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CONGRUENCES OF SURFACES IN $\mathbb{C}^2$

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A. ŠVEC [1] studied transitive layers of hypersurfaces in $\mathbb{C}^2$; in what follows, I am going to describe transitive two-parametric systems of surfaces in $\mathbb{C}^2$.

As usually, let us write $\mathbb{C}^2 = (\mathbb{R}^4, J)$, $J: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ being the endomorphism satisfying $J^2 = -\text{id}$ and defined by $J(v) = iv$. In $\mathbb{C}^2 = (\mathbb{R}^4, J)$, consider a congruence, i.e., a 2-parametric system of 2-dimensional surfaces $\mathcal{L}$ such that through each point $m \in \mathbb{R}^4$ there passes exactly one surface $V^2 \in \mathcal{L}$. Let $T_m(V_m^2)$ be the tangent plane of V_m^2 at m ; in what follows, we restrict ourselves to congruences $\mathcal{L}$ such that

$$(1) \quad T_m(V_m^2) \cap J T_m(V_m^2) = \{0\}$$

for each point $m \in \mathbb{R}^4$. At each point $m \in \mathbb{R}^4$, let us choose a frame $\{v_1, v_2, v_3, v_4\}$ such that

$$(2) \quad v_1, v_2 \in T_m(V_m^2), \quad v_3 = Jv_1, \quad v_4 = Jv_2.$$

Choosing another frames $\{w_1, w_2, w_3, w_4\}$ satisfying equations analogous to (2), we have

$$(3) \quad \begin{aligned} w_1 &= \alpha v_1 + \beta v_2, & w_2 &= \gamma v_1 + \delta v_2, \\ w_3 &= \alpha v_3 + \beta v_4, & w_4 &= \gamma v_3 + \delta v_4, \\ & & \alpha\delta - \beta\gamma &\neq 0. \end{aligned}$$

Thus the congruence $\mathcal{L}$ induces on $\mathbb{R}^4$ a G -structure, being given by (3).

Let us write

$$(4) \quad \begin{aligned} [v_1, v_2] &= a_1 v_1 + a_2 v_2, \\ [v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4, \\ [v_1, v_4] &= c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4, \\ [v_2, v_3] &= d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4, \\ [v_2, v_4] &= e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4, \\ [v_3, v_4] &= f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4. \end{aligned}$$

From the Jacobi identities

$$(5) \quad [v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0;$$

$$i, j, k = 1, \dots, 4;$$

we get

$$(6) \quad \begin{aligned} v_1 d_1 + d_2 a_1 + d_3 b_1 + d_4 c_1 - v_2 b_1 - b_3 d_1 - b_4 e_1 + v_3 a_1 - a_2 d_1 &= 0, \\ v_1 d_2 + d_3 b_2 + d_4 c_2 - v_2 b_2 + b_1 a_2 - b_3 d_2 - b_4 e_2 + v_3 a_2 - a_1 b_2 &= 0, \\ v_1 d_3 + d_4 c_3 - v_2 b_3 - b_4 e_3 - a_1 b_3 - a_2 d_3 &= 0, \\ v_1 d_4 + d_3 b_4 + d_4 c_4 - v_2 b_4 - b_3 d_4 - b_4 e_4 - a_1 b_4 - a_2 d_4 &= 0; \end{aligned}$$

$$(7) \quad \begin{aligned} v_1 e_1 + e_2 a_1 + e_3 b_1 + e_4 c_1 - v_2 c_1 - c_3 d_1 - c_4 e_1 + v_4 a_1 - a_2 e_1 &= 0, \\ v_1 e_2 + e_3 b_2 + e_4 c_2 - v_2 c_2 + c_1 a_2 - c_3 d_2 - c_4 e_2 + v_4 a_2 - a_1 c_2 &= 0, \\ v_1 e_3 + e_3 b_3 + e_4 c_3 - v_2 c_3 - c_3 d_3 - c_4 e_3 - a_1 c_3 - a_2 e_3 &= 0, \\ v_1 e_4 + e_3 b_4 - v_2 c_4 - c_3 d_4 - a_1 c_4 - a_2 e_4 &= 0; \end{aligned}$$

$$(8) \quad \begin{aligned} v_1 f_1 + f_2 a_1 + f_3 b_1 + f_4 c_1 - v_3 c_1 + c_2 d_1 - c_4 f_1 + v_4 b_1 - b_2 e_1 - \\ - b_3 f_1 &= 0, \\ v_1 f_2 + f_2 a_2 + f_3 b_2 + f_4 c_2 - v_3 c_2 + c_1 b_2 + c_2 d_2 - c_4 f_2 + v_4 b_2 - \\ - b_1 c_2 - b_2 e_2 - b_3 f_2 &= 0, \\ v_1 f_3 + f_4 c_3 + c_1 b_3 + c_2 d_3 - c_4 f_3 + v_4 b_3 - b_1 c_3 - b_2 e_3 - v_3 c_3 &= 0, \\ v_1 f_4 + f_3 b_4 - v_3 c_4 + c_1 b_4 + c_2 d_4 + v_4 b_4 - b_1 c_4 - b_2 e_4 - b_3 f_4 &= 0; \end{aligned}$$

$$(9) \quad \begin{aligned} v_2 f_1 - f_1 a_1 + f_3 d_1 + f_4 e_1 - v_3 e_1 + e_1 b_1 + e_2 d_1 - e_4 f_1 + v_4 d_1 - \\ - d_1 c_1 - d_2 e_1 - d_3 f_1 &= 0, \\ v_2 f_2 - f_1 a_2 + f_3 d_2 + f_4 e_2 - v_3 e_2 + e_1 b_2 - e_4 f_2 + v_4 d_2 - d_1 c_2 - \\ - d_3 f_2 &= 0, \\ v_2 f_3 + f_4 e_3 - v_3 e_3 + e_1 b_3 + e_2 d_3 - e_4 f_3 + v_4 d_3 - d_1 c_3 - d_2 e_3 &= 0, \\ v_2 f_4 + f_3 d_4 - v_3 e_4 + e_1 b_4 + e_2 d_4 + v_4 d_4 - d_1 c_4 - d_2 e_4 - d_3 e_3 &= 0. \end{aligned}$$

Analogously, we have

$$(10) \quad \begin{aligned} [w_1, w_2] &= A_1 w_1 + A_2 w_2, \\ [w_1, w_3] &= B_1 w_1 + B_2 w_2 + B_3 w_3 + B_4 w_4, \\ [w_1, w_4] &= C_1 w_1 + C_2 w_2 + C_3 w_3 + C_4 w_4, \end{aligned}$$

$$[w_2, w_3] = D_1 w_1 + D_2 w_2 + D_3 w_3 + D_4 w_4,$$

$$[w_2, w_4] = E_1 w_1 + E_2 w_2 + E_3 w_3 + E_4 w_4,$$

$$[w_3, w_4] = F_1 w_1 + F_2 w_2 + F_3 w_3 + F_4 w_4$$

with the integrability conditions

$$\begin{aligned}
 (11) \quad & w_1 D_1 + D_2 A_2 + D_3 B_1 + D_4 C_1 - w_2 B_1 - B_3 D_1 - B_4 E_1 + w_3 A_1 - \\
 & - A_2 D_1 = 0, \\
 & w_1 D_2 + D_3 B_2 + D_4 C_2 - w_2 B_2 + B_1 A_2 - B_3 D_2 - B_4 E_2 + w_3 A_2 - \\
 & - A_1 B_2 = 0, \\
 & w_1 D_3 + D_4 C_3 - w_3 B_3 - B_4 E_3 - A_1 B_3 - A_2 D_3 = 0, \\
 & w_1 D_4 + D_3 B_4 + D_4 C_4 - w_2 B_4 - B_3 D_4 - B_4 E_4 + A_1 B_4 - A_2 D_4 = 0, \\
 & w_1 E_1 + E_2 A_1 + E_3 B_1 + E_4 C_1 - w_2 C_1 - C_3 D_1 - C_4 E_1 + \\
 & + w_4 A_1 - A_2 E_1 = 0, \\
 & w_1 E_2 + E_3 B_2 + E_4 C_2 - w_2 C_2 + C_1 A_2 - C_3 D_2 - C_4 E_2 + \\
 & + w_4 A_2 - A_1 C_2 = 0, \\
 & w_1 E_3 + E_3 B_3 + E_4 C_3 - w_2 C_3 - C_3 D_3 - C_4 E_3 - A_1 C_3 - A_2 C_3 = 0, \\
 & w_1 E_4 + E_3 B_4 + E_4 C_4 - w_2 C_4 - C_3 D_4 - C_4 E_4 - A_1 C_4 - A_2 C_4 = 0, \\
 & w_1 F_1 + F_2 A_1 + F_3 B_1 + F_4 C_1 - w_3 C_1 + C_2 D_1 - C_4 F_1 + \\
 & + w_4 B_1 - B_2 E_1 - B_3 F_1 = 0, \\
 & w_1 F_2 + F_2 A_2 + F_3 B_2 + F_4 C_2 - w_3 C_2 + C_2 D_2 - C_4 F_2 + \\
 & + C_1 B_2 + w_4 B_2 - B_1 C_2 - B_2 E_2 - B_3 F_2 = 0, \\
 & w_1 F_3 + F_4 C_3 + C_1 B_3 + C_2 D_3 - C_4 F_3 + w_4 B_3 - B_1 C_3 - B_2 E_3 - \\
 & - w_3 C_3 = 0, \\
 & w_1 F_4 + F_3 B_4 - w_3 C_4 + C_1 B_4 + C_2 D_4 + w_4 B_4 - B_1 C_4 - \\
 & - B_2 E_4 - B_3 F_4 = 0, \\
 & w_2 F_1 - F_1 A_1 + F_3 D_1 + F_4 E_1 - w_3 E_1 + E_1 B_1 + E_2 D_1 - E_4 F_1 + \\
 & + w_4 D_1 - D_1 C_1 - D_2 E_1 - D_3 F_1 = 0, \\
 & w_2 F_2 - F_1 A_2 + F_3 D_2 + F_4 E_2 - w_3 E_2 + E_1 B_2 - E_4 E_2 + \\
 & + w_4 D_2 - D_1 C_2 - D_3 F_2 = 0,
 \end{aligned}$$

$$\begin{aligned}
& w_2 F_3 + F_4 E_3 - V_3 E_3 + E_1 B_3 + E_2 D_3 - E_4 F_3 + V_4 D_3 - \\
& - D_1 C_3 - D_2 E_3 = 0, \\
& w_2 F_4 + F_3 D_4 - w_3 E_4 + E_1 B_4 + E_2 D_4 + w_4 D_4 - D_1 C_4 - \\
& - D_2 E_4 - D_3 F_3 = 0.
\end{aligned}$$

Comparing (4) with (10), we get

$$\begin{aligned}
(12) \quad [w_1, w_2] &= [\alpha v_1 + \beta v_2, \gamma v_1 + \delta v_2] = w_1 \gamma \cdot v_1 + w_2 \delta \cdot v_2 - w_2 \alpha \cdot v_1 - \\
&- w_2 \beta \cdot v_2 - \gamma \beta (a_1 v_1 + a_2 v_2) + \alpha \delta (a_1 v_1 + a_2 v_2) = \\
&= A_1 w_1 + A_2 w_2 = A_1 (\alpha v_1 + \beta v_2) + A_2 (\gamma v_1 + \delta v_2),
\end{aligned}$$

$$\begin{aligned}
(13) \quad [w_1, w_3] &= [\alpha v_1 + \beta v_2, \alpha v_3 + \beta v_4] = w_1 \alpha \cdot v_3 + w_1 \beta \cdot v_4 + \\
&+ \alpha^2 (b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4) + \alpha \beta (d_1 v_1 + d_2 v_2 + \\
&+ d_3 v_3 + d_4 v_4) + \beta \alpha (c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4) + \\
&+ \beta^2 (e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4) - w_3 \alpha \cdot v_1 - w_3 \beta \cdot v_2 = \\
&= B_1 w_1 + B_2 w_2 + B_3 w_3 + B_4 w_4 = B_1 (\alpha v_1 + \beta v_2) + \\
&+ B_2 (\gamma v_1 + \delta v_2) + B_3 (\alpha v_3 + \beta v_4) + B_4 (\gamma v_3 + \delta v_4),
\end{aligned}$$

$$\begin{aligned}
(14) \quad [w_1, w_4] &= [\alpha v_1 + \beta v_2, \gamma v_3 + \delta v_4] = w_1 \gamma \cdot v_3 + w_1 \delta \cdot v_4 + \\
&+ \alpha \gamma (b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4) + \beta \gamma (d_1 v_1 + d_2 v_2 + \\
&+ d_3 v_3 + d_4 v_4) - w_4 \alpha \cdot v_1 - w_4 \beta \cdot v_2 + \alpha \delta (c_1 v_1 + c_2 v_2 + \\
&+ c_3 v_3 + c_4 v_4) + \beta \delta (e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4) = \\
&= C_1 w_1 + C_2 w_2 + C_3 w_3 + C_4 w_4 = C_1 (\alpha v_1 + \beta v_2) + \\
&+ C_2 (\gamma v_1 + \delta v_2) + C_3 (\alpha v_3 + \beta v_4) + C_4 (\gamma v_3 + \delta v_4),
\end{aligned}$$

$$\begin{aligned}
(15) \quad [w_2, w_3] &= [\gamma v_1 + \delta v_2, \alpha v_3 + \beta v_4] = w_2 \alpha \cdot v_3 + w_2 \beta \cdot v_4 + \\
&+ \alpha \gamma (b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4) + \alpha \delta (d_1 v_1 + d_2 v_2 + \\
&+ d_3 v_3 + d_4 v_4) - w_3 \delta \cdot v_2 + \beta \gamma (c_1 v_1 + c_2 v_2 + c_3 v_3 + \\
&+ c_4 v_4) + \beta \delta (e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4) = D_1 w_1 + \\
&+ D_2 w_2 + D_3 w_3 + D_4 w_4 = D_1 (\alpha v_1 + \beta v_2) + \\
&+ D_2 (\gamma v_1 + \delta v_2) + D_3 (\alpha v_3 + \beta v_4) + D_4 (\gamma v_3 + \delta v_4),
\end{aligned}$$

$$\begin{aligned}
(16) \quad [w_2, w_4] &= [\gamma v_1 + \delta v_2, \gamma v_3 + \delta v_4] = w_2 \gamma \cdot v_3 + w_2 \delta \cdot v_4 + \\
&\quad + \gamma^2(b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4) + \gamma \delta(d_1 v_1 + d_2 v_2 + \\
&\quad + d_3 v_3 + d_4 v_4) + \gamma \delta(c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4) + \\
&\quad + \delta^2(e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4) - w_4 \gamma \cdot v_1 - w_4 \delta \cdot v_2 = \\
&= E_1 w_1 + E_2 w_2 + E_3 w_3 + E_4 w_4 = E_1(\alpha v_1 + \beta v_2) + \\
&\quad + E_2(\gamma v_1 + \delta v_2) + E_3(\alpha v_3 + \beta v_4) + E_4(\gamma v_3 + \delta v_4), \\
(17) \quad [w_3, w_4] &= [\alpha v_3 + \beta v_4, \gamma v_3 + \delta v_4] = w_3 \gamma \cdot v_3 + w_3 \delta \cdot v_4 - \\
&\quad - \gamma \beta(f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4) + \alpha \delta(f_1 v_1 + f_2 v_2 + \\
&\quad + f_3 v_3 + f_4 v_4) - w_4 \alpha \cdot v_3 - w_4 \beta \cdot v_4 = F_1 w_1 + F_2 w_2 + \\
&\quad + F_3 w_3 + F_4 w_4 = F_1(\alpha v_1 + \beta v_2) + F_2(\gamma v_1 + \delta v_2) + \\
&\quad + F_3(\alpha v_3 + \beta v_4) + F_4(\gamma v_3 + \delta v_4).
\end{aligned}$$

From (12)–(17),

$$\begin{aligned}
(18) \quad w_1 \alpha &= -\alpha^2 b_3 - \alpha \beta d_3 - \beta \alpha c_3 - \beta^2 e_3 + B_3 \alpha + B_4 \gamma, \\
w_2 \alpha &= -\alpha \gamma b_3 - \alpha \delta d_3 - \beta \gamma c_3 - \beta \delta e_3 + D_3 \alpha + D_4 \gamma, \\
w_3 \alpha &= \alpha^2 b_1 + \alpha \beta d_1 + \beta \alpha c_1 + \beta^2 e_1 - B_1 \alpha - B_2 \gamma, \\
w_4 \alpha &= \alpha \gamma b_1 + \beta \gamma d_1 + \alpha \delta c_1 + \delta \beta e_1 - C_1 \alpha - C_2 \gamma, \\
w_1 \beta &= -\alpha^2 b_4 - \alpha \beta d_4 - \beta \alpha c_4 - \beta^2 e_4 + B_3 \beta + B_4 \delta, \\
w_2 \beta &= -\alpha \gamma b_4 - \alpha \delta d_4 - \beta \gamma c_4 - \beta \delta e_4 + D_3 \beta + D_4 \delta, \\
w_3 \beta &= \alpha^2 b_2 + \alpha \beta d_2 + \beta \alpha c_2 + \beta^2 e_2 - B_1 \beta - B_2 \delta, \\
w_4 \beta &= \alpha \gamma b_2 + \beta \gamma d_2 + \alpha \delta c_2 + \delta \beta e_2 - C_1 \beta - C_2 \delta, \\
w_1 \gamma &= -\alpha \gamma b_3 - \beta \gamma d_3 - \alpha \delta c_3 - \beta \delta e_3 + C_3 \alpha + C_4 \gamma, \\
w_2 \gamma &= -\gamma^2 b_3 - \gamma \delta d_3 - \gamma \delta c_3 - \delta^2 e_3 + E_3 \alpha + E_4 \gamma, \\
w_3 \gamma &= \alpha \gamma b_1 + \alpha \delta d_1 + \beta \gamma c_1 + \beta \delta e_1 - D_1 \alpha - D_2 \gamma, \\
w_4 \gamma &= \gamma^2 b_1 + \gamma \delta d_1 + \gamma \delta c_1 + \delta^2 e_1 - E_1 \alpha - E_2 \gamma, \\
w_1 \delta &= -\alpha \gamma b_4 - \beta \gamma d_4 - \alpha \delta c_4 - \beta \delta e_4 + C_3 \beta + C_4 \delta, \\
w_2 \delta &= -\gamma^2 b_4 - \gamma \delta d_4 - \gamma \delta c_4 - \delta^2 e_4 + E_3 \beta + E_4 \delta, \\
w_3 \delta &= \alpha \gamma b_2 + \alpha \delta d_2 + \beta \gamma c_2 + \beta \delta e_2 - D_1 \beta - D_2 \delta, \\
w_4 \delta &= \gamma^2 b_2 + \gamma \delta d_2 + \gamma \delta c_2 + \delta^2 e_2 - E_1 \beta - E_2 \delta;
\end{aligned}$$

$$\begin{aligned}
(19) \quad w_1\gamma - w_2\alpha &= \beta\gamma a_1 - \alpha\delta a_1 + A_1\alpha + A_2\gamma, \\
w_1\delta - w_2\beta &= \beta\gamma a_2 - \alpha\delta a_2 + A_1\beta + A_2\delta, \\
w_3\gamma - w_4\alpha &= \beta\gamma f_3 - \alpha\delta f_3 + F_3\alpha + F_4\gamma, \\
w_3\delta - w_4\beta &= \beta\gamma f_4 - \alpha\delta f_4 + F_3\beta + F_4\delta;
\end{aligned}$$

$$\begin{aligned}
(20) \quad (\alpha\delta - \beta\gamma)f_1 &= \alpha F_1 + \gamma F_2, \\
(\alpha\delta - \beta\gamma)f_2 &= \beta F_1 + \delta F_2.
\end{aligned}$$

From (18), we obtain

$$\begin{aligned}
(21) \quad w_1\gamma - w_2\alpha &= -\beta\gamma d_3 + \alpha\delta d_3 - \alpha\delta c_3 + \beta\gamma c_3 + c_3\alpha - \\
&\quad - D_3\alpha + C_4\gamma - D_4\gamma, \\
w_1\delta - w_2\beta &= -\beta\gamma d_4 + \alpha\delta d_4 - \alpha\delta c_4 + \beta\gamma c_4 + c_3\beta - \\
&\quad - D_3\beta + C_4\delta - D_4\delta, \\
w_3\gamma - w_4\alpha &= \alpha\delta d_1 - \beta\gamma d_1 + \beta\gamma c_1 - \alpha\delta c_1 - D_1\alpha + \\
&\quad + C_1\alpha - D_2\gamma + C_2\gamma, \\
w_3\delta - w_4\beta &= \alpha\delta d_2 - \beta\gamma d_2 + \beta\gamma c_2 - \alpha\delta c_2 - D_1\beta + \\
&\quad + C_1\beta - D_2\delta + C_2\delta.
\end{aligned}$$

Multiplying the first equations in (19) and (21) by δ , the second ones by γ , the third ones by β and the fourth ones by α resp., we get

$$\begin{aligned}
(22) \quad \delta(\beta\gamma - \alpha\delta)a_1 + \alpha\delta A_1 + \gamma\delta A_2 &= \delta(\alpha\delta - \beta\gamma)d_3 + \\
&\quad + \delta(\beta\gamma - \alpha\delta)c_3 + \alpha\delta(C_3 - D_3) + \gamma\delta(C_4 - D_4), \\
\gamma(\beta\gamma - \alpha\delta)a_2 + \beta\gamma A_1 + \gamma\delta A_2 &= \gamma(\alpha\delta - \beta\gamma)d_4 + \\
&\quad + \gamma(\beta\gamma - \alpha\delta)c_4 + \beta\gamma(C_3 - D_3) + \gamma\delta(C_4 - D_4), \\
\beta(\beta\gamma - \alpha\delta)f_3 + \alpha\beta F_3 + \beta\gamma F_4 &= \beta(\alpha\delta - \beta\gamma)d_1 + \\
&\quad + \beta(\beta\gamma - \alpha\delta)c_1 - \alpha\beta(C_1 - D_1) + \beta\gamma(C_2 - D_2), \\
\alpha(\beta\gamma - \alpha\delta)f_4 + \alpha\beta F_3 + \alpha\delta F_4 &= \alpha(\alpha\delta - \beta\gamma)d_2 + \\
&\quad + \alpha(\beta\gamma - \alpha\delta)c_2 - \alpha\beta(C_1 - D_1) + \alpha\delta(C_2 - D_2)
\end{aligned}$$

and

$$\begin{aligned}
(23) \quad A_1 + D_3 - C_3 &= \delta(a_1 + d_3 - c_3) - \gamma(a_2 + d_4 - c_2), \\
F_4 + D_2 - C_2 &= \alpha(f_4 + d_2 - c_2) - \beta(f_3 + d_1 - c_1).
\end{aligned}$$

Analogously, multiplying the first equations in (19) and (21) by β , the second ones by α , the third ones by δ and the fourth ones by γ resp. we get

$$\begin{aligned}
 (24) \quad & \beta(\beta\gamma - \alpha\delta) a_1 + \beta\alpha A_1 + \beta\gamma A_2 = \beta(\alpha\delta - \beta\gamma) d_3 + \\
 & + \beta(\beta\gamma - \alpha\delta) c_3 + \beta\alpha(C_3 - D_3) + \beta\gamma(C_4 - D_4), \\
 & \alpha(\beta\gamma - \alpha\delta) a_2 + \alpha\beta A_1 + \alpha\delta A_2 = \alpha(\alpha\delta - \beta\gamma) d_4 + \\
 & + \alpha(\beta\gamma - \alpha\delta) c_4 + \alpha\beta(C_3 - D_3) + \alpha\delta(C_4 - D_4), \\
 & \delta(\beta\gamma - \alpha\delta) f_3 + \alpha\delta F_3 + \gamma\delta F_4 = \delta(\alpha\delta - \beta\gamma) d_1 + \\
 & + \delta(\beta\gamma - \alpha\delta) c_1 + \alpha\delta(C_1 - D_1) + \gamma\delta(C_2 - D_2), \\
 & \gamma(\beta\gamma - \alpha\delta) f_4 + \beta\gamma F_3 + \gamma\delta F_4 = \gamma(\alpha\delta - \beta\gamma) d_2 + \\
 & + \gamma(\beta\gamma - \alpha\delta) c_2 + \beta\gamma(C_1 - D_1) + \gamma\delta(C_2 - D_2)
 \end{aligned}$$

and

$$\begin{aligned}
 (25) \quad & A_2 + D_4 - C_4 = \alpha(a_2 + d_4 - c_4) - \beta(a_1 + d_3 - c_3), \\
 & F_3 + D_1 - C_1 = \delta(f_3 + d_1 - c_1) - \gamma(f_4 + d_2 - c_2).
 \end{aligned}$$

Thus we see that we may choose the frames $\{w_1, w_2, w_3, w_4\}$ in such a way that

$$(26) \quad A_1 + D_3 - C_3 = F_4 + D_2 - C_2 = 0.$$

Suppose that the frames $\{v_1, v_2, v_3, v_4\}$ have been chosen in this way, too. Then

$$(27) \quad a_1 + d_3 - c_3 = f_4 + d_2 - c_2 = 0$$

From (23) and (25),

$$\begin{aligned}
 (28) \quad & 0 = -\gamma(a_2 + d_4 - c_4), \\
 & A_2 + D_4 - C_4 = \alpha(a_2 + d_4 - c_4), \\
 & 0 = -\beta(f_3 + d_1 - c_1), \\
 & F_3 + D_1 - C_1 = \delta(f_3 + d_1 - c_1).
 \end{aligned}$$

The functions $a_2 + d_4 - c_4, f_3 + d_1 - c_1$ are thus relative invariants. Let us restrict ourselves to the "general" case

$$(29) \quad a_2 + d_4 - c_4 \neq 0, \quad f_3 + d_1 - c_1 \neq 0.$$

Then it is possible to choose the frames in such a way that

$$(30) \quad a_2 + d_4 - c_4 = 1, \quad f_3 + d_1 - c_1 = 1.$$