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ON OVALOIDS IN E^4

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One of the main tools used for characterizations of hyperspheres in E^n is the integral formula (1.14.1) of [1]. For a simple situation to be described below, we are going to rewrite this formula and to prove several more profound consequences of it.

Let $M^3 \subset E^4$ be an hypersurface satisfying: (i) *on M^3, there is a system of lines of curvature,* (ii) *the principal curvatures are positive,* (iii) *the boundary ∂M^3 of M^3 consists of umbilical points.*

In a suitable domain of M^3, consider the moving orthonormal frames $\{M, v_1, v_2, v_3, v_4\}$ such that v_1, v_2, v_3 are tangent to the lines of curvature. Then

$$\begin{aligned}
(1)\qquad \mathrm{d}M &= \omega^1 v_1 + \omega^2 v_2 + \omega^3 v_3 ,\\
\mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 ,\\
\mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 ,\\
\mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 ,\\
\mathrm{d}v_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3
\end{aligned}$$

with the usual integrability conditions, and we may write

$$(2)\qquad \omega_1^4 = a\omega^1 , \quad \omega_2^4 = b\omega^2 , \quad \omega_3^4 = c\omega^3 ,$$

$a > 0$, $b > 0$, $c > 0$ being the principal curvatures. From (2), we get

$$\begin{aligned}
(3)\qquad & \mathrm{d}a \wedge \omega^1 + (a-b)\,\omega_1^2 \wedge \omega^2 + (a-c)\,\omega_1^3 \wedge \omega^3 = 0 ,\\
& (a-b)\,\omega_1^2 \wedge \omega^1 + \mathrm{d}b \wedge \omega^2 + (b-c)\,\omega_2^3 \wedge \omega^3 = 0 ,\\
& (a-c)\,\omega_1^3 \wedge \omega^1 + (b-c)\,\omega_2^3 \wedge \omega^2 + \mathrm{d}c \wedge \omega^3 = 0
\end{aligned}$$

and the existence of functions $a_1, \ldots, c_3, e$ such that

$$\begin{aligned}
(4)\qquad \mathrm{d}a &= a_1\omega^1 + a_2\omega^2 + a_3\omega^3 , \quad (a-b)\,\omega_1^2 = a_2\omega^1 + b_1\omega^2 + e\omega^3 ,\\
\mathrm{d}b &= b_1\omega^1 + b_2\omega^2 + b_3\omega^3 , \quad (a-c)\,\omega_1^3 = a_3\omega^1 + e\omega^2 + c_1\omega^3 ,\\
\mathrm{d}c &= c_1\omega^1 + c_2\omega^2 + c_3\omega^3 , \quad (b-c)\,\omega_2^3 = e\omega^1 + b_3\omega^2 + c_2\omega^3 .
\end{aligned}$$

The *curvatures* of M^3 be defined by

$$H = a + b + c\,, \quad L = ab + ac + bc\,, \quad K = abc\,. \tag{5}$$

On M^3, consider the invariant 2-form

$$\tau = (b-c)^2\,\omega^1 \wedge \omega_2^3 - (a-c)^2\,\omega^2 \wedge \omega_1^3 + (a-b)^2\,\omega^3 \wedge \omega_1^2\,; \tag{6}$$

we have

$$\begin{aligned} d\tau = \{&2\varphi(b_1, c_1, a_1) + 2\varphi(a_2, c_2, b_2) + 2\varphi(a_3, b_3, c_3) + 6e^2 + \\ &+ (a-b)^2\,ab + (a-c)^2\,ac + (b-c)^2\,bc\}\,\omega^1 \wedge \omega^2 \wedge \omega^3\,, \end{aligned} \tag{7}$$

where

$$\varphi(x, y, z) = x^2 + y^2 - xy - xz - yz\,. \tag{8}$$

Theorem 1. *Let $M^3 \subset E^4$ satisfy* (i)–(iii) *and* (iv): *there is, on M^3,*

$$A\,\mathrm{d}a + B\,\mathrm{d}b + C\,\mathrm{d}c = 0\,, \tag{9}$$

A, B, C being functions such that

$$\begin{aligned} &A > 0\,, \quad B > 0\,, \quad C > 0\,; \quad \varrho(A, B, C) \geqq 0\,, \quad \varrho(A, C, B) \geqq 0\,, \\ &\varrho(B, C, A) \geqq 0\,; \quad \varrho(X, Y, Z) := 4(X + Z)(Y + Z) - (X + Y - Z)^2\,. \end{aligned} \tag{10}$$

Then M^3 is a (part of a) hypersphere.

Proof. From (9), we get

$$Aa_1 = -Bb_1 - Cc_1\,, \quad Bb_2 = -Aa_2 - Cc_2\,, \quad Cc_3 = -Aa_3 - Bb_3\,. \tag{11}$$

Then

$$\begin{aligned} A\varphi(b_1, c_1, a_1) &= (A + B)\,b_1^2 + (B + C - A)\,b_1c_1 + (A + C)\,c_1^2 \geqq 0\,, \\ B\varphi(a_2, c_2, b_2) &= (A + B)\,a_2^2 + (A + C - B)\,a_2c_2 + (B + C)\,c_2^2 \geqq 0\,, \\ C\varphi(a_3, b_3, c_3) &= (A + C)\,a_3^2 + (A + B - C)\,a_3b_3 + (B + C)\,b_3^2 \geqq 0 \end{aligned} \tag{12}$$

as a consequence of (10). From the Stokes formula $\int_{\partial M} \tau = \int_M d\tau$, we get $a = b = c$. QED.

Theorem 2. *Let $M^3 \subset E^4$ satisfy* (i)–(iii) *and* (iv): *there is a function $F(H, L, K)$ such that, on M^3,*

$$\begin{aligned} &\sigma(b, c) > 0\,, \quad \sigma(a, c) > 0\,; \quad \sigma(a, b) > 0\,; \\ &\sigma(\xi, \eta) := F_H + (\xi + \eta)\,F_L + \xi\eta F_K\,; \end{aligned} \tag{14}$$

$$(15) \qquad \varkappa(a, b, c) \geqq 0\,, \quad \varkappa(a, c, b) \geqq 0\,, \quad \varkappa(b, c, a) \geqq 0\,,$$
$$\varkappa(u, v, w) := 15F_H^2 + 4(2u^2 + 2v^2 + 5uv + 3uw + 3vw)\,F_L^2 +$$
$$+ (3u^2v^2 - u^2w^2 - v^2w^2 + 6u^2vw + 6uv^2w + 2uvw^2)\,F_K^2 +$$
$$+ 12(2u + 2v + w)\,F_HF_L + 6(3uv + uw + vw)\,F_HF_K +$$
$$+ 4(3u^2v + u^2w + 3uv^2 + v^2w + 7uvw)\,F_LF_K\,.$$

Then M^3 is a (part of a) hypersphere.

Proof. From (13), we get (9) with

$$(16) \qquad A = \sigma(b, c)\,, \quad B = \sigma(a, c)\,, \quad C = \sigma(a, b)$$

and the conditions (10) turn out to be exactly (14) and (15). QED.

Let us prove just two consequences of our Theorem 2.

Corollary 1. *Let $M^3 \subset E^4$ satisfy* (i)–(iii) *and* (iv): *we have, on M^3,*

$$(17) \qquad f(H, L, rHL + K) = 0\,,$$

$r \in \mathbb{R}$ satisfying $83r \geqq 6\sqrt{(3)} - 5$ and $f(\alpha, \beta, \gamma)$ being a function with one of its derivatives positive and the other two non-negative. Then M^3 is a (part of a) hypersphere.

Proof. Set

$$(18) \qquad F(H, L, K) = f(H, L, rHL + K)\,.$$

Then

$$(19) \qquad F_H = f_\alpha + rLf_\gamma\,, \quad F_L = f_\beta + rHf_\gamma\,, \quad F_K = f_\gamma\,,$$

$$(20) \qquad \sigma(\xi, \eta) = f_\alpha + (\xi + \eta)\,f_\beta + \{r(\xi + \eta)\,(2H - \xi - \eta) + (r + 1)\,\xi\eta\}\,f_\gamma\,,$$

and we have (14). Further,

$$(21) \qquad \varkappa(u, v, w) = \mu_1 f_\alpha^2 + \mu_2 f_\beta^2 + \mu_3 f_\alpha f_\beta + \mu_4 f_\alpha f_\gamma + \mu_5 f_\beta f_\gamma +$$
$$+ \{\mu_6 + (83r^2 + 10r - 1)\,w^2(u^2 + v^2)\}\,f_\gamma^2$$

with $\mu_i = \mu_i(u, v, w) \geqq 0$ for $u \geqq 0$, $v \geqq 0$, $w \geqq 0$, and (15) follow easily. QED.

Corollary 2. *Let $M^3 \subset E^4$ satisfy* (i)–(iii) *and* (iv): *we have, on M^3,*

$$(22) \qquad K = \text{const.}\,, \quad 4HK \geqq L^2\,.$$

Then M^3 is a (part of a) hypersphere.