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ISOMETRIC PARTS OF OPERATORS AND THE CRITICAL EXPONENT

VLASTIMIL PTÁK, Praha

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In the theory of linear operators in Hilbert space, the idea of splitting off a subspace on which the given operator possesses some distinguished property, has proved useful. The unitary part of a contraction is an example [6]. The normal part of an arbitrary bounded operator may be investigated in a similar manner [1]. Recently, E. DURSZT [2] has described the unitary part of an arbitrary bounded linear operator on a Hilbert space. In the present remark we collect some simple results concerning the analogous question of identifying isometric parts of operators; of course, it is not to be expected that results of the same order of completeness may be obtained in this case; nevertheless, under additional assumptions the results are satisfactory.

It is not difficult to describe, for each bounded linear operator T on a Hilbert space H, a subspace $\varphi(T)$, invariant with respect to T, on which T is isometric. It is easy to describe the basic properties of this subspace; we collect them in the first section. The second section shows how they may be used to characterize linear operators whose spectral radius equals their norm; in particular, this approach gives another simple explanation of the fact that the critical exponent of the n-dimensional Hilbert space is exactly n. Section three contains examples to show that some of the restrictions imposed in section one cannot be removed.

1. ISOMETRIC PARTS

In the first section, we collect some elementary facts concerning subspaces on which the given operator is isometric.

(1,1) *Let E be a linear space, T a linear operator on E. If F is a subspace of E, denote by $\varphi(F, T)$ the intersection*

$$F \cap T^{-1}F \cap T^{-2}F \cap \dots$$

Then $\varphi(F, T)$ is invariant with respect to T and contains every subspace of F which is invariant with respect to T.

Proof. It suffices to prove the second assertion. If $E_0 \subset F$ and E_0 is invariant with respect to T, we have, for each $k \geqq 1$ $T^k E_0 \subset E_0 \subset F$ whence $E_0 \subset T^{-k}F$. It follows that $E_0 \subset \varphi(F, T)$.

(1,2) *Let T be a bounded linear operator on a Hilbert space H. Set*

$$\varphi(T) = \varphi(\operatorname{Ker}(I - T^*T), T).$$

Then $\varphi(T)$ is invariant with respect to T and T restricted to $\varphi(T)$ is an isometry. If T is a contraction then $\varphi(T)$ contains every closed subspace H_0 invariant with respect to T such that $T \mid H_0$ is isometric.

Proof. If $x \in \varphi(T)$, we have, in particular $x \in \operatorname{Ker}(I - T^*T)$ whence $(x, x) = (T^*Tx, x) = (Tx, Tx)$. Hence T is isometric on $\varphi(T)$. Now suppose H_0 is a closed subspace of H invariant with respect to T and such that the restriction of T to H_0 is isometric. Now let T be a contraction. We have then, for each $x \in H_0$

$$((I - T^*T)\,x, x) = (x, x) - (T^*Tx, x) = (x, x) - (Tx, Tx) = 0\,;$$

since $I - T^*T \geqq 0$, this implies $(I - T^*T)\,x = 0$. We have thus $H_0 \subset \operatorname{Ker}(I - T^*T)$; at the same time, H_0 is invariant with respect to T so that, by (1,1), it follows that $H_0 \subset \varphi(T)$.

It is natural to ask whether $\varphi(T)$ reduces T. This, unfortunately, is not true in general. We have, however, the following important particular case.

(1,3) *Let T be a contraction on a Hilbert space H. Suppose $H_0 \subset H$ is invariant with respect to T and T isometric on H_0. If H_0 is finite-dimensional then H_0 reduces T.*

Proof. We begin by showing that T maps H_0 onto itself. First of all, T is injective on H_0 since $x \in H_0$ and $Tx = 0$ implies $|x| = |Tx| = 0$. The space H_0 being finite-dimensional, it follows from the injectiveness of T on H_0 that $TH_0 = H_0$. Given $x \in H_0$, we have $(T^*Tx, x) = (Tx, Tx) = (x, x)$ so that $((I - T^*T)\,x, x) = 0$. Since $I - T^*T \geqq 0$ it follows that $(I - T^*T)\,x = 0$. This shows that $H_0 \subset \operatorname{Ker}(I - T^*T)$. If $y \in H_0 = TH_0$, there exists a $h_0 \in H_0$ such that $y = Th_0$. Hence $T^*y = T^*Th_0 = h_0 \in H_0$. The space H_0 is thus invariant also with respect to T^*.

In the general case, we have

(1,4) *Let T be a bounded linear operator on a Hilbert space H. Then*

$$T^*(\varphi(T) \cap \operatorname{Ker}(I - TT^*)) \subset \varphi(T).$$

Proof. Suppose that $x \in \varphi(T)$ and $TT^*x = x$. We are to prove, for each $k = 0, 1, 2, \ldots$, the equation $(I - T^*T)\,T^kT^*x = 0$. For $k = 0$, we have $(I - T^*T)\,T^*x = T^*(I - TT^*)\,x = 0$. For $k > 0$, we have $(I - T^*T)\,T^kT^*x = (I - T^*T)\,.\,T^{k-1}(TT^*x) = (I - T^*T)\,T^{k-1}x = 0$. It follows that $T^*x \in \varphi(T)$ and the proof is complete.

For the sake of completeness we include the following theorem due to E. Durszt [2].

(1,5) *Let T be a bounded linear operator on a Hilbert space H. Then $\varphi(T) \cap \varphi(T^*)$ is invariant with respect to both T and T^*.*

If we denote by $\varkappa$ the family of all closed subspaces of H which reduce T and on which T is unitary, then

1° $\varphi(T) \cap \varphi(T^*) \in \varkappa$,

2° *every H_0 in $\varkappa$ is contained in $\varphi(T) \cap \varphi(T^*)$.*

Proof. According to the preceding lemma, we have

$$T^*(\varphi(T) \cap \varphi(T^*)) \subset T^*(\varphi(T) \cap \operatorname{Ker}(I - TT^*)) \subset \varphi(T);$$

by symmetry

$$T(\varphi(T) \cap \varphi(T^*)) \subset \varphi(T^*).$$

Together, these inclusions prove the first assertion.

Since $\varphi(T) \cap \varphi(T^*) \subset \operatorname{Ker}(I - T^*T) \cap \operatorname{Ker}(I - TT^*)$ the restriction of T to $\varphi(T) \cap \varphi(T^*)$ is unitary. Now suppose that $H_0 \in \varkappa$. Since H_0 reduces T, we have $(T \mid H_0)^* = T^* \mid H_0$. Since $T \mid H_0$ is unitary, it follows that $T^*Tx = TT^*x = x$ for all $x \in H_0$. Thus $H_0 \subset \operatorname{Ker}(I - T^*T)$ and $H_0 \subset \operatorname{Ker}(I - TT^*)$. By (1,1), the first inclusion, together with $TH_0 \subset H_0$, gives $H_0 \subset \varphi(T)$. The second inclusion and $T^*H_0 \subset H_0$ yields $H_0 \subset \varphi(T^*)$. The proof is complete.

(1,6) *Let T be a bounded linear operator in a Hilbert space H, n a natural number. Then*

$$\{x \in H;\ |x| = |Tx| = \ldots = |T^n x|\} \supset K \cap T^{-1}K \cap \ldots \cap T^{-(n-1)}K$$

where $K = \operatorname{Ker}(I - T^*T)$.

If T is a contraction, the two sets are equal; in particular, the set on the left-hand side is a subspace.

Proof. Let us show first that the subspace on the right-hand side always is contained in the set on the left-hand side, without assuming $|T| \leqq 1$. Indeed, $x \in K$ implies $(x, x) - (Tx, Tx) = (x - T^*Tx, x) = ((I - T^*T)x, x) = 0$. If $k > 0$ and $x \in T^{-k}K$ then $(T^k x, T^k x) - (T^{k+1}x, T^{k+1}x) = (T^k x - T^*T^{k+1}x, T^k x) = ((I - T^*T)T^k x, T^k x) = 0$. It follows that $x \in K \cap T^{-1}K \cap \ldots \cap T^{-(n-1)}K$ implies $|x| = |Tx| = \ldots = |T^n x|$.

Now assume that T is a contraction. Then

$$I - T^{*n}T^n = (I - T^*T) + T^*(I - T^*T)T + \ldots + T^{*n-1}(I - T^*T)T^{n-1}$$

and each of the summands is a nonnegative operator. Suppose now that $|x| = |T^n x|$. Then

$$0 = ((I - T^{*n}T^n)x, x) = (R_0 x, x) + (R_1 x, x) + \ldots + (R_{n-1}x, x)$$

where $R_k = T^{*k}(I - T^*T) T^k$. Since each R_k is nonnegative, it follows that $(R_k x, x) = 0$ for each $k = 0, 1, \ldots, n - 1$. Now $((I - T^*T) T^k x, T^k x) = (R_k x, x) = 0$ whence $(I - T^*T) T^k x = 0$ so that $x \in T^{-k}K$ for each k, $0 \leqq k \leqq n - 1$. The proof is complete.

(1,7) *Let T be a linear operator on a Hilbert space of dimension n. Then*

$$\varphi(T) = K \cap T^{-1}K \cap \ldots \cap T^{-(n-1)}K$$

Proof. Apply the Cayley-Hamilton theorem.

2. THE CRITICAL EXPONENT

The following theorem has been proved first by the present author in [7]. The original proof is geometrically intuitive and has not lost its interest although several new proofs have been published recently [3], [4], [9], [10]. The results of the preceding section make it possible to give a very simple proof. Let us remark that condition 4° does not appear explicitly in the author's original paper; the original proof, however, is based on singling out a nontrivial reducing subspace on which the operator is unitary.

(2,1) **Theorem.** *Let A be a linear operator on a Hilbert space of dimension n. Then the following conditions are equivalent.*

1° $|A| = |A|_\sigma$

2° $|A| = |A^2|^{1/2} = |A^3|^{1/3} = \ldots$

3° $|A| = |A^n|^{1/n}$

4° $A = |A| \begin{pmatrix} U & 0 \\ 0 & B \end{pmatrix}$

where U is unitary and B is a contraction; more precisely: there is a nontrivial subspace H_0 of H such that both H_0 and $H_0^\perp$ are invariant with respect to A, $|A|^{-1} A$ restricted to H_0 is unitary and, restricted to $H_0^\perp$, is a contraction.

Proof. Assume 1°. Then, for each natural number p,

$$|A| = |A|_\sigma = |A^p|_\sigma^{1/p} \leq |A^p|^{1/p} \leq |A| .$$

The implication 2° → 3° is immediate and so is 4° → 1°. The proof will be complete if we show that 3° implies 4°. Hence assume 3°; the operator $T = |A|^{-1} A$ has norm one.

The dimension of H being n, we have, using (1,7) and (1,6)

$$\varphi(T) = K \cap \ldots \cap T^{-(n-1)}K = \{x \in H;\ |x| = |T^n x|\} .$$

Since $1 = |T| = |T^n|$, there exists a vector $x \neq 0$ for which $|x| = |T^n x|$. It follows that the subspace $\varphi(T)$ is nontrivial. According to (1,3) it reduces T and $T | \varphi(T)$ is isometric, hence unitary. The proof is complete.

3. EXAMPLES

In this section we present simple examples to show that some of the hypotheses made in section one are essential for the validity of the results.

(3,1) Let H be an infinite dimensional Hilbert space with an orthonormal basis $e_0, e_1, e_2, \ldots$. Let U be the isometric shift defined by $Ue_k = e_{k+1}$ for $k = 0, 1, 2, \ldots$. It follows that $U^* e_0 = 0$ and $U^* e_i = e_{i-1}$ for $i = 1, 2, \ldots$. Hence $U^* U = I$ and $(I - UU^*) x = (x, e_0) e_0$. It follows that $\operatorname{Ker}(I - U^* U) = H$ whence $\varphi(U) = H$. On the other hand $x \in \operatorname{Ker}(I - UU^*)$ is equivalent to $(x, e_0) = 0$. If $x \in \varphi(U^*)$ then $U^{*k} x \in \operatorname{Ker}(I - UU^*)$ for each $k = 0, 1, 2, \ldots$. Now $U^{*k} x \in \operatorname{Ker}(I - UU^*)$ implies $(U^{*k} x, e_0) = 0$ whence $(x, e_k) = (x, U^k e_0) = (U^{*k} x, e_0) = 0$. This shows that $\varphi(U^*) = 0$. In this example $\varphi(U) = H$ so that $\varphi(U)$ is invariant with respect to U^*; nevertheless $\varphi(U^*) = 0$.

(3,2) Let H be a Hilbert space with an orthonormal basis e_k indexed by the set of all integers. Define a linear operator T by the equations

$$Te_k = e_{k+1} \quad \text{for} \quad k \geqq 0\,, \qquad Te_k = \tfrac{1}{2} e_{k+1} \quad \text{for} \quad k < 0\,.$$

Clearly T is a contraction; it is not difficult to show that

$$T^* e_j = e_{j-1} \quad \text{for} \quad j \geqq 1\,, \qquad T^* e_j = \tfrac{1}{2} e_{j-1} \quad \text{for} \quad j < 1\,.$$

Denote by P^+ the orthogonal projection on the closed linear span of the sequence $e_0, e_1, e_2, \ldots$ and by P^- the complementary projection so that $I = P^+ + P^-$. We have

$$T^* T = P^+ + \tfrac{1}{4} P^-\,, \quad TT^* = P^+ - \tfrac{1}{2} E_0 + \tfrac{1}{4} P^-$$

where $E_0 x = (x, e_0) e_0$. It follows that $(I - T^* T) x = 0$ if and only if $P^+ x = x$. This space being invariant with respect to T, we have $\varphi(T) = R(P^+)$. Clearly $\varphi(T)$ is not invariant with respect to T^*. Also $(I - TT^*) x = 0$ if and only if $(x, e_j) = 0$ for all $j \leqq 0$. Now suppose $x \in \varphi(T^*)$ and let j be a given integer. If $j \leqq 0$ then $(x, e_j) = 0$ since $x \in \operatorname{Ker}(I - TT^*)$. If $j > 0$, we have $T^{*j} x \in \operatorname{Ker}(I - TT^*)$ whence $(T^{*j} x, e_0) = 0$ so that

$$(x, e_j) = (x, T^j e_0) = (T^{*j} x, e_0) = 0\,.$$

It follows that $x = 0$. Hence $\varphi(T^*) = 0$. This example shows that the restriction to finite-dimensional subspaces H_0 is essential in lemma (1,3).

(3,3) Consider a two-dimensional Hilbert space H with an orthonormal basis e_1, e_2.