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BOUNDARY VALUE PROBLEMS WITH JUMPING NONLINEARITIES

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1. INTRODUCTION

Consider the nonlinear two point boundary value problem

$$(1.1) \qquad \begin{gathered} u''(\tau) + \psi(u(\tau)) = p(\tau)\,, \\ u(0) = u(\pi) = 0\,, \end{gathered}$$

where ψ is a continuous real-valued function defined on $(-\infty, \infty)$. If

$$\lim_{\xi \to \infty} \frac{\psi(\xi)}{\xi} = \lim_{\xi \to -\infty} \frac{\psi(\xi)}{\xi} = A$$

we shall say that the nonlinearity ψ is not jumping. The results obtained under various assumptions may be summarized as follows:

I. If $A = \infty$ (see [3]) or $A \neq n^2$, n is positive integer (see e.g. [6], [10]) then (1.1) has a solution for any right hand side p.

The assumption $A \neq n^2$ means that A is not an eigenvalue of the boundary value problem

$$(1.2) \qquad \begin{gathered} u''(\tau) + \lambda\, u(\tau) = 0\,, \\ u(0) = u(\pi) = 0\,. \end{gathered}$$

II. If $A = n^2$ (see e.g. [4], [5], [9], [13]) then necessary and sufficient conditions on p have been given for (1.1) to be solvable.

If

$$\lim_{\xi \to \infty} \frac{\psi(\xi)}{\xi} \neq \lim_{\xi \to -\infty} \frac{\psi(\xi)}{\xi}$$

we shall say that the nonlinearity ψ is jumping.

III. Under some assumptions it is proved (see [10], for partial differential analogue see [8]) that if

$$n^2 < \lim_{\xi \to \infty} \frac{\psi(\xi)}{\xi} < (n+1)^2 ,$$

$$n^2 < \lim_{\xi \to -\infty} \frac{\psi(\xi)}{\xi} < (n+1)^2$$

(n is positive integer) then (1.1) is solvable for any right hand side p.

In this case the nonlinearity may be jumping but it does not jump over an eigenvalue of (1.2). To the author's best knowledge, the first result about solvability of (1.1) with the nonlinearity jumping over an eigenvalue of (1.2) is proved by A. AMBROSETTI and G. PRODI (see [1], for partial differential analogue see [2], a generalization is given in [12]).

IV (see [1]). Let ψ be a continuous function of class C^2 satisfying the following conditions:

(i) $\psi(0) = 0$;

(ii) $\psi''(\xi) > 0, \quad \xi \in (-\infty, \infty)$;

(iii) $0 < \lim_{\xi \to -\infty} \frac{\psi(\xi)}{\xi} < 1$;

(iv) $1 < \lim_{\xi \to \infty} \frac{\psi(\xi)}{\xi} < 4$.

Then there exists in $C\langle 0, \pi \rangle$ a closed connected C^1-manifold M of codimension 1, such that $C\langle 0, \pi \rangle \setminus M$ consists of exactly two connected components A_1, A_2 with the following properties:

(a) if $p \in A_1$ then (1.1) has no solution;

(b) if $p \in A_2$ then (1.1) has exactly two solutions;

(c) if $p \in M$ then (1.1) has a unique solution.

In the last case the nonlinearity ψ jumps over the least eigenvalue of (1.2). This paper deals with the following cases:

(α) the nonlinearity ψ jumps over an eigenvalue of (1.2) which is not the least (see 2.15, 2.17);

(β) the nonlinearity jumps over more than one eigenvalue of (1.2) (see 2.18);

(γ) the nonlinearity jumps from an eigenvalue of (1.2) to another one (see 2.16);

(δ) the nonlinearity jumps off an eigenvalue (but not to another) (see 2.15–2.17, 3.9–3.11).

The paper serves also as an example that the assumptions (iii) and (iv) are essential for the assertion in IV since if the nonlinearity jumps over an eigenvalue which is not the least we obtain solvability of (1.1) for arbitrary right hand side.

The proofs are based on the properties of the Leray-Schauder degree which are for the reader's convenience recalled in 2.2. Routine and tedious calculations play by no means a merely trifling part in the proofs. From this point of view it seems that it is not possible to obtain the partial differential analogue of the results given here in the same way. Thus the problem of solvability of boundary value problems for nonlinear partial differential equations with a nonlinearity of one of the types $(\alpha)-(\delta)$ remains open. On the other hand, the partial differential analogues of I–IV (except the case $A = \infty$) are known. Other open problems are formulated in the ends of both sections.

2. JUMPING OUTSIDE THE LEAST EIGENVALUE

2.1. Notation and terminology. Unless otherwise stated, we shall suppose that X and Y are real Banach spaces with the norms $\| \|_X$ and $\| \|_Y$, respectively. Let F be a mapping with the domain $D \subset X$ and values in Y $(F : D \subset X \to Y)$. Then F is said to be completely continuous on D if for each bounded subset $M \subset D$, $F(M)$ is compact and F is continuous on D.

2.2. Leray-Schauder degree. Let $K_\varrho = \{x \in X;\ \|x\|_X < \varrho\}$ and let $F : \bar{K}_\varrho \subset X \to X$ (the bar denotes the closure) be a completely continuous mapping. Denote by Id the identity mapping in X, i.e. $Id(x) = x$ for every $x \in X$. Let $x - F(x) \neq 0_X$ (0_X means the zero element of X) for each $x \in X$ with $\|x\|_X = \varrho$. Then it is possible to define the Leray-Schauder degree $\mathrm{d}[Id - F; K_\varrho, 0_X]$ of the mapping $Id - F$ with respect to K_ϱ and the point 0_X so that (see e.g. [7]):

I. $\mathrm{d}[Id; K_\varrho, 0_X] = 1$;

II. $\mathrm{d}[Id - F; K_\varrho, 0_X] \neq 0$ implies that there exists at least one $x_0 \in K_\varrho$ such that $x_0 = F(x_0)$.

III. Let $G : \bar{K}_\varrho \subset X \to X$ be also a completely continuous mapping. Suppose that for each $x \in \bar{K}_\varrho$, $\|x\|_X = \varrho$ and $t \in \langle 0, 1 \rangle$ it is $x - F(x) - t\,G(x) \neq 0_X$. Then $\mathrm{d}[Id - F; K_\varrho, 0_X] = \mathrm{d}[Id - F - G; K_\varrho, 0_X]$.

IV. Suppose that for arbitrary $k \in \langle 0, 1 \rangle$ the equation

$$x = \frac{1}{1+k}\,F(x) - \frac{k}{1+k}\,F(-x)$$

has only the trivial solution. Then

$$\mathrm{d}[Id - F; K_\varrho, 0_X] \neq 0\,.$$

I – III imply immediately

V (Schauder fixed point theorem). *Let* $F(\overline{K}_\varrho) \subset \overline{K}_\varrho$. *Then there exists at least one* $x_0 \in \overline{K}_\varrho$ *such that* $F(x_0) = x_0$.

2.3. Lemma. *Let* J *be an isomorphism between* X *and* Y. *Let* $S : X \to Y$, $R : X \to Y$ *be completely continuous mappings. Suppose that there exist* $\alpha \geqq 0$, $\beta \geqq 0$, $\gamma \in \langle 0, 1)$ *such that*

$$\|R(x)\|_Y \leqq \alpha + \beta \|x\|_X^\gamma \tag{2.3.1}$$

for every $x \in X$.

For every $x \in X$ *and* $t \geqq 0$, *let*

$$S(tx) = t\, S(x)\,. \tag{2.3.2}$$

Then the equation

$$J(x) - S(x) + R(x) = y \tag{2.3.3}$$

is solvable for any right hand side $y \in Y$ *provided the equations*

$$J(x) = \frac{1}{1+k} S(x) - \frac{k}{1+k} S(-x) \tag{2.3.4}$$

have for each $k \in \langle 0, 1 \rangle$ *the trivial solution only.*

Proof. Since the operator J is an isomorphism between X and Y it is sufficient to show that the equation

$$x - J^{-1}\, S(x) + J^{-1}\, R(x) = \eta \tag{2.3.5}$$

is solvable for any $\eta \in X$, where J^{-1} is the inverse of J. First of all we notice that there exists $c > 0$ such that

$$\|x - J^{-1}\, S(x)\|_X \geqq c\|x\|_X \tag{2.3.6}$$

for every $x \in X$. Let $\eta \in X$ be arbitrary but fixed. There exists $\varrho > 0$ such that

$$c\varrho > (\alpha + \beta\varrho^\gamma)\, \|J^{-1}\| + \|\eta\|_X\,. \tag{2.3.7}$$

Put $K_\varrho = \{x \in X;\ \|x\|_X < \varrho\}$. It is easy to see that the mappings $J^{-1}S$ and $J^{-1}R$ are completely continuous. From (2.3.6), (2.3.7) according to III (see 2.2) we have

$$\mathrm{d}[x - J^{-1}\, S(x) + J^{-1}\, R(x) - \eta;\ K_\varrho, 0_X] = \mathrm{d}[x - J^{-1}\, S(x);\ K_\varrho, 0_X]\,.$$

Now it is sufficient to show (see the property II from 2.2) that

$$\mathrm{d}[x - J^{-1}\, S(x);\ K_\varrho, 0_X] \neq 0\,.$$

This fact follows immediately from our assumption that the equations (2.3.4) have for each $k \in \langle 0, 1 \rangle$ only the trivial solution and form the property IV of the Leray-Schauder degree.

2.4. Notation. In the sequel, $L_p(0, \pi)$ $(p \geqq 1)$ will denote the space of all measurable real-valued functions u such that $|u|^p$ is integrable, with the usual norm

$$\|u\|_{L_p} = \left\{ \int_0^\pi |u(\tau)|^p \, \mathrm{d}\tau \right\}^{1/p}$$

and with the inner product

$$(u, v) = \int_0^\pi u(\tau)\, v(\tau) \, \mathrm{d}\tau$$

if $p = 2$.

For $k = 0, 1, 2, \ldots,$ $C^k\langle 0, \pi \rangle$ will denote the space of all functions which are k-times continuously differentiable on $(0, \pi)$ and such that the derivatives can be extended continuously to $\langle 0, \pi \rangle$. With the usual norm

$$\|u\|_{C^k} = \sup_{0 \leqq r \leqq k} \sup_{x \in (0,\pi)} |u^{(r)}(x)| ,$$

$C^k\langle 0, \pi \rangle$ is a Banach space. $C^k_0\langle 0, \pi \rangle$ will denote the subspace of $C^k\langle 0, \pi \rangle$ consisting of all functions which are zero at 0 and π.

Denote by $W^1_2\langle 0, \pi \rangle$ the Sobolev space of all real-valued absolutely continuous functions u on the interval $\langle 0, \pi \rangle$ whose derivatives u' (which exist almost everywhere) are elements of $L_2(0, \pi)$. Put $\mathring{W}^1_2\langle 0, \pi \rangle = W^1_2\langle 0, \pi \rangle \cap C^0_0\langle 0, \pi \rangle$. It is easy to see that $\mathring{W}^1_2\langle 0, \pi \rangle$ is a Hilbert space with the inner product

$$\langle u, v \rangle = (u', v') , \quad u, v \in \mathring{W}^1_2\langle 0, \pi \rangle$$

and that the imbedding $i : u \mapsto u$ is a completely continuous mapping from $\mathring{W}^1_2\langle 0, \pi \rangle$ into $C^0_0\langle 0, \pi \rangle$.

Let g be a continuous real-valued function defined on $(-\infty, \infty)$ and such that there exist $c_1 \geqq 0$, $c_2 \geqq 0$ and $\gamma \in \langle 0, 1)$ such that

$$|g(\xi)| \leqq c_1 + c_2 |\xi|^\gamma \tag{2.4.1}$$

for every $\xi \in (-\infty, \infty)$.

2.5. Definition. Let $p \in L_1(0, \pi)$, μ, ν real numbers. The function $u_0 \in \mathring{W}^1_2\langle 0, \pi \rangle$ is said to be a *weak solution* of the boundary value problem

$$u''(\tau) + \mu\, u^+(\tau) - \nu\, u^-(\tau) + g(u(\tau)) = p(\tau) , \tag{2.5.1}$$
$$u(0) = u(\pi) = 0$$

(where $u^+(\tau) = \max\{u(\tau), 0\}$, $u^-(\tau) = \max\{-u(\tau), 0\}$) if for every $v \in \mathring{W}_2^1\langle 0, \pi\rangle$ the integral identity

$$(2.5.2) \quad -\int_0^\pi u_0'(\tau)\, v'(\tau)\, \mathrm{d}\tau + \mu \int_0^\pi u_0^+(\tau)\, v(\tau)\, \mathrm{d}\tau - \nu \int_0^\pi u_0^-(\tau)\, v(\tau)\, \mathrm{d}\tau + \\ + \int_0^\pi g(u_0(\tau))\, v(\tau)\, \mathrm{d}\tau = \int_0^\pi p(\tau)\, v(\tau)\, \mathrm{d}\tau$$

holds.

The existence of a weak solution of the boundary value problem (2.5.1) will be proved (under some assumptions) by means of Lemma 2.3 the assumptions of which will be verified in 2.6 and 2.8.

Let $u \in \mathring{W}_2^1\langle 0, \pi\rangle$ and $p \in L_1(0, \pi)$ be fixed. It is easy to see that

$$j_u : v \longmapsto -\int_0^\pi u'(\tau)\, v'(\tau)\, \mathrm{d}\tau\,,$$

$$s_u : v \longmapsto -\mu \int_0^\pi u^+(\tau)\, v(\tau)\, \mathrm{d}\tau + \nu \int_0^\pi u^-(\tau)\, v(\tau)\, \mathrm{d}\tau\,,$$

$$r_u : v \longmapsto \int_0^\pi g(u(\tau))\, v(\tau)\, \mathrm{d}\tau\,,$$

$$\xi : v \longmapsto \int_0^\pi p(\tau)\, v(\tau)\, \mathrm{d}\tau$$

are continuous linear functionals on the space $\mathring{W}_2^1\langle 0, \pi\rangle = X = Y$. By the Riesz representation theorem there exist uniquely determined elements $J(u)$, $S(u)$, $R(u)$, $y \in X$ such that

$$\langle J(u), v\rangle = j_u(v)\,, \quad \langle S(u), v\rangle = s_u(v)\,, \quad \langle R(u), v\rangle = r_u(v)\,, \quad \langle y, v\rangle = \xi(v)$$

for any $v \in X$.

2.6. Lemma. a) *The mapping J is an isomorphism on $\mathring{W}_2^1\langle 0, \pi\rangle$;*

b) *the mappings S and R are completely continuous;*

c) *there exist $\alpha \geqq 0$, $\beta \geqq 0$, $\gamma \in \langle 0, 1)$ such that*

$$\|R(u)\|_{\mathring{W}_2^1} \leqq \alpha + \beta \|u\|_{\mathring{W}_2^1}^{\gamma}$$

for every $u \in \mathring{W}_2^1\langle 0, \pi\rangle$;

d) $S(tu) = t\, S(u)$ *for every $u \in \mathring{W}_2^1\langle 0, \pi\rangle$ and $t \geqq 0$.*

Proof. The assertions a), c) and d) are obvious. The assertion b) follows immediately from the complete continuity of the imbedding from $\mathring{W}_2^1\langle 0, \pi\rangle$ into $C_0^0\langle 0, \pi\rangle$.

2.7. Lemma. *Let $p \in C^0\langle 0, \pi\rangle$ and let $u_0 \in \mathring{W}_2^1\langle 0, \pi\rangle$ be a weak solution of the boundary value problem (2.5.1). Then $u_0 \in C_0^2\langle 0, \pi\rangle$ and the equation in (2.5.1) is satisfied at every $\tau \in \langle 0, \pi\rangle$.*

(This regularity result can be obtained immediately by integrating by parts in the integral identity (2.5.2).)

Denote by $\mathbb{N}$ the set of all positive integers.

2.8. Lemma. *The boundary value problem*

$$u''(\tau) + \bar{\mu}\, u^+(\tau) - \bar{\nu}\, u^-(\tau) = 0\,, \qquad u(0) = u(\pi) = 0 \tag{2.8.1}$$

has a nontrivial weak solution if and only if one from the following conditions is satisfied:

a) $\bar{\nu} = 1$, *$\bar{\mu}$ is arbitrary;*

b) *$\bar{\nu}$ is arbitrary,* $\bar{\mu} = 1$;

c) $\bar{\mu} > 1,\ \bar{\nu} > 1,\ \omega_1(\bar{\nu}, \bar{\mu}) = \dfrac{\sqrt{\bar{\mu}}\sqrt{\bar{\nu}}}{\sqrt{\bar{\mu}} + \sqrt{\bar{\nu}}} \in \mathbb{N}$;

d) $\bar{\mu} > 1,\ \bar{\nu} > 1,\ \omega_2(\bar{\nu}, \bar{\mu}) = \dfrac{\sqrt{\bar{\nu}}(\sqrt{\bar{\mu}} - 1)}{\sqrt{\bar{\mu}} + \sqrt{\bar{\nu}}} \in \mathbb{N}$;

e) $\bar{\mu} > 1,\ \bar{\nu} > 1,\ \omega_3(\bar{\nu}, \bar{\mu}) = \dfrac{\sqrt{\bar{\mu}}(\sqrt{\bar{\nu}} - 1)}{\sqrt{\bar{\mu}} + \sqrt{\bar{\nu}}} \in \mathbb{N}$.

Proof. Let $u_0 \in \mathring{W}_2^1\langle 0, \pi\rangle$ be a nontrivial weak solution of (2.8.1). In virtue of the assertion of Lemma 2.7 it is $u_0 \in C_0^2\langle 0, \pi\rangle$ and u_0 is a nontrivial classical solution of (2.8.1). According to the Uniqueness Theorem for ordinary differential equations the function u_0 has a finite number of zero points in the interval $(0, \pi)$. If u_0 has no zero point in $(0, \pi)$ then we obtain either a) or b). If u_0 has a zero point in $(0, \pi)$ then $\bar{\nu} > 0$, $\bar{\mu} > 0$ and the function u_0 is periodic with the period

$$\pi\left(\frac{1}{\sqrt{\bar{\nu}}} + \frac{1}{\sqrt{\bar{\mu}}}\right)$$

(since on the interval where the function u_0 is positive there exists a constant $m > 0$ such that $u_0(\tau) = m \sin \sqrt{(\bar{\mu})}\, \tau$ and analogously $u_0(\tau) = n \sin \sqrt{(\bar{\nu})}\, \tau$ with a suitable constant $n < 0$ on the interval where $u_0(\tau) < 0$). Hence one of the equations

$$k\left(\frac{1}{\sqrt{\bar{\nu}}} + \frac{1}{\sqrt{\bar{\mu}}}\right) = 1\,,$$

$$k\left(\frac{1}{\sqrt{\bar{\nu}}} + \frac{1}{\sqrt{\bar{\mu}}}\right) + \frac{1}{\sqrt{\bar{\mu}}} = 1\,,$$

$$k\left(\frac{1}{\sqrt{\bar{\nu}}} + \frac{1}{\sqrt{\bar{\mu}}}\right) + \frac{1}{\sqrt{\bar{\nu}}} = 1$$

must have a positive integer k for a solution. Thus one of the conditions c)–e) is fulfilled.

Conversely, if one of the conditions a)–e) is fulfilled then it is easy to construct a nontrivial classical solution of (2.8.1) and thus also a nontrivial weak solution.

Let $k \in \langle 0, 1\rangle$ and let us consider the equation (2.3.4) in our special case, i.e. we shall seek the nontrivial $u \in \mathring{W}_2^1\langle 0, \pi\rangle$ such that the integral identity

$$-\int_0^\pi u'(\tau)\, v'(\tau)\, d\tau = -\frac{\mu + k\nu}{1 + k}\int_0^\pi u^+(\tau)\, v(\tau)\, d\tau + \frac{\nu + k\mu}{1 + k}\int_0^\pi u^-(\tau)\, v(\tau)\, d\tau$$

holds for all $v \in \mathring{W}_2^1\langle 0, \pi\rangle$. (According to the assertion of Lemma 2.7, $u \in C_0^2\langle 0, \pi\rangle$ and satisfies at every point $\tau \in \langle 0, \pi\rangle$ the equation

$$u''(\tau) + \frac{\mu + k\nu}{1 + k}\, u^+(\tau) - \frac{\nu + k\mu}{1 + k}\, u^-(\tau) = 0\,.)$$

2.9. Theorem. *Let $\mu < 1$, $\nu < 1$, let a continuous function g satisfy the condition* (2.4.1). *Then the boundary value problem* (2.5.1) *is weakly solvable for every* $p \in L_1(0, \pi)$.

Proof. If $\mu < 1$ and $\nu < 1$ then also

$$\frac{\mu + k\nu}{1 + k} < 1\,, \quad \frac{\nu + k\mu}{1 + k} < 1\,.$$

With respect to the assertion of Lemma 2.8, the equations (2.3.4) have for arbitrary $k \in \langle 0, 1\rangle$ the trivial solution only. The other assumptions of Lemma 2.3 are verified in Lemma 2.6. Thus the assertion of Theorem follows from Lemma 2.3.

2.10. Remark. The assertion of the previous theorem is well-known. It follows immediately from the Leray-Lions theorem (see [11]).

Let $\mu > 1$, $\nu > 1$ and put

$$\Phi_i(\nu, \mu) = \max_{t \in \langle 0,1 \rangle} \omega_i \left(\frac{\nu + t\mu}{1 + t}, \frac{\mu + t\nu}{1 + t} \right),$$

$$\varphi_i(\nu, \mu) = \min_{t \in \langle 0,1 \rangle} \omega_i \left(\frac{\nu + t\mu}{1 + t}, \frac{\mu + t\nu}{1 + t} \right)$$

for $i = 1, 2, 3$.

Analogously as Theorem 2.9, we obtain immediately from the above lemmas the following result.

2.11. Theorem. *Let $\mu > 1$, $\nu > 1$ and let the continuous function g satisfy the condition (2.4.1). Then the boundary value problem (2.5.1) is weakly solvable for every $p \in L_1(0, \pi)$ provided*

$$\langle \varphi_i(\nu, \mu), \Phi_i(\nu, \mu) \rangle \cap \mathbb{N} = \emptyset \qquad (2.11.1)_i$$

for $i = 1, 2, 3$.

2.12. Corollary. *Let $\mu = \nu = m^2$, $m \notin \mathbb{N}$ and let the continuous function g satisfy the condition (2.4.1). Then the boundary value problem (2.5.1) is weakly solvable for every $p \in L_1(0, \pi)$.*

2.13. Remark. It is possible to obtain the assertion of the previous corollary also from the so-called "*Fredholm alternative for nonlinear operators*" (see e.g. [6, Chapter II]).

2.14. Remark. Let $1 < \mu < \nu$, $k \doteq \nu + \mu$. By elementary calculation we obtain:

$$\Phi_1(\nu, \mu) = \max_{\varrho \in \langle \mu, k/2 \rangle} \frac{\sqrt{\varrho}\sqrt{(k - \varrho)}}{\sqrt{\varrho} + \sqrt{(k - \varrho)}} = \frac{1}{2}\sqrt{\left(\frac{\mu + \nu}{2}\right)};$$

$$\varphi_1(\nu, \mu) = \min_{\varrho \in \langle \mu, k/2 \rangle} \frac{\sqrt{\varrho}\sqrt{(k - \varrho)}}{\sqrt{\varrho} + \sqrt{(k - \varrho)}} = \frac{\sqrt{\nu}\sqrt{\mu}}{\sqrt{\nu} + \sqrt{\mu}};$$

$$\Phi_2(\nu, \mu) = \max_{\varrho \in \langle \mu, k/2 \rangle} \frac{\sqrt{(k - \varrho)}\,(\sqrt{\varrho} - 1)}{\sqrt{\varrho} + \sqrt{(k - \varrho)}} = \frac{1}{2}\left(\sqrt{\left(\frac{\mu + \nu}{2}\right)} - 1\right);$$

$$\varphi_2(\nu, \mu) = \min_{\varrho \in \langle \mu, k/2 \rangle} \frac{\sqrt{(k - \varrho)}\,(\sqrt{\varrho} - 1)}{\sqrt{\varrho} + \sqrt{(k - \varrho)}} = \frac{\sqrt{\nu}\,(\sqrt{\mu} - 1)}{\sqrt{\nu} + \sqrt{\mu}}.$$

In the same way we have

$$\Phi_3(\nu, \mu) = \max_{\varrho \in \langle \mu, k/2 \rangle} z(\varrho)$$

and

$$\varphi_3(\nu, \mu) = \min_{\varrho \in \langle \mu, k/2 \rangle} z(\varrho),$$

where

$$z : \varrho \mapsto \frac{\sqrt{\varrho}\,(\sqrt{(k-\varrho)} - 1)}{\sqrt{\varrho} + \sqrt{(k-\varrho)}}.$$

It is

$$z'(\varrho) = \frac{(k-\varrho)^{3/2} - \varrho^{3/2} - k}{2(\sqrt{\varrho} + \sqrt{(k-\varrho)})^2 \sqrt{\varrho}\,\sqrt{(k-\varrho)}}$$

on $\langle \mu, k/2 \rangle$. If $\nu(\sqrt{\nu} - 1) \leqq \mu(\sqrt{\mu} + 1)$ then $z'(\varrho) \leqq 0$ and

$$\Phi_3(\nu, \mu) = \frac{\sqrt{\mu}\,(\sqrt{\nu} - 1)}{\sqrt{\mu} + \sqrt{\nu}},$$

$$\varphi_3(\nu, \mu) = \frac{1}{2}\left(\sqrt{\left(\frac{\nu + \mu}{2}\right)} - 1\right).$$

If $\nu(\sqrt{\nu} - 1) > \mu(\sqrt{\mu} + 1)$ then there exists exactly one $\varrho_0 \in (\mu, k/2)$ such that $z'(\varrho_0) = 0$ and

$$z(\varrho_0) = \max_{\varrho \in \langle \mu, k/2 \rangle} z(\varrho) = \varrho_0^{3/2}/k.$$

Thus

$$\Phi_3(\nu, \mu) = z(\varrho_0),$$

$$\varphi_3(\nu, \mu) = \min\left\{\frac{1}{2}\left(\sqrt{\left(\frac{\nu + \mu}{2}\right)} - 1\right), \quad \frac{\sqrt{\mu}\,(\sqrt{\nu} - 1)}{\sqrt{\mu} + \sqrt{\nu}}\right\}.$$

It follows from the previous calculation that the conditions $(2.11.1)_i$ $(i = 1, 2, 3)$ assume that form

(2.14.1) $$\left\langle \frac{\sqrt{\nu}\,\sqrt{\mu}}{\sqrt{\nu} + \sqrt{\mu}}, \ \frac{1}{2}\sqrt{\left(\frac{\mu + \nu}{2}\right)} \right\rangle \cap \mathbb{N} = \emptyset$$

and

(2.14.2) $$\left\langle \frac{\sqrt{\nu}\,(\sqrt{\mu} - 1)}{\sqrt{\mu} + \sqrt{\nu}}, \ A(\nu, \mu) \right\rangle \cap \mathbb{N} = \emptyset,$$

where

$$A(\nu, \mu) = \begin{cases} \dfrac{\sqrt{\mu}\,(\sqrt{\nu} - 1)}{\sqrt{\mu} + \sqrt{\nu}} & \text{if } \nu(\sqrt{\nu} - 1) \leqq \mu(\sqrt{\mu} + 1) \\ z(\varrho_0) \ \text{ if } \ \nu(\sqrt{\nu} - 1) > \mu(\sqrt{\mu} + 1), \end{cases}$$

for $\varphi_2(\nu, \mu) \leqq \varphi_3(\nu, \mu) \leqq \Phi_2(\nu, \mu) \leqq \Phi_3(\nu, \mu)$. It is easy to see that

$$z(\varrho_0) = \varrho_0^{3/2}/k < \frac{1}{2}\sqrt{\left(\frac{\nu + \mu}{2}\right)}.$$

Since $\varphi_2(\nu, \mu) \leqq \varphi_1(\nu, \mu)$ and $\Phi_3(\nu, \mu) < \Phi_1(\nu, \mu)$, we shall consider the condition

$$\left\langle \frac{\sqrt{\nu}\,(\sqrt{\mu} - 1)}{\sqrt{\mu} + \sqrt{\nu}},\ \frac{1}{2}\sqrt{\left(\frac{\nu + \mu}{2}\right)}\right\rangle \cap \mathbb{N} = \emptyset \tag{2.14.3}$$

instead of (2.14.1) and (2.14.2) if $\nu(\sqrt{\nu} - 1) > \mu(\sqrt{\mu} + 1)$. It is possible to consider the condition (2.14.3) also if

$$\nu(\sqrt{\nu} - 1) \leqq \mu(\sqrt{\mu} + 1)\,.$$

2.15. Corollary. *Let a continuous function g satisfy the condition* (2.4.1). *Let* $m > 1$. *Then there exists* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *the boundary value problem* (2.5.1) *is weakly solvable for every* $p \in L_1(0, \pi)$ *provided* $\mu = m^2$, $\nu = (m + \varepsilon)^2$.

Proof. It is sufficient (see 2.11 and 2.14) to show that there exists ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ we have

$$(m + \varepsilon)^2\,(m + \varepsilon - 1) \leqq m^2(m + 1)\,,$$
$$(m/2, (m + \varepsilon)/2) \cap \mathbb{N} = \emptyset\,,$$
$$((m - 1)/2,\ (m + \varepsilon - 1)/2) \cap \mathbb{N} = \emptyset\,.$$

The existence of ε_0 with the previous properties is trivial.

2.16. Corollary. *Let a continuous function g satisfy the condition* (2.4.1). *Suppose that* $\varepsilon \geqq 0$, $\delta \geqq 0$, $\varepsilon + \delta < 1$ *and let n be an odd positive integer*, $n \geqq 3$. *Put* $\mu = (n + \varepsilon)^2$, $\nu = (n + 1 - \delta)^2$. *Then the boundary value problem* (2.5.1) *has a weak solution for arbitrary* $p \in L_1(0, \pi)$.

Proof. According to 2.11 and 2.14 it is sufficient to verify the condition (2.14.3). It is easy to see that

$$\left\langle \frac{(n + 1 - \delta)\,(n + \varepsilon - 1)}{2n + 1 - \delta + \varepsilon},\ \frac{1}{2}\sqrt{\left(\frac{(n + \varepsilon)^2 + (n + 1 - \delta)^2}{2}\right)}\right\rangle \cap \mathbb{N} \subset$$
$$\subset \left(\frac{n + \varepsilon - 1}{2},\ \frac{n + 1 - \delta}{2}\right) \cap \mathbb{N} = \emptyset$$

for n is odd.

2.17. Corollary. *Let a continuous function g satisfy the condition* (2.4.1). *Suppose that* $1 > \varepsilon \geqq 0$, $1 > \delta \geqq 0$, $\varepsilon + \delta > 0$. *Put* $\mu = (n - \varepsilon)^2$, $\nu = (n + \delta)^2$, *where n is an odd positive integer, $n \geqq 3$. Moreover, let*

$$\delta(n - 1 - 2\varepsilon) > \varepsilon(n + 1)\,. \tag{2.17.1}$$

Then the boundary value problem (2.5.1) *has a weak solution for arbitrary* $p \in L_1(0, \pi)$.

Proof. Similarly as in the proof of 2.16 we have

$$\left\langle \frac{(n + \delta)(n - \varepsilon - 1)}{2n + \delta - \varepsilon}, \frac{1}{2}\sqrt{\left(\frac{(n - \varepsilon)^2 + (n + \delta)^2}{2}\right)} \right\rangle \cap \mathbb{N} \subset$$

$$\subset \left(\frac{n - 1}{2}, \frac{n + \delta}{2}\right) \cap \mathbb{N} = \emptyset$$

for n is odd.

2.18. Example. In the previous corollaries we solved the problem when the nonlinearity jumps over one eigenvalue (see 2.15 and 2.17) or jumps from an eigenvalue to the next one (see 2.16). 2.11 and 2.14 imply also the weak solvability of (2.5.1) if the nonlinearity jumps over two eigenvalues. An example is provided by the case $\mu = 2.75^2$, $\nu = 4.75^2$ for

$$\left\langle \frac{4.75 \times 1.75}{2.75 + 4.75}, \frac{1}{2}\sqrt{\left(\frac{2.75^2 + 4.75^2}{2}\right)} \right\rangle \cap \mathbb{N} \subset (1, 2) \cap \mathbb{N} = \emptyset\,.$$

2.19. Remark. *Let g be a continuous function satisfying* (2.4.1). *Let the couple μ, ν satisfy the assumptions from one of the parts* 2.15–2.18. *Then the boundary value problem*

$$u''(\tau) + \nu\, u^+(\tau) - \mu\, u^-(\tau) + g(u(\tau)) = p(\tau)\,, \qquad u(0) = u(\pi) = 0 \tag{2.19.1}$$

has a weak solution for arbitrary $p \in L_1(0, \pi)$.

Proof. Let $p \in L_1(0, \pi)$. The continuous function $\xi \mapsto -g(-\xi)$ satisfies the condition (2.4.1). By 2.15–2.18, the boundary value problem

$$u''(\tau) + \mu\, u^+(\tau) - \nu\, u^-(\tau) - g(-u(\tau)) = p(\tau)\,,$$

$$u(0) = u(\pi) = 0$$

has a weak solution $u_0 \in \mathring{W}_2^1\langle 0, \pi\rangle$. Thus the function $-u_0$ is a weak solution of (2.19.1).

2.20. Open problems. a) Are the assertions of 2.16 and 2.17 true also if n is an even integer?

b) Is true that the boundary value problem (2.5.1) is weakly solvable for arbitrary $p \in L_1(0, \pi)$ if only $\mu > 1$, $\nu > 1$, $\mu \neq \nu$?

3. JUMPING OFF THE LEAST EIGENVALUE

3.1. Notation. Let g be a continuous function defined on $(-\infty, \infty)$ and suppose that there exist $c_1 \geqq 0$, $c_2 \geqq 0$ such that

$$|g(\xi)| \leqq c_1 + c_2|\xi| \tag{3.1.1}$$

for every $\xi \in (-\infty, \infty)$. Let $\nu > 1$ be a real number. Define the mappings $L : C_0^2\langle 0, \pi\rangle \to C^0\langle 0, \pi\rangle$, $N : C_0^2\langle 0, \pi\rangle \to C^0\langle 0, \pi\rangle$ by the formulas

$$L : u \mapsto u'' + u\,, \tag{3.1.2}$$

$$N : u \mapsto \nu_1 u^-(\tau) - g(u(\tau))\,, \tag{3.1.3}$$

where $\nu_1 = \nu - 1$.

To obtain the (classical) solution of the boundary value problem

$$u''(\tau) + u^+(\tau) - \nu u^-(\tau) + g(u(\tau)) = p(\tau)\,, \qquad u(0) = u(\pi) = 0 \tag{3.1.4}$$

for $p \in C^0\langle 0, \pi\rangle$ we have to show that the operator equation

$$L(u) = N(u) + p \tag{3.1.5}$$

is solvable in $C_0^2\langle 0, \pi\rangle$.

3.2. Lemma. a) *The mapping L defined by the formula* (3.1.2) *is linear and continuous. The null-space* Ker $[L]$ *of L is a linear hull generated by the function* $\tau \mapsto \sin \tau$, *i.e.,*

$$\operatorname{Ker}[L] = \{\lambda \sin \tau;\ \lambda \in (-\infty, \infty)\}\,.$$

b) *The image* Im $[L]$ *of L is*

$$\left\{z \in C^0\langle 0, \pi\rangle;\ \int_0^\pi z(\tau) \sin \tau \,d\tau = 0\right\}.$$

c) *The mappings*

$$Q : z \mapsto \frac{2 \sin \tau}{\pi} \int_0^\pi z(\xi) \sin \xi \,d\xi\,, \quad z \in C^0\langle 0, \pi\rangle\,, \tag{3.2.1}$$

$$P : x \mapsto \frac{2 \sin \tau}{\pi} \int_0^\pi x(\xi) \sin \xi \,d\xi\,, \quad x \in C_0^2\langle 0, \pi\rangle \tag{3.2.2}$$

are continuous linear projections in the spaces considered and

$$\operatorname{Im}[P] = \operatorname{Ker}[L], \quad \operatorname{Im}[Q^c] = \operatorname{Im}[L],$$

where $Q^c = Id - Q$.

d) *The mapping N is completely continuous. Moreover,*

$$\|N(u)\|_{C^0} \leqq c_1 + (c_2 + v_1)\|u\|_{C_0^2}$$

for every $u \in C_0^2\langle 0, \pi\rangle$.

(The assertions a)–c) are well-known. The proof of d) is trivial.)

3.3. Remark. The restriction $\tilde{L}$ of the operator L to

$$X_1 = (Id - P)(C_0^2\langle 0, \pi\rangle)$$

is one-to-one and according to the Closed Graph Theorem the mapping $\tilde{L}$ is an isomorphism between X_1 and $\operatorname{Im}[L]$. Denote its inverse by K (the so-called right inverse of L). Considering the same norm on X_1 as in $C_0^2\langle 0, \pi\rangle$, let $\|K\|$ be the norm of K.

3.4. Lemma. *Let there exist* $t_0 \in (-\infty, \infty)$ *and* $v_0 \in X_1$ *such that*

(3.4.1) $$QN(t \sin \tau + v(\tau)) + Q(p) = 0,$$

(3.4.2) $$KQ^cN(t \sin \tau + v(\tau)) + K\,Q^c(p) = v$$

hold with $t = t_0$, $v = v_0$.

Then $u_0(\tau) = t_0 \sin \tau + v_0(\tau)$ *is a solution of the equation* (3.1.5).

Proof. $L(u_0) - N(u_0) - p = L(v_0) - Q\,N(u_0) - Q^c\,N(u_0) - Q(p) - Q^c(p) = 0_X$.

3.5. Lemma. *Let* $v \in X_1$. *Define*

(3.5.1) $$\varphi_v : t \mapsto v_1 \int_0^\pi (t \sin \tau + v(\tau))^- \sin \tau \,d\tau - \int_0^\pi g(t \sin \tau + v(\tau)) \sin \tau \,d\tau$$

(*i.e.*, $QN(t \sin \tau + v(\tau)) = (2 \sin \tau/\pi)\,\varphi_v(t)$).

Let

(3.5.2) $$t_1 < t_2 \Rightarrow g(t_1) \leqq g(t_2).$$

Then:

(a) φ_v *is continuous*;

(b) $\lim_{t \to \infty} \varphi_v(t) = -2\,g(\infty)$, $(g(\infty) = \lim_{\xi \to \infty} g(\xi))$;

(c) $\lim_{t \to -\infty} \varphi_v(t) = \infty$.

(The assertions follow immediately from the classical theorems about interchanging the integration and the limit process.)

3.6. Remark. From the previous lemma we see that a necessary condition for the solvability of (3.4.1) is

$$\int_0^\pi p(\tau) \sin \tau \, d\tau \leqq 2\, g(\infty)$$

provided $g(\infty)$ is finite.

3.7. Lemma. *Suppose in addition that the function g satisfies the following condition*:

(3.7.1) $$g(\infty) < \infty \; ; \quad t_1 \neq t_2 \, , \quad g(t_1) = g(t_2) \Rightarrow g(t_1) = g(\infty) \, .$$

Let $p \in C^0\langle 0, \pi \rangle$ *and*

(3.7.2) $$\int_0^\pi p(\tau) \sin \tau \, d\tau < 2\, g(\infty) \, .$$

Then for every $v \in X_1$ *there exists exactly one* $t(v) \in (-\infty, \infty)$ *such that*

(3.7.3) $$\varphi_v(t(v)) + \int_0^\pi p(\tau) \sin \tau \, d\tau = 0 \, .$$

The mapping $v \mapsto t(v)$ *is continuous.*

Proof. The existence of t such that

$$\varphi_v(t) + \int_0^\pi p(\tau) \sin \tau \, d\tau = 0$$

is evident. Suppose $t_1 < t_2$. Obviously $\varphi_v(t_1) \geqq \varphi_v(t_2)$. Let

(3.7.4) $$\varphi_v(t_1) + \int_0^\pi p(\tau) \sin \tau \, d\tau = \varphi_v(t_2) + \int_0^\pi p(\tau) \sin \tau \, d\tau = 0 \, .$$

Thus $\varphi_v(t_1) = \varphi_v(t_2)$ which implies

(3.7.5) $$\int_0^\pi (t_1 \sin \tau + v(\tau))^- \sin \tau \, d\tau = \int_0^\pi (t_2 \sin \tau + v(\tau))^- \sin \tau \, d\tau$$

and

(3.7.6) $$\int_0^\pi g(t_1 \sin \tau + v(\tau)) \sin \tau \, d\tau = \int_0^\pi g(t_2 \sin \tau + v(\tau)) \sin \tau \, d\tau \, .$$

From (3.7.5) we have $0 < t_1 \sin \tau + v(\tau) < t_2 \sin \tau + v(\tau)$ and from (3.7.6) we have $g(t_1 \sin \tau + v(\tau)) = g(\infty)$. Substituting into (3.7.4) we obtain

$$-2\, g(\infty) = \int_0^\pi p(\tau) \sin \tau \, d\tau = 0$$

which contradicts (3.7.2).

The continuity of $v \mapsto t(v)$ is obvious.

3.8. Lemma. *There exist* $c_3(\nu_1) \geqq 0$ *and* $c_4 \geqq 0$ *such that*

$$|t(v)| \leqq c_3(\nu_1) + c_4 \|v\|_{C_0^2}$$

for every $v \in X_1$.

Proof. Since there exists $M > 0$ such that

$$\left| \frac{v(\tau)}{\sin \tau} \right| \leqq M \|v\|_{C_0^2}$$

for $\tau \in (0, \pi)$ and $v \in C_0^2\langle 0, \pi \rangle$ we have

$$\nu_1 \int_0^\pi (t(v) \sin \tau - M \|v\|_{C_0^2} \sin \tau)^- \sin \tau \, d\tau \geqq$$

$$\geqq \int_0^\pi g(t(v) \sin \tau - M \|v\|_{C_0^2} \sin \tau) \sin \tau \, d\tau - \int_0^\pi p(\tau) \sin \tau \, d\tau \, .$$

Let $t(v) > M \|v\|_{C_0^2}$. Then

$$\int_0^\pi g(\lambda \sin \tau) \sin \tau \, d\tau = \int_0^\pi p(\tau) \sin \tau \, d\tau \geqq \int_0^\pi g((t(v) - M \|v\|_{C_0^2}) \sin \tau) \sin \tau \, d\tau$$

and thus $t(v) \leqq \lambda + M \|v\|_{C_0^2}$ for every $v \in X_1$.

On the other hand,

$$-\nu_1\, t(v) \int_0^\pi \sin^2 \tau \, d\tau = -\nu_1 \int_0^\pi (t(v) \sin \tau + v(\tau)) \sin \tau \, d\tau \leqq$$

$$\leqq \nu_1 \int_0^\pi (t(v) \sin \tau + v(\tau))^- \sin \tau \, d\tau \leqq 2\, g(\infty) - \int_0^\pi p(\tau) \sin \tau \, d\tau$$

and

$$t(v) \geqq \frac{2}{\pi \nu_1} \left\{ \int_0^\pi p(\tau) \sin \tau \, d\tau - 2\, g(\infty) \right\} .$$

3.9. Theorem. *Suppose* (3.1.1), (3.5.2), (3.7.1), (3.7.2) *and*

$$\|K\| \, \|Q^c\| \, (c_2 + \nu_1)(c_4 + 1) < 1 . \tag{3.9.1}$$

Then the boundary value problem (3.1.4) *has a solution.*

Proof. The foregoing consideration shows that it is sufficient to prove that the mapping

$$F : v \mapsto K Q^c N(t(v) \sin \tau + v(\tau)) + K \, Q^c(p)$$

has at least one fixed point in X_1. Obviously, F is completely continuous. Moreover,

$$\|F(v)\|_{C_0{}^2} \leqq c_7 + \|K\| \, \|Q^c\| \, (c_2 + \nu_1)(c_4 + 1) \, \|v\|_{C_0{}^2}$$

$$(c_7 = \|K \, Q^c(p)\|_{C_0{}^2} + \|K\| \, \|Q^c\| \, c_1 + (c_2 + \nu_1) \, c_3(\nu_1))$$

for every $v \in X_1$.

Now it is easy to see that there exists $\varrho > 0$ such that

$$\|F(v)\|_{C_0{}^2} \leqq \varrho$$

for each $v \in X_1$, $\|v\|_{C_0{}^2} \leqq \varrho$. The Schauder fixed point theorem (see 2.2) implies our assertion.

3.10. Corollary. *Suppose* (3.1.1), (3.5.2), (3.7.1), (3.9.1) *and*

$$g(0) \neq g(\infty) . \tag{3.10.1}$$

Then the condition (3.7.2) *is necessary and sufficient for the solvability of* (3.1.4).

Proof. Let

$$\int_0^\pi p(\tau) \sin \tau \, d\tau = 2 \, g(\infty)$$

and suppose that $u_0(\tau) = t_0 \sin \tau + v_0(\tau)$ is a solution of (3.1.4). From

$$\nu_1 \int_0^\pi (t_0 \sin \tau + v_0(\tau))^- \sin \tau \, d\tau = \int_0^\pi g(t_0 \sin \tau + v_0(\tau)) \sin \tau \, d\tau - 2 \, g(\infty) \leqq 0$$

it follows that $(t_0 \sin \tau + v_0(\tau))^- = 0$ and $t_0 \sin \tau + v_0(\tau) \geqq 0$. Thus

$$\int_0^\pi g(t_0 \sin \tau + v_0(\tau)) \sin \tau \, d\tau = 2 \, g(\infty) = \int_0^\pi g(\infty) \sin \tau \, d\tau$$

and $g(t_0 \sin \tau + v_0(\tau)) = g(\infty)$ for every $\tau \in \langle 0, \pi \rangle$. The last fact is in contradiction with (3.10.1).