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ON SOME LINEAR VOLTERRA DELAY EQUATIONS

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The L^p-solutions of the equation

(I) $$x(t) = a(t) + \int_0^t B(t, s)\, x(\mu(s))\, ds$$

with $\mu(t) \leqq t$ are investigated. R. K. MILLER has constructed the *resolvent kernel* of (I) with $\mu(t) = t$ in [9] using *Picard successive approximation method*. Using this kernel, an explicit formula for the solution x of (I) corresponding to the right-hand side a is available. Similarly, we shall find the resolvent kernel R of (I) in general case solving the *resolvent equation*

(R) $$R(t, s) = B(t, s) + \int_s^t B(t, u)\, R(\mu(u), s)\, du$$

or

(R′) $$R(t, s) = B(t, s) + \int_s^t R(t, u)\, B(\mu(u), s)\, du\,.$$

This kernel enables us to express the solution x of (I) using the explicit *resolvent formula*

(X) $$x(t) = a(t) + \int_0^t R(t, s)\, a(\mu(s))\, ds\,.$$

Modifying this method, similar results for continuous solutions and for the solutions of more complicated equation

($\hat{\text{I}}$) $$x(t) = a(t) + \int_0^t \sum_\alpha B^\alpha(t, s)\, x(\mu^\alpha(s))\, ds$$

will be shown. The continuous dependence of x on the kernel B and the delay function μ is investigated in the second part. The equations considered comprise the linear cases of the differential delay equations introduced by L. E. ELŚGOLĆ and S. B. Norkin in [6] and many of the cases introduced by A. B. Myškis in [10]. We may also find close relationships to some linear cases of the functional differential equations investigated by J. Hale in [8].

1. EXISTENCE AND UNICITY THEOREMS

1. Notation. We shall fix an integer $n \geqq 1$, real numbers τ, T; $\tau \leqq 0 < T$; real numbers p, q; $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$; $1/p + 1/q = 1$. $|\cdot|$ will be the Euclidean norm of matrices. We put $J = \langle \tau, T \rangle$. We shall write

$$\|f\|_r = \left(\int_J |f|^r \right)^{1/r}, \quad 1 \leqq r < \infty ;$$

$$\|f\|_\infty = \operatorname{vrai\,sup}_t |f(t)|$$

for a matrix-valued (Lebesgue) measurable function f on J. $f \circ \mu$ will be the composition of functions f, μ;

$$(B \circ \mu)(t, s) = B(\mu(t), s) ; \quad t, s \in J ;$$

$$\|B\|_{r,s} = \left[\int_J \left(\int_J |B(t, u)|^s \, du \right)^{r/s} dt \right]^{1/r} ; \quad 1 < r < \infty , \quad 1 < s < \infty ;$$

$$\|B\|^{r,s} = \left[\int_J \left(\int_J |B(t, u)|^r \, dt \right)^{s/r} du \right]^{1/s} ; \quad 1 < r < \infty, \quad 1 < s < \infty ;$$

$$\|B\|_{r,\infty} = \left[\int_J (\operatorname{vrai\,sup}_s |B(t, s)|)^r \, dt \right]^{1/r} , \quad 1 < r < \infty ;$$

$$\|B\|_{\infty,s} = \operatorname{vrai\,sup}_t \left(\int_J |B(t, u)|^s \, du \right)^{1/s} , \quad 1 < s < \infty .$$

for a measurable function B on the cartesian product $J \times J$ and a function $\mu : J \to J$.

2. μ-assumptions. Let

(2,1) $\mu : J \to J$;

(2,2) μ be a measurable function on J;

(2,3) $\tau \leqq \mu(t) \leqq t$ for all $t \in J$.

3. B-assumptions. Let

(3,1) B be a finite complex $n \times n$-matrix-valued function defined for all points of the interval $J \times J$;

(3,2) $B(t, s) = 0$ for $s < 0$ or $s > t$;

(3,3) B be measurable on $J \times J$;

(3,4) $B \circ \mu$ be measurable on $J \times J$;

(3,5) $|B(t, s)| \leqq g(t)\, h(s)$; $t, s \in J$;

(3,6) $|B(\mu(t), s)| \leqq g(t)\, h(s)$; $t, s \in J$,

where the real functions g, h satisfy $\|g\|_p < \infty$, $\|h\|_q < \infty$. We shall sometimes use weaker assumptions

(3,7) $\|B(t, \cdot)\|_q < \infty$, $\|B(\mu(\cdot), t)\|_p < \infty$, $t \in J$;

(3,8) $\|B\|_{p,q} < \infty$;

(3,9) $\|B \circ \mu\|_{p,q} < \infty$;

(3,10) $\|B \circ \mu\|^{p,q} < \infty$;

instead of (3,5–6), if $1 < p, q < \infty$.

4. ***a*-assumptions.** Let

(4,1) a be an n-dimensional vector function (column-matrix) defined on J;

(4,2) a be measurable on J;

(4,3) $\|a\|_p < \infty$;

(4,4) $a \circ \mu$ be measurable on J;

(4,5) $\|a \circ \mu\|_p < \infty$.

5. Definition. $\mathscr{M}$ will denote the set of all μ satisfying (2,1–3). Let $\mu \in \mathscr{M}$.

$$\mathbf{B} = \mathbf{B}_n^{p,\mu}(J)$$

will be the set of all B satisfying (3,1–6) and, if $1 < p$,

$$\mathbf{\hat{B}} = \mathbf{\hat{B}}_n^{p,\mu}(J)$$

the set of all B satisfying (3,1–4), (3,7–10).

$$\mathbf{L} = \mathbf{L}_n^{p,\mu}(J)$$

will be the set of the functions a satisfying (4,1–5). We shall write shortly $\mathbf{L}^p$ if $\mu(t) = t$. The solution of the equation (I) will be a function $x \in \mathbf{L}$ satisfying (I) for all $t \in J$.

6. Remark. We get immediately $\mathbf{B} \subset \mathbf{\hat{B}}$ for $1 < p < \infty$ from Definition 5. It follows from the equation (I) that its solution x is independent of the values $\mu(t)$ for $t < 0$. We have introduced these values only for easier formulations.

We find some essential differences comparing the equation (I) with the classical case $\mu(t) = t$. Let us note that supposing measurability or integrability of a function f, we do not generally get the same property for the composition $f \circ \mu$. Hence the assumptions (3,4), (3,6) e.t.c. are necessary. It is also worth mentioning that changing a value $x(t)$ of a solution x of (I) at one point we may get a function that does not satisfy (I) for the elements of a nonzero set, for some μ. Hence it is not sufficient to define x, a, $B(\cdot, s)$ only almost everywhere. However, the functions μ, $B(t, \cdot)$ may be defined only almost everywhere.

7. Theorem. *Let $\mu \in \mathcal{M}$, $B \in \hat{\mathbf{B}}$. Then there exists a resolvent kernel $R \in \hat{\mathbf{B}}$ satisfying* (R), (R') *for all $t, s \in J$.*

Proof. We prove the theorem using Picard successive approximation method. (Cf. [9].) We introduce these approximations by

(7,1) $$R_0(t, s) = B(t, s)\,,$$

$$R_{\nu+1}(t, s) = \int_s^t B(t, u)\, R_\nu(\mu(u), s)\, du; \quad \nu = 0, 1, 2, \ldots; \quad t, s \in J\,.$$

We shall prove by induction that:

(i) the definition (7,1) is a good one;

(ii) $$R_{\nu+1} \in \hat{\mathbf{B}}\,,$$

(iii) $$R_{\nu+1}(t, s) = \int_s^t R_\nu(t, u)\, B(\mu(u), s)\, du\,,$$

(iv) $$|R_{\nu+1}(t, s)| \leqq \xi(t)^{1/q}\, \eta(s)^{1/p} [\zeta(t, s)^\nu / \nu!]^{1/p}$$

where

$$\xi(t) = \int_0^t |B(t, s)|^q\, ds\,, \quad \eta(s) = \int_s^t |B(\mu(t), s)|^p\, dt\,, \quad \zeta(t, s) = \int_s^t \xi(\mu(u))^{p/q}\, du$$

for $t, s \in J$; $\nu = 0, 1, 2, \ldots$. Let firstly $\nu = 0$. Using the Hölder inequality we obtain

$$\int_s^t |B(t, u)|\, |B(\mu(u), s)|\, du \leqq$$

$$\leqq \left(\int_s^t |B(t, u)|^q\, du\right)^{1/q} \left(\int_s^t |B(\mu(u), s)|^p\, du\right)^{1/p} \leqq \xi(t)^{1/q}\, \eta(s)^{1/p}\,.$$

Hence the definition (7,1) is a good one, (iv) holds, R_1 satisfies (3,1–4) and (3,7–10) and we obtain (ii). Clearly (iii) holds as well. Let now $\nu \geqq 1$ and let the assertions (i)–(iv) hold for the indices $\alpha \leqq \nu - 1$. Using the induction predicate and the Hölder inequality we get

$$\int_s^t |B(t, u)|\, |R_\nu(\mu(u), s)|\, du \leqq \left(\int_s^t |B(t, u)|^q\, du\right)^{1/q} \left(\int_s^t |R_\nu(\mu(u), s)|^p\, du\right)^{1/p} \leqq$$

$$\leqq \xi(t)^{1/q} \left[\int_s^t \xi(\mu(u))^{p/q} \frac{\zeta(\mu(u), s)^{\nu-1}}{(\nu - 1)!}\, du\right]^{1/p} \eta(s)^{1/p} \leqq$$

$$\leqq \xi(t)^{1/q} \left[\int_s^t \xi(\mu(u))^{p/q} \frac{1}{(\nu - 1)!} \left[\int_s^u \xi(\mu(v))^{p/q}\, dv\right]^{\nu-1} du\right]^{1/p} \eta(s)^{1/p} \leqq$$

$$\leqq \xi(t)^{1/q} \left[\frac{\zeta(t, s)^\nu}{\nu!}\right]^{1/p} \eta(s)^{1/p}.$$

Now we follow the argument of the first induction step. (iii) follows from the relations

$$R_{\nu+1}(t, s) = \int_s^t B(t, u) \int_s^u R_{\nu-1}(\mu(u), v)\, B(\mu(v), s)\, dv\, du =$$

$$= \int_s^t \int_v^t B(t, u)\, R_{\nu-1}(\mu(u), v)\, du\, B(\mu(v), s)\, dv = \int_s^t R_\nu(t, v)\, B(\mu(v), s)\, dv.$$

Let us put

$$\tilde{R}(t, s) = \sum_{\nu=1}^{\infty} R_\nu(t, s). \tag{7,2}$$

The function $\tilde{R}$ is defined on the whole interval $J \times J$ and satisfies

$$|\tilde{R}(t, s)| \leqq c\, \xi(t)^{1/q}\, \eta(s)^{1/p}.$$

Similarly,

$$|\tilde{R}(\mu(t), s)| \leqq c\, \xi(\mu(t))^{1/q}\, \eta(s)^{1/p}.$$

Now, if $R = \tilde{R} + B$ then $R \in \tilde{B}$ and using the Lebesgue theorem we get

$$\int_s^t B(t, u)\, R(\mu(u), s)\, du = \int_s^t \sum_{\nu=0}^{\infty} B(t, u)\, R_\nu(\mu(u), s)\, du =$$

$$= \sum_{\nu=0}^{\infty} \int_s^t B(t, u)\, R_\nu(\mu(u), s)\, du = \sum_{\nu=1}^{\infty} R_\nu(t, s) = R(t, s) - B(t, s).$$

Hence (R) holds. We prove (R′) similarly using (iii).

8. Remark. There exists $R \in \mathbf{B}$ satisfying (R), (R′) and the inequalities

$$|R(t, s)|, |R(\mu(t), s)| \leqq \varkappa\, g(t)\, h(s)$$

for $B \in \mathbf{B}$, where $\varkappa$ depends only on the functions g, h corresponding to B according to the relations (3,5–6). The proof is similar to that of Theorem 7. We obtain for the successive approximations

$$|R_\nu(t, s)|, |R_\nu(\mu(t), s)| \leqq g(t)\, h(s)\, f_\nu(t, s)$$

where

$$f_\nu(t, s) = \begin{cases} \dfrac{1}{\nu!^{1/q}} \left(\displaystyle\int_s^T g^p\right)^{\nu/p} \left(\displaystyle\int_0^t h^q\right)^{\nu/q}, & 1 < p < \infty, \\ \dfrac{1}{\nu!} \left(\displaystyle\int_0^t h\right)^\nu, & p = \infty, \\ \dfrac{1}{\nu!} \left(\displaystyle\int_0^t g\right)^\nu, & p = 1. \end{cases}$$

9. Theorem. *Let $\mu \in \mathcal{M}$, $B \in \mathbf{B}$ or $B \in \tilde{\mathbf{B}}$; $a \in \mathbf{L}$. Then there exists a unique solution $x \in \mathbf{L}$ of the equation* (I). *This solution is given by* (X), *where R is the corresponding resolvent kernel.*

Proof. Let us define x by (X). Then

$$\int_0^t B(t, s)\, x(\mu(s))\, ds = \int_0^t B(t, s) \left[a(\mu(s)) + \int_0^{\mu(s)} R(\mu(s), u)\, a(\mu(u))\, du \right] ds =$$

$$= \int_0^t B(t, u)\, a(\mu(u))\, du + \int_0^t \int_u^t B(t, s)\, R(\mu(s), u)\, ds\, a(\mu(u))\, du =$$

$$= \int_0^t R(t, u)\, a(\mu(u))\, du$$

in virtue of (R). Hence x fulfils (I). If the couple x, a satisfies (I) it satisfies also (X) because $-B$ is the resolvent kernel corresponding to $-R$. So we get the unicity.

10. Example. For $t - \mu(t) \geqq c > 0$, $t \in J$, we get a finite number of approximations for the resolvent kernel evaluation. We may also simply compute the solution x provided that μ is a step function e.t.c.

11. Remark. Let $1 < p \leqq \infty$, $B \in \mathbf{B}_n^{p,\mu}(J)$,

$$(11,1) \qquad \int_J |B(t, s) - B(u, s)|^q\, ds \to 0, \quad u \to t; \quad u, t \in J.$$

Then the resolvent kernel also fulfils (11,1) and for continuous a the solution x of (I) is continuous as well. Using *Carathéodory condition* for the measurability of a composed function (see e.g. M. M. VAJNBERG [12]) we may prove the following more general assertion.

12. Theorem. *Let $\mu \in \mathcal{M}$, let the kernel B satisfy* (3,1 – 2) *and*

(12,1) *$B(t, \cdot)$ is measurable on J for all $t \in J$;*

(12,2) $|B(t, s)| \leqq h(s)$; *$t, s \in J$ where $h \in \mathbf{L}^1$;*

(12,3) $$\int_J |B(u, s) - B(t, s)| \, ds \to 0 \quad \text{for} \quad u \to t \, ; \quad u, t \in J \, .$$

Let a be continuous on J. Then the equation (I) *has a unique solution x continuous on J.*

13. Remark. A function B continuous on $J \times J$ satisfies (12,1 – 3).

14. The more general case. Now we generalize the previous results to the equation $(\hat{\mathrm{I}})$. Let $\mathscr{A}$ be a countable set, $\mu^\alpha \in \mathcal{M}$ for all $\alpha \in \mathscr{A}$. Let $\alpha_0 \in \mathscr{A}$, $\mu^{\alpha_0}(t) = t$, $t \in J$. Let B^α be a kernel satisfying (3,1 – 2) for all $\alpha \in \mathscr{A}$ and let
(14,1) $B^\alpha(\mu^\beta(t), s)$ be a measurable function on $J \times J$;

(14,2) $$|B^\alpha(\mu^\beta(t), s)| \leqq g^\beta(t) \, h^\alpha(s) \, ; \quad t, s \in J \, ;$$

for all $\alpha, \beta \in \mathscr{A}$ where

$$G \equiv (\sum \|g^\beta\|_p^p)^{1/p} < \infty \, , \quad H \equiv (\sum_\beta \|h^\beta\|_q^q)^{1/q} < \infty \, .$$

(We put $G = \sup_\beta \|g^\beta\|_\infty$ if $p = \infty$ e.t.c.) We denote $B = \{B^\alpha\}$ the system of this kernels, $\hat{\mathbf{B}}$ the family of this systems. Let $\hat{\mathbf{L}}$ be the set of functions satisfying (4,1 – 2) and the assumptions:

(14,3) $a \circ \mu^\alpha$ is measurable for all $\alpha \in \mathscr{A}$;

(14,4) $$\|a\|_{p,\mu} \equiv (\sum_\alpha \|a \circ \mu^\alpha\|_p^p)^{1/p} < \infty \, .$$

(We put $\|a\|_{p,\mu} = \sup_\alpha \|a \circ \mu^\alpha\|_\infty$ if $p = \infty$.) We shall consider the equation

$(\hat{\mathrm{I}})$ $$x(t) = a(t) + \int_0^t \sum_\alpha B^\alpha(t, s) \, x(\mu^\alpha(s)) \, ds$$

and seek a system $R = \{R^\alpha\}$ of resolvent kernels satisfying the resolvent equations

$$(\hat{\mathrm{R}}) \qquad R^\alpha(t, s) = B^\alpha(t, s) + \int_s^t \sum_\beta B^\beta(t, u)\, R^\alpha(\mu^\beta(u), s)\, du\,,$$

$$(\hat{\mathrm{R}}') \qquad R^\alpha(t, s) = B^\alpha(t, s) + \int_s^t \sum_\beta R^\alpha(t, u)\, B^\beta(\mu^\beta(u), s)\, du$$

for all $\alpha \in \mathscr{A}$. The corresponding resolvent formula for the solution x will be of the form

$$(\hat{\mathrm{X}}) \qquad x(t) = a(t) + \int_0^t \sum_\beta R^\beta(t, s)\, a(\mu^\beta(s))\, ds\,, \quad t \in J\,.$$

15. Theorem. *Let $B = \{B^\alpha\} \in \hat{\mathbf{B}}$. Then there exists a system $R = \{R^\alpha\} \in \hat{\mathbf{B}}$ of resolvent kernels satisfying* (R), (R′) *and the inequalities*

$$|R^\alpha(\mu^\beta(t), s)| \leqq c\, g^\beta(t)\, h^\alpha(s)\,; \quad t, s \in J\,; \quad \alpha, \beta \in \mathscr{A}\,;$$

where the constant c depends only on the functions g^γ, h^δ, $\gamma, \delta \in \mathscr{A}$.

Proof. We may define the systems of resolvent kernels by the formulas

$$R_0^\alpha(t, s) = B^\alpha(t, s)\,,$$

$$R_{\nu+1}^\alpha(t, s) = \int_s^t \sum_\beta B^\beta(t, u)\, R^\alpha(\mu^\beta(u), s)\, du\,; \quad t, s \in J\,; \quad \alpha \in \mathscr{A}\,; \quad \nu = 0, 1, 2, \ldots$$

similarly to the case of the equation (I) (see Remark 8). These systems belong to $\hat{\mathbf{B}}$ and it holds

$$|R_\nu^\alpha(\mu^\beta(t), s)| \leqq g^\beta(t)\, h^\alpha(s)\, w_\nu(t)$$

where

$$w_\nu(t) = \begin{cases} \left[\dfrac{\left(G \sum_\beta \int_0^t (h^\beta)^q\right)^\nu}{\nu!}\right]^{1/q} & \text{if} \quad 1 < p \leqq \infty\,, \\[2ex] \dfrac{\left(H \sum_\beta \int_0^t g^\beta\right)^\nu}{\nu!} & \text{if} \quad p = 1\,. \end{cases}$$

The system of resolvent kernels satisfying the assertion of the theorem may be defined by

$$R^\alpha = \sum_{\nu=0}^\infty R_\nu^\alpha\,, \quad \alpha \in \mathscr{A}\,.$$

16. Theorem. *Let $B = \{B^\alpha\} \in \hat{\mathbf{B}}$. $R = \{R^\alpha\}$ be the corresponding system of re-*

solvent kernels, $a \in \hat{\mathbf{L}}$. Then the equation $(\hat{\mathrm{I}})$ has a unique solution $x \in \hat{\mathbf{L}}$. This solution is given by the resolvent formula $(\hat{\mathrm{X}})$.

Proof is analogous to that of Theorem 9.

2. CONTINUITY

1. Lemma. Let $b \in \mathbf{B}_1^{p,\mu}$; $f, z \in \mathbf{L}_1^{p,\mu}$; $b \geqq 0$; let r be the resolvent kernel corresponding to the kernel b. Then $r \geqq 0$ and the inequality

$$(1,1) \qquad z(t) \leqq f(t) + \int_0^t b(t, s)\, z(\mu(s))\, ds\,, \quad t \in J\,,$$

implies

$$(1,2) \qquad z(t) \leqq f(t) + \int_0^t r(t, s)\, f(\mu(s))\, ds\,, \quad t \in J\,.$$

Moreover, there exists a constant $c > 0$ dependent only on the functions g_b, h_b so that (1,1) and the assumption $z \geqq 0$ imply

$$(1,3) \qquad \|z\|_p \leqq \|f\|_p + c\|f \circ \mu\|_p\,, \quad \|z \circ \mu\|_p \leqq c\|f \circ \mu\|_p\,.$$

Proof. We obtain $r \geqq 0$ from $b \geqq 0$ and the successive approximation method. Hence and from (1,1) it follows

$$(1,4) \qquad z(t) + \int_0^t r(t, u)\, z(\mu(u))\, du \leqq f(t) + \int_0^t r(t, u)\, f(\mu(u))\, du +$$
$$+ \int_0^t b(t, s)\, z(\mu(s))\, ds + \int_0^t r(t, u) \int_0^{\mu(u)} b(\mu(u), s)\, z(\mu(s))\, ds\, du\,.$$

Let us denote the last integral by U. Replacing the upper limit $\mu(u)$ by u and using the resolvent formula we get after simple calculation

$$U = \int_0^t [r(t, s) - b(t, s)]\, z(\mu(s))\, ds\,.$$

Hence and from (1,4), (1,2) follows. (1,2) implies (using also $z \geqq 0$)

$$\|z\|_p \leqq \|f\|_p + \|f \circ \mu\|_p\, \|h_r\|_q\, \|g_r\|_p\,,$$
$$\|z \circ \mu\|_p \leqq \|f \circ \mu\|_p + \|f \circ \mu\|_p\, \|h_r\|_q\, \|g_r\|_p\,.$$

We obtain (1,3) from here and Remark 8 of the first part.

2. Lemma. Let $B \in \mathbf{B}_n^{p,\mu}$, $a \in \mathbf{L}_n^{p,\mu}$. Then $|B| \in \mathbf{B}_1^{p,\mu}$; $|B|$, $|B(\mu(\cdot), \cdot)| \in \mathbf{B}_1^{p,1}$, $|a| \in \mathbf{L}_1^{p,1}$

and we may choose $g_{|B|} = g_B$, $h_{|B|} = h_B$. If, moreover, x is a solution of (I), then there exists a constant $c > 0$ depending only on g_B, h_B so that

$$\|x\|_p \leqq \|a\|_p + c\|a \circ \mu\|_p \,, \quad \|x \circ \mu\|_p \leqq c\|a \circ \mu\|_p \,.$$

Proof follows from Lemma 1.

3. Lemma. Let $\lambda, \mu \in \mathcal{M}$, $a \in \mathbf{L}_n^{p,\mu} \cap \mathbf{L}_n^{p,\lambda}$, $B \in \mathbf{B}_n^{p,\mu} \cap \mathbf{B}_n^{p,\lambda}$, $g \in \mathbf{L}^p$, $h \in \mathbf{L}^q$,

$$(3,1) \qquad |B(t, s)|, |B(\mu(t), s)|, |B(\lambda(t), s)| \leqq g(t)\, h(s) \,; \quad t, s \in J \,;$$

let x be a solution of (I), y a solution of the equation

$$(3,2) \qquad y(t) = a(t) + \int_0^t B(t, s)\, y(\lambda(s))\, \mathrm{d}s \,, \quad t \in J \,.$$

Then there exists a constant c depending only on the functions g, h so that it holds

$$(3,3) \quad \|x - y\|_p \leqq c[\|a \circ \mu - a \circ \lambda\|_p + \|B \circ \mu - B \circ \lambda\|_{p,q}(\|a \circ \mu\|_p + \|a \circ \lambda\|_p)] \,,$$

$$(3,4) \qquad \|x \circ \mu - y \circ \lambda\|_p \leqq$$

$$\leqq c[\|a \circ \mu - a \circ \lambda\|_p + \|B \circ \mu - B \circ \lambda\|_{p,q}(\|a \circ \mu\|_p + \|a \circ \lambda\|_p)] \,.$$

Proof. We get

$$(3,5) \qquad x(t) - y(t) = \int_0^t B(t, s)\, [x(\mu(s)) - y(\lambda(s))]\, \mathrm{d}s \,,$$

$$(3,6) \qquad x(\mu(t)) - y(\lambda(t)) = a(\mu(t)) - a(\lambda(t)) +$$

$$+ \int_0^{\mu(t)} B(\mu(t), s)\, x(\mu(s))\, \mathrm{d}s - \int_0^{\lambda(t)} B(\lambda(t), s)\, y(\lambda(s))\, \mathrm{d}s =$$

$$= a(\mu(t)) - a(\lambda(t)) + \int_0^{\mu(t)} [B(\mu(t), s) - B(\lambda(t), s)]\, x(\mu(s))\, \mathrm{d}s +$$

$$+ \int_0^{\mu(t)} B(\lambda(t), s)\, [x(\mu(s)) - y(\lambda(s))]\, \mathrm{d}s - \int_{\mu(t)}^{\lambda(t)} B(\lambda(t), s)\, y(\lambda(s))\, \mathrm{d}s$$

from (I) and (3,2). Let us put (for $t, s \in J$)

$$z(t) = |x(\mu(t)) - y(\lambda(t))| \,, \quad b(t, s) = |B(\lambda(t), s)| \,,$$

$$f_1(t) = |a(\mu(t)) - a(\lambda(t))| \,,$$

$$f_2(t) = \int_0^t |B(\mu(t), s) - B(\lambda(t), s)| \, |x(\mu(s))| \, ds ,$$

$$f_3(t) = \left| \int_{\mu(t)}^{\lambda(t)} B(\lambda(t), s) \, y(\lambda(s)) \, ds \right| ,$$

$$f = f_1 + f_2 + f_3 .$$

We get

$$z(t) \leqq f(t) + \int_0^t b(t, s) \, z(s) \, ds , \quad t \in J ,$$

from (3,6). Clearly $z, f \in \mathbf{L}_1^{p,1}$; $b \in \mathbf{B}_1^{p,1}$. Using Lemma 1 and (3,1) we get

$$\|z\|_p \leqq c\|f\|_p \tag{3,7}$$

(c denotes constants depending only on g, h). Using Lemma 2 we obtain

$$\|f_2\|_p \leqq \|B \circ \mu - B \circ \lambda\|_{p,q} \|x \circ \mu\|_p \leqq c\|B \circ \mu - B \circ \lambda\|_{p,q} \|a \circ \mu\|_p . \tag{3,8}$$

$f_3(t) = 0$ if $\lambda(t) \leqq \mu(t)$ because $B(\lambda(t), s) = 0$ for $s > \lambda(t)$. For $\lambda(t) > \mu(t)$ it holds

$$f_3(t) = \left| \int_{\mu(t)}^{\lambda(t)} [B(\mu(t), s) - B(\lambda(t), s)] \, y(\lambda(s) \, ds \right| \leqq$$

$$\leqq \int_0^t |B(\mu(t), s) - B(\lambda(t), s)| \, |y(\lambda(s))| \, ds .$$

Using this and Lemma 2 we get

$$\|f_3\|_p \leqq \|B \circ \mu - B \circ \lambda\|_{p,q} \|y \circ \lambda\|_p \leqq c\|B \circ \mu - B \circ \lambda\|_{p,q} \|a \circ \lambda\|_p . \tag{3,9}$$

(3,5) implies

$$\|x - y\|_p \leqq c\|x \circ \mu - y \circ \lambda\|_p = c\|z\|_p . \tag{3,10}$$

(3,3–4) follows from (3,7–10).

4. Assumptions. Let

$$\mu_\nu \in \mathcal{M} , \quad B \in \mathbf{B}_n^{p,\mu_\nu}(J) , \quad a \in \mathbf{L}_n^{p,\mu_\nu}(J) ,$$

let x_ν be the solution of (I) with $\mu = \mu_\nu$ for $\nu = 0, 1, 2, \ldots$.

5. Assumptions. Let

$$\|a \circ \mu_2\|_p \leqq \alpha < \infty ; \tag{5,1}$$

$$|B(t, s)|, |B(\mu_\nu(t), s)| \leqq g(t) \, h(s) ; \quad t, s \in J ; \tag{5,2}$$

for $v = 0, 1, 2, \ldots$ where $g \in \mathbf{L}^p(J)$, $h \in \mathbf{L}^q(J)$;

$$\|a \circ \mu_v - a \circ \mu_0\|_p \to 0\,, \quad v \to \infty\,; \tag{5,3}$$

$$\|B \circ \mu_v - B \circ \mu_0\|_{p,q} \to 0\,, \quad v \to \infty\,, \tag{5,4}$$

6. Corollary. *Let the assumptions 4,5 hold. Then* $\|x_v - x_0\| \to 0$, $\|x_v \circ \mu_v - x_0 \circ \mu_0\| \to 0$ if $v \to \infty$.

Proof follows from Lemma 3.

7. Theorem. *Let the assumptions 4 hold. Let* $p < \infty$,

$$|a(u) - a(v)| \leqq A\,|u - v|^{1/p}\,;\, u, v \in J\,; \tag{7,1}$$

$$|B(u, s) - B(v, s)| \leqq \beta(s)\,|u - v|^{1/p}\,; \quad s, u, v \in J\,, \tag{7,2}$$

where A *is a constant,* $\beta \in \mathbf{L}^q$;

$$\|\mu_v - \mu_0\|_1 \to 0\,, \quad v \to \infty\,; \tag{7,3}$$

$$\sup_v |\mu_v - \mu_0| \leqq \mu \in \mathbf{L}^1\,. \tag{7,4}$$

Then $x_v \to x_0$ for $v \to \infty$.

Proof. (7,1–4) imply the assumptions 5. Now we apply Corollary 6.

8. Lemma. *Let* $B, K \in \mathbf{B}_n^{p,\mu}$, $a \in \mathbf{L}_n^{p,\mu}$, *let* x *be a solution of* (I), y *a solution of*

$$y(t) = a(t) + \int_0^t K(t, s)\, y(\mu(s))\, \mathrm{d}s\,, \quad t \in J\,. \tag{8,1}$$

Then there exists a constant c *depending only on the functions* g_B, h_B, g_K, h_K *so that*

$$\|x - y\|_p \leqq c[\|B - K\|_{p,q} + \|B \circ \mu - K \circ \mu\|_{p,q}]\,\|a \circ \mu\|_p\,, \tag{8,2}$$

$$\|x \circ \mu - y \circ \mu\|_p \leqq c\|B \circ \mu - K \circ \mu\|_{p,q}\,\|a \circ \mu\|_p \tag{8,3}$$

hold.

Proof. It follows

$$|x(t) - y(t)| \leqq \int_J |B(t, s)|\,|x(\mu(s)) - y(\mu(s))|\,\mathrm{d}s + \int_J |B(t, s) - K(t, s)|\,|y(\mu(s))|\,\mathrm{d}s\,, \quad t \in J\,; \tag{8,4}$$