Časopis pro pěstování matematiky, roč. 101 (1976), Praha

THE LYAPUNOV STABILITY OF THE TIMOSHENKO TYPE EQUATION

JAROSLAV BARTÁK, Praha

(Received October 25, 1974)

The purpose of this paper is the investigation of the global exponential stability, respectively the stability of the zero solution of the equation

$$(1)\qquad u''''(t) + a\,u'''(t) + (b_1 A^{1/2} + b_2 I)\,u''(t) + (c_1 A^{1/2} + c_2 I)\,u'(t) + \\ + (d_1 A + d_2 A^{1/2} + d_3 I)\,u(t) = 0$$

where A is a selfadjoint, strictly positive linear operator in a Hilbert space H; I is the identity operator in H; $a, b_1, b_2, c_1, c_2, d_1, d_2, d_3$ are real constants.

Under the solution of (1) we understand a function u from the space $\mathscr{U} = \{u : \langle 0, \infty) \to H \mid u^{(j)} \in C(\mathscr{D}(u), \mathscr{D}(A^{(4-j)/4})),\ j = 0, 1, 2, 3\}$, fulfilling the equation (1) on $\langle 0, \infty)$.

Let us define the norm $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$ by the relation

$$\|u(t)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} = \\ = \|(u(t), u'(t), u''(t), u'''(t))\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} = \\ = [\|A\,u(t)\|^2 + \|A^{3/4}\,u'(t)\|^2 + \|A^{1/2}\,u''(t)\|^2 + \|A^{1/4}\,u'''(t)\|^2]^{1/2}$$

for $u \in \mathscr{U}$ and $t \in \langle 0, \infty)$, ($\|\cdot\|$ is the norm in the space H).

Definition 1. We say that the solution $v(t)$ of the equation (1) is *stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$ if to arbitrarily chosen $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ so that the following implication holds:

$$\|u(0) - v(0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \leqq \delta(\varepsilon) \Rightarrow \\ \Rightarrow \|u(t) - v(t)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \leqq \varepsilon$$

for $t \geqq 0$ and for every solution $u(t)$ of the equation (1).

Definition 2. We say that the solution $v(t)$ of the equation (1) is *exponentially stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$ if there exist positive numbers δ, K, α so that the following implication holds:

$$\|u(0) - v(0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \leqq \delta \Rightarrow$$

$$\Rightarrow \|u(t) - v(t)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \leqq K e^{-\alpha t} \, .$$

$$. \, \|u(0) - v(0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$$

for $t \geqq 0$ and for every solution $u(t)$ of the equation (1).

If $\delta = +\infty$ in addition, we speak about the *global exponential stability*.

Let $u(t)$ be a solution of (1) and let the following initial conditions be fulfilled:

$$u(0) = \varphi_0 \, , \quad u'(0) = \varphi_1 \, , \quad u''(0) = \varphi_2 \, , \quad u'''(0) = \varphi_3 \, , \tag{2}$$

where $\varphi_i \in \mathscr{D}(A^{1-i/4})$, $i = 0, \ldots, 3$.

Let us assume that

(3) the solution of (1) fulfilling (2) is unique.

The problem of the uniqueness is studied in [1], [2].

Let us denote $E(s)$ a spectral resolution of the identity corresponding to the operator A, $\delta = \inf \sigma(A)$. By the assumptions on the operator A, we have

$$\delta > 0 \, . \tag{4}$$

Let us write the solution of (1) fulfilling (2) in the form (we shall show that this is possible)

$$u(t) = \sum_{i=0}^{3} \int_{\delta}^{\infty} m_i(t, s) \, \mathrm{d}E(s) \, \varphi_i \, , \tag{5}$$

where $m_i(t, s)$, $(i = 0, \ldots, 3)$ are solutions of

$$m''''(t, s) + a\, m'''(t, s) + (b_1 s^{1/2} + b_2)\, m''(t, s) + (c_1 s^{1/2} + c_2)\, m'(t, s) + \tag{6}$$

$$+ (d_1 s + d_2 s^{1/2} + d_3)\, m(t, s) = 0$$

fulfilling the initial conditions

$$m_i^{(k)}(0, s) = \delta_i^k \, , \quad i, k = 0, \ldots, 3 \, , \quad s \geqq \delta \, . \tag{7}$$

The symbol of derivative means the derivative with respect to the variable t; $s \geqq \delta$ is a parameter.

Suppose that $\lambda_i = \lambda_i(s)$, $i = 1, \ldots, 4$ are solutions of

$$\lambda^4(s) + a\,\lambda^3(s) + (b_1 s^{1/2} + b_2)\,\lambda^2(s) + (c_1 s^{1/2} + c_2)\,\lambda(s) + \\ + d_1 s + d_2 s^{1/2} + d_3 = 0\,. \tag{8}$$

For the sake of simplification we shall further use the following notation

$$b = b_1 s^{1/2} + b_2\,, \quad c = c_1 s^{1/2} + c_2\,, \quad d = d_1 s + d_2 s^{1/2} + d_3\,. \tag{9}$$

Then

$$m_0(t, s) = \sum_{i=1}^{4} \frac{\lambda_i^3 + a\lambda_i^2 + b\lambda_i + c}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,, \tag{10$_0$}$$

$$m_1(t, s) = \sum_{i=1}^{4} \frac{\lambda_i^2 + a\lambda_i + b}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,, \tag{10$_1$}$$

$$m_2(t, s) = \sum_{i=1}^{4} \frac{\lambda_i + a}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,, \tag{10$_2$}$$

$$m_3(t, s) = \sum_{i=1}^{4} \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,, \tag{10$_3$}$$

if $\lambda_i - \lambda_j \neq 0$ for $i \neq j$.

It will be advantageous to express the functions $m_i(t, s)$ in the following form:

$$m_0(t, s) = (\lambda_1^3 + a\lambda_1^2 + b\lambda_1 + c) \int_0^t e^{\lambda_1(t-\tau)} \int_0^\tau e^{\lambda_2(\tau-\sigma)} \,. \\ . \int_0^\sigma e^{\lambda_3(\sigma-\varrho)} e^{\lambda_4 \varrho}\, d\varrho\, d\sigma\, d\tau + [\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 + \\ + a(\lambda_1 + \lambda_2) + b] \int_0^t e^{\lambda_2(t-\sigma)} \int_0^\sigma e^{\lambda_3(\sigma-\varrho)} e^{\lambda_4 \varrho}\, d\varrho\, d\sigma + \\ + (\lambda_1 + \lambda_2 + \lambda_3 + a) \int_0^t e^{\lambda_3(t-\varrho)} e^{\lambda_4 \varrho}\, d\varrho + e^{\lambda_4 t}\,, \tag{11$_0$}$$

$$(11_1)\qquad m_1(t, s) = (\lambda_1^2 + a\lambda_1 + b)\int_0^t e^{\lambda_1(t-\tau)}\int_0^\tau e^{\lambda_2(\tau-\sigma)}\,.$$

$$.\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,d\tau + (\lambda_1 + \lambda_2 + a)\,.$$

$$.\int_0^t e^{\lambda_2(t-\sigma)}\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma + \int_0^t e^{\lambda_3(t-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,,$$

$$(11_2)\qquad m_2(t, s) = (\lambda_1 + a)\int_0^t e^{\lambda_1(t-\tau)}\int_0^\tau e^{\lambda_2(\tau-\sigma)}\,.$$

$$.\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,d\tau + \int_0^t e^{\lambda_2(t-\sigma)}\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,,$$

$$(11_3)\qquad m_3(t, s) = \int_0^t e^{\lambda_1(t-\tau)}\int_0^\tau e^{\lambda_2(\tau-\sigma)}\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,d\tau\,.$$

Lemma 1. *Let the following conditions be fulfilled:*

$$(12)\qquad a > 0\,,$$

$$(13)\qquad c_1 s^{1/2} + c_2 > 0 \quad \text{for} \quad s \geqq \delta\,, \quad c_1 > 0\,,$$

$$(14)\qquad d_1 s + d_2 s^{1/2} + d_3 > 0 \quad \text{for} \quad s \geqq \delta\,, \quad d_1^2 + d_2^2 > 0\,,$$

$$(15)\qquad a(b_1 s^{1/2} + b_2)(c_1 s^{1/2} + c_2) - a^2(d_1 s + d_2 s^{1/2} + d_3) -$$

$$- (c_1 s^{1/2} + c_2)^2 > 0 \quad \text{for} \quad s \geqq \delta\,,$$

$$(16)\qquad ab_1 c_1 - a^2 d_1 - c_1^2 > 0\,.$$

Then there exists a constant $\omega > 0$ such that

$$(17)\qquad \operatorname{Re} \lambda_i(s) \leqq -\omega$$

for all solutions $\lambda_i(s)$ of the equation (8) and all $s \geqq \delta$.

Proof. We can easily derive by means of the Hurwitz theorem that the necessary and sufficient conditions that the inequality $\operatorname{Re} \lambda_i(s) \leqq -\omega$ (for $s \geqq \delta$) holds are

$$(18_1)\qquad -4\omega + a > 0\,,$$

$$(18_2)\qquad (-4\omega + a)(6\omega^2 - 3a\omega + b) - (-4\omega^3 + 3a\omega^2 - 2b\omega + c) > 0\,,$$

$$(18_3) \qquad (-4\omega + a)(6\omega^2 - 3a\omega + b)(-4\omega^3 + 3a\omega^2 - 2b\omega + c) -$$
$$- (-4\omega + a)^2 (\omega^4 - a\omega^3 + b\omega^2 - c\omega + d) -$$
$$- (-4\omega^3 + 3a\omega^2 - 2b\omega + c)^2 > 0 ,$$

$$(18_4) \qquad \omega^4 - a\omega^3 + b\omega^2 - c\omega + d > 0 ;$$

the inequalities (18) must be fulfilled for all $s \geqq \delta$. It follows from (12) that the condition (18_1) holds for sufficiently small $\omega > 0$. (18_2) follows immediately from (13), (14), (18_3), (18_4). The condition (18_4) is also fulfilled for sufficiently small $\omega > 0$ because of (14). Further it follows from (16) that there exists $S_0 \geqq \delta$ such that (18_3) holds for $s \geqq S_0$. Using (15) we can guarantee also (18_3) on the interval $[\delta, S_0]$, if we consider sufficiently small $\omega > 0$ only.

Lemma 1A. *Suppose that it holds* (12), (13), (14), (15). *Then*

$$(19) \qquad \operatorname{Re} \lambda_i(s) \leqq 0$$

for all solutions $\lambda_i(s)$ *of the equation* (8) *and all* $s \geqq \delta$.

Proof. It can be proved that to each $S_0 \geqq \delta$ there exists $\omega = \omega(S_0) > 0$ such that (17) holds for all solutions $\lambda_i(s)$ of the equation (8) and all $s \in [\delta, S_0]$ similarly as in the proof of Lemma 1. This proves Lemma 1A.

Lemma 2. *There exists a constant* $A_1 > 0$ *such that for each solution* $\lambda_i(s)$ *of the equation* (8) (*which can be written in the form*

$$(20) \qquad \lambda^4(s) + a\,\lambda^3(s) + b\,\lambda^2(s) + c\,\lambda(s) + d = 0$$

when we use the notation (9)) *it holds*

$$(21) \qquad |\lambda_i(s)| \leqq A_1 s^{1/4}$$

for $s \geqq \delta$.

Proof. If we put

$$(22) \qquad \lambda = y - \frac{a}{4}$$

we can transform the equation (20) to

$$(23) \qquad y^4 + ey^2 + fy + g = 0$$

where

$$e = b - \frac{3}{8}a^2 , \quad f = \frac{a^3}{8} - \frac{ab}{2} + c , \quad g = -\frac{3}{256}a^4 + \frac{a^2 b}{16} - \frac{ac}{4} + d .$$

All solutions of the equation (23) are:

$$(24_1)\qquad y_1 = \tfrac{1}{2}(z_1^{1/2} + z_2^{1/2} + z_3^{1/2}),$$

$$(24_2)\qquad y_2 = \tfrac{1}{2}(z_1^{1/2} - z_2^{1/2} - z_3^{1/2}),$$

$$(24_3)\qquad y_3 = \tfrac{1}{2}(-z_1^{1/2} + z_2^{1/2} - z_3^{1/2}),$$

$$(24_4)\qquad y_4 = \tfrac{1}{2}(-z_1^{1/2} - z_2^{1/2} + z_3^{1/2}),$$

where z_1, z_2, z_3 are solutions of a cubic equation

$$(25)\qquad z^3 + 2ez^2 + (e^2 - 4g)\,z - f^2 = 0\,.$$

We choose values of the square roots such that $z_1^{1/2} \,.\, z_2^{1/2} \,.\, z_3^{1/2} = -f$. Let us put

$$(26)\qquad z = x - \frac{2}{3}\,e\,.$$

Then the equation (25) can be transformed to

$$(27)\qquad x^3 + 3px + 2q = 0$$

where

$$p = -\frac{e^2}{9} - \frac{4g}{3}\,,\quad q = -\frac{e^3}{27} + \frac{4eg}{3} - \frac{f^2}{2}\,.$$

Let us denote

$$(28)\qquad u = \sqrt[3]{(-q + \sqrt{(q^2 + p^3)})}\,,\quad v = \sqrt[3]{(-q - \sqrt{(q^2 + p^3)})}\,.$$

The square roots are chosen such that $uv = -p$.

Further let us put $\varepsilon = e^{2\pi i/3}$. Then solutions of the equation (27) are

$$(29_1)\qquad x_1 = u + v\,,$$

$$(29_2)\qquad x_2 = \varepsilon u + \varepsilon^2 v\,,$$

$$(29_3)\qquad x_3 = \varepsilon^2 u + \varepsilon v\,.$$

Substituting for p, q to (28), we get

$$(30)\qquad u = K_u s^{1/2} + o(s^{1/2})\,,\quad v = K_v s^{1/2} + o(s^{1/2})\,,$$

where K_u, K_v are constants and $o(f(s))$ means any expression such that

$$\lim_{s \to +\infty} \frac{o(f(s))}{f(s)} = 0\,.$$

We get from (22), (24), (26), (29), (30)

(31) $$\lambda_i(s) = K_i s^{1/4} + o(s^{1/4}), \, i = 1, \dots 4,$$

K_i are constants. We can easily find with help of (4) that

(32) to each $S_0 \geqq \delta$ there exists a constant $K(S_0)$ such that $|\lambda_i(s)| \leqq K(S_0)\,\delta^{1/4}$ for $s \in [\delta, S_0]$, $i = 1, \dots, 4$.

The assertion of the lemma follows immediately from (31), (32).

Lemma 3. *Suppose that*

(33) $$d_1 \neq 0,$$

(34) $$b_1^2 - 4d_1 \neq 0.$$

Then there exist constants $\Lambda_2 > 0$, $S_0 \geqq \delta$ such that

(35) $$|\lambda_i(s) - \lambda_j(s)| \geqq \Lambda_2 s^{1/4} \quad \text{for} \quad s \geqq S_0, \quad i \neq j, \quad i, j = 1, \dots, 4.$$

Proof. We use all notations from the proof of Lemma 2. Then

(36) $$\lambda_1 - \lambda_2 = z_2^{1/2} + z_3^{1/2}, \quad \lambda_2 - \lambda_3 = z_1^{1/2} - z_2^{1/2},$$
$$\lambda_1 - \lambda_3 = z_1^{1/2} + z_3^{1/2}, \quad \lambda_2 - \lambda_4 = z_1^{1/2} - z_3^{1/2},$$
$$\lambda_1 - \lambda_4 = z_1^{1/2} + z_2^{1/2}, \quad \lambda_3 - \lambda_4 = z_2^{1/2} - z_3^{1/2}.$$

So if (35) is to be proved it suffices to prove

(37) $$(z_i^{1/2} + z_j^{1/2})\, s^{-1/4} \xrightarrow{(s \to +\infty)} {}^1K_{ij} \neq 0, \quad \text{for} \quad i \neq j,$$
$$(z_i^{1/2} - z_j^{1/2})\, s^{-1/4} \xrightarrow{(s \to +\infty)} {}^2K_{ij} \neq 0, \quad \text{for} \quad i \neq j;$$

the existence of finite limits is clear, cf. (31).

The conditions (37) will be fulfilled, if

(38) $$\pm \lim_{s \to +\infty} z_i^{1/2} s^{-1/4} \neq \lim_{s \to +\infty} z_j^{1/2} s^{-1/4}, \quad \text{for } i \neq j$$

(the existence of finite limits is clear again).

Using (26) we get the following sufficient condition that (38) is fulfilled

(39) $$\lim_{s \to +\infty} x_i s^{-1/2} \neq \lim_{s \to +\infty} x_j s^{-1/2}, \quad \text{for } i \neq j, \, i, j = 1, 2, 3.$$

Let us denote

$$\bar{p} = -\frac{b_1^2}{9} - \frac{4}{3}d_1, \quad \bar{q} = -\frac{b_1^3}{27} + \frac{4}{3}b_1 d_1,$$

$$\bar{u} = \sqrt[3]{(-\bar{q} + \sqrt{(\bar{q}^2 + \bar{p}^3)})}, \quad \bar{v} = \sqrt[3]{(-\bar{q} - \sqrt{(\bar{q}^2 + \bar{p}^3)})},$$

then

$$\lim_{s\to+\infty} x_1 s^{-1/2} = \bar{u} + \bar{v}, \tag{40}$$

$$\lim_{s\to+\infty} x_2 s^{-1/2} = \varepsilon\bar{u} + \varepsilon^2\bar{v},$$

$$\lim_{s\to+\infty} x_3 s^{-1/2} = \varepsilon^2\bar{u} + \varepsilon\bar{v}.$$

It follows from (40): the condition (39) is fulfilled if

$$\bar{q}^2 + \bar{p}^3 \neq 0. \tag{41}$$

We can easily find that (41) follows from (33), (34).
This proves the lemma.

Proposition 1. *Suppose that* (12)–(16), (33), (34) *hold. Then there exist constants* $L > 0$, $\omega > 0$ *such that*

$$\left|m_i^{(k)}(t, s)\, s^{(i-k)/4}\right| \leqq L e^{-\omega t} \tag{42}$$

for $t \geqq 0$, $s \geqq \delta$, $i = 0, \ldots, 3$, $k = 0, \ldots, 4$.

Proof. It follows from (10), (17), (21), (35) that (42) is fulfilled for $s \geqq S_0$. If we take into consideration the boundedness of $\lambda_i(s)$ for $s \in [\delta, S_0]$ and use (11), we easily prove that (42) holds on $[\delta, S_0]$, too.

Proposition 1A. *Suppose that* (12)–(15), (33), (34) *hold. Then there exists a constant* $L > 0$ *such that*

$$\left|m_i^{(k)}(t, s)\, s^{(i-k)/4}\right| \leqq L \tag{43}$$

for $t \geqq 0$, $s \geqq \delta$, $i = 0, \ldots, 3$, $k = 0, \ldots, 4$.

Proof. It is similar to the proof of Proposition 1.

It follows immediately from Proposition 1A:

Theorem 1. *Let* (12)–(15), (33), (34) *be fulfilled. Then the function* $u(t)$, *defined by the relation* (5), *is the solution of the equation* (1) *and fulfils the initial conditions* (2).

Theorem 2. *Let* (12)–(16), (33), (34) *be fulfilled. Then the zero solution of the equation* (1) *is globally exponentially stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$.

Proof. Using (42) we get from (5)

$$(44_0)\qquad \|Au(t)\|^2 \leqq 4\left\{\int_\delta^\infty |m_0(t,s)|^2\, s^2\, \mathrm{d}\|E(s)\,\varphi_0\|^2 + \int_\delta^\infty |m_1(t,s)\, s^{1/4}|^2 \,.\right.$$

$$. \, s^{3/2}\, \mathrm{d}\|E(s)\,\varphi_1\|^2 + \int_\delta^\infty |m_2(t,s)\, s^{1/2}|^2\, s\, \mathrm{d}\|E(s)\,\varphi_2\|^2 +$$

$$\left. + \int_\delta^\infty |m_3(t,s)\, s^{3/4}|^2\, s^{1/2}\, \mathrm{d}\|E(s)\,\varphi_3\|^2\right\} \leqq 4[Le^{-\omega t}]^2 \,.$$

$$. \left(\|A\varphi_0\|^2 + \|A^{3/4}\varphi_1\|^2 + \|A^{1/2}\varphi_2\|^2 + \|A^{1/4}\varphi_3\|^2\right) =$$

$$= 4[Le^{-\omega t}]^2\, \|u(0)\|^2_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \,.$$

We can prove similarly

$$(44_k)\qquad \|A^{1-k/4}\, u^{(k)}(t)\|^2 \leqq 4\, L[e^{-\omega t}]^2\, \|u(0)\|^2_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}\,,$$

$$k = 1, 2, 3\,.$$

If we add (44_0)–(44_3), we get the global exponential stability of the zero solution.

Theorem 3. *Let* (12)–(15), (33), (34) *be fulfilled. Then the zero solution of the equation* (1) *is stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$.

The proof is similar to that of Theorem 2.

Remark 1. *Suppose that $v(t)$ is any solution of the equation* (1). *Then under the assumptions of Theorem* 2, *respectively Theorem* 3, *$v(t)$ is globally exponentially stable, respectively stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$.

Proof. Let $u(t)$ be a solution of (1). Then the function $w(t) = u(t) - v(t)$ satisfies equation (1), too. Now our assertion immediately follows from Theorem 2, respectively Theorem 3.

Example. The following problem is often investigated:

$$(45)\qquad \varepsilon_1\varepsilon_2\, u_{tttt}(t,x) + a\varepsilon_1\varepsilon_2\, u_{ttt}(t,x) - (\varepsilon_1 + \varepsilon_2)\, u_{ttxx}(t,x) +$$

$$+ (1 + c\varepsilon_1\varepsilon_2)\, u_{tt}(t,x) - a\varepsilon_2\, u_{txx}(t,x) + a\, u_t(t,x) + u_{xxxx}(t,x) -$$

$$- c\varepsilon_2\, u_{xx}(t,x) + c\, u(t,x) = 0\,,$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $a > 0$, c are real constants,

$$u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0.$$