Define a linear operator T on H by setting $Te_1 = -e_1$, $Te_2 = \sqrt{(3)}\, e_1$. It follows that $T^2 = -T$. Consider the following unit vectors

$$f = \tfrac{1}{2}(-e_1 + \sqrt{(3)}\, e_2)\,, \quad g = \tfrac{1}{2}(e_1 + \sqrt{(3)}\, e_2)\,.$$

It is easy to verify the equations

$$T^*e_1 = 2f\,, \quad T^*e_2 = 0\,, \quad Tg = e_1\,, \quad Tf = 2e_1\,, \quad T^*g = f\,, \quad T^*f = -f\,.$$

It follows that $\varphi(T) \subset \operatorname{Ker}(I - T^*T) = 0$ in spite of the fact that the line through e_1 is invariant with respect to T and T is an isometry on it. Also, $\varphi(T^*) \subset$ $\subset \operatorname{Ker}(I - TT^*) = 0$ although the line generated by f is invariant with respect to T^* and T^* is an isometry on it. For each natural number n we have

$$T^n g = (-1)^{n+1}\, e_1\,, \quad T^{*n} g = (-1)^{n+1} f\,.$$

This example shows that the two sets in lemma (1,6) may be different; this is, of course, only possible for operators of norm greater than one. Also, we see that there may exist invariant subspaces not contained in $\varphi(T)$ on which T is isometric. The same is true for T^*.

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Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).