$ such that $a_i \in A_{\gamma_i}$ $(i = 1, \ldots, n)$ imply $f(a_1, \ldots, a_n) \in A_{\gamma_0}$.

Clearly, if Θ is a congruence on an algebra $\mathfrak{A}$, then the system of all classes of the partition of A induced by Θ forms a compatible covering of $\mathfrak{A}$.

Definition 2. Let $\mathfrak{A} = (A, F)$ be an algebra, $S = \{A_\gamma, \gamma \in \Gamma\}$ a covering of $\mathfrak{A}$. The binary relation $T(S)$ defined on A by the rule

$$a\ T(S)\ b \text{ if and only if there exists } \gamma_0 \in \Gamma \text{ such that } a, b \in A_{\gamma_0}$$

is called *induced* by S.

It is clear that $T(S)$ is a tolerance relation on A for an arbitrary covering S of $\mathfrak{A}$. If S is a partition of A, then $T(S)$ is an equivalence on A.

Lemma 1. *Let $\mathfrak{A} = (A, F)$ be an algebra and S a compatible covering of $\mathfrak{A}$. Then the relation $T(S)$ induced by S is a compatible tolerance relation on $\mathfrak{A}$.*

The proof is clear and follows directly from Definition 1.

Let L be a lattice. By $\vee$ or $\wedge$ the operation join or meet on L, respectively, is denoted. Denote by $\leqq$ the lattice ordering on L. If $a, b \in L$ are incomparable, i.e. neither $a \leqq b$ nor $b \leqq a$, then we symbolize it by $a \parallel b$. By the symbol $J(a)$ we denote the *principal ideal of L generated by a*.

Notation. Let L be a lattice, $a \in L$, and J be an ideal of L. Denote $a \vee J = \{a \vee j; j \in J\}$.

Theorem 1. *Let L be a lattice. Then the following two conditions are equivalent:*

(a) *L is modular;*

(b) *for each ideal J of L and each element $a \in L$ the set $a \vee J$ is a convex sublattice of L.*

Proof. Let (a) be valid, $a \in L$, and let J be an ideal of L. Let $j \in J$ and $x \in [a, a \vee j]$. From a result of Croisot [2] it follows that $x \in a \vee J$. Hence $a \vee J$ is a convex subset of L. Let $x, y \in a \vee J$. Then there exist $i_1, i_2 \in J$ such that $x = a \vee i_1$, $y = a \vee i_2$. Thus

$$x \vee y = (a \vee i_1) \vee (a \vee i_2) = a \vee (i_1 \vee i_2) \in a \vee J\,,$$

$$x \wedge y = (a \vee i_1) \wedge (a \vee i_2) = a \vee (i_1 \wedge (a \vee i_2)) = a \vee i \in a \vee J\,,$$

where $i = i_1 \wedge (a \vee i_2)$. Hence (b) holds.

Conversely, assume that (a) does not hold. It is well-known that then L must contain a five-element non-modular sublattice $\{x_0, x_1, x_2, x_3, x_4\}$ such that $x_0 < x_2 < x_1$, $x_0 < x_3 < x_4 < x_1$. Put $J = J(x_2)$, $a = x_3$. Then clearly $x_1, x_3 \in a \vee J$. Suppose that $x_4 = a \vee j$ for some $j \in J$. Then $j \leqq x_4$ and from $j \in J(x_2)$ we have $j \leqq x_2$. Hence $j \leqq x_2 \wedge x_4 = x_0$. Thus

$$x_4 = a \vee j \leqq a \vee x_0 = x_3 \vee x_0 = x_3\,,$$

which is a contradiction with $x_3 < x_4$. Hence $a \vee J$ fails to be a convex subset in L.

Lemma 2. *Let L be a lattice and J an ideal of L. Then $S_J = \{a \vee J,\ a \in L\}$ is a covering of L.*

Proof. Let $a \in L$, $x \in J$. Then $a \wedge x \in J$, thus $a = a \vee (x \wedge a) \in a \vee J$.

Definition 3. Let L be a lattice and J an ideal of L. The covering $S_J = \{a \vee J,\ a \in L\}$ is called *induced by J* and the tolerance relation $T(S_J)$ induced by S_J is called *tolerance on L induced by the ideal J*. For the sake of brevity, denote by $T_J = T(S_J)$ the tolerance induced by J.

2.

Now, we have two natural problems: the first, for which ideal J of L the relation T_J is a compatible tolerance on L, and the second, for which J is a compatible tolerance which is not a congruence on L. This first problem is considered in what follows for the case of modular lattices.

Definition 4. Let L be a lattice and $c \in L$. If for each $a, b \in L$ c fulfils the identity

$$(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c\,,$$

c is called a *semi-distributive element.*

Theorem 2. *Let L be a modular lattice and $j \in L$ a semi-distributive element of L. If J is the principal ideal of L generated by j, then T_J is a compatible tolerance relation on L.*

Proof. By Lemma 2, T_J is a tolerance relation on L. It remains to prove that T_J is compatible on L. If the covering $S_J = \{x \vee J,\ x \in L\}$ induced by J is a compatible covering of L, then, by Lemma 1, T_J is a compatible tolerance on L. Accordingly, it suffices to prove only the compatibility of S_J.

Let $a, b \in L$, $x \in a \vee J$, $y \in b \vee J$. Then there exist $i_1, i_2 \in J$ such that $x = a \vee i_1$, $y = b \vee i_2$. Evidently, $x \vee y = (a \vee b) \vee i$ where $i = i_1 \vee i_2 \in J$, thus $x \vee y \in (a \vee b) \vee J$.

Further, we have

$$(1^\circ) \qquad x \wedge y = (a \vee i_1) \wedge (b \vee i_2) \geqq (a \wedge b) \vee (i_1 \wedge i_2) \in (a \wedge b) \vee J\,.$$

As $J = J(j)$, it is $i \leqq j$ for each $i \in J$. Then

$$x \wedge y = (a \vee i_1) \wedge (b \vee i_2) \leqq (a \vee j) \wedge (b \vee j)\,.$$

However, j is a semi-distributive element, thus

$$(2^\circ) \qquad x \wedge y \leqq (a \wedge b) \vee j \in (a \wedge b) \vee J\,.$$