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WEAK-CONTINUITY AND CLOSED GRAPHS

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I. INTRODUCTION

The concept of weak-continuity was first introduced by N. LEVINE [4]. In 1968, M. K. SINGAL and A. R. SINGAL [7] defined almost-continuous functions and showed that every continuous function is almost-continuous and every almost-continuous function is weakly-continuous, but the converses are not necessarily true in general. Recently, P. E. LONG and L. L. HERRINGTON [6] have obtained several properties concerning almost-continuous functions and have given two sufficient conditions for almost-continuous functions to be continuous. The purpose of the present note is to give some sufficient conditions for weakly-continuous functions to be continuous.

II. DEFINITIONS

Let S be a subset of a topological space X. The closure of S and the interior of S are denoted by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$, respectively. Throughout this note, X and Y denote topological spaces, and by $f : X \to Y$ we represent a function f of a space X into a space Y.

Definition 1. A function $f : X \to Y$ is said to be *almost-continuous* [7] (resp. *weakly-continuous* [4]) if for each point $x \in X$ and each open set $V \subset Y$ containing $f(x)$, there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Int}_Y(\mathrm{Cl}_Y(V))$ (resp. $f(U) \subset \mathrm{Cl}_Y(V)$).

Definition 2. A subset S of a space X is said to be *N-closed relative to X* (briefly *N-closed*) [1] if for each cover $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ of S by open sets of X, there exists a finite subfamily $\mathscr{A}_0 \subset \mathscr{A}$ such that

$$S \subset \bigcup\{\mathrm{Cl}_X(U_\alpha) \mid \alpha \in \mathscr{A}_0\}.$$

Definition 3. A space X is said to be *rim-compact* [8, p. 276] if each point of X has a base of neighborhoods with compact frontiers.

III. WEAK-CONTINUITY AND CLOSED GRAPHS

It is well known that if $f : X \to Y$ is continuous and Y is Hausdorff, then the graph $G(f)$ is closed in the product space $X \times Y$. P. E. Long and L. L. Herrington showed that "continuous" in this result can be replaced by "almost-continuous" [6, Theorem 9]. Moreover, we shall show that "almost-continuous" can be replaced by "weakly-continuous".

Theorem 1. *If $f : X \to Y$ is weakly-continuous and Y is Hausdorff, then f has the following property:*

(P) *For each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap \mathrm{Int}_Y(\mathrm{Cl}_Y(V)) = \emptyset$.*

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W containing y and $f(x)$, respectively. Thus, we have $\mathrm{Int}_Y(\mathrm{Cl}_Y(V)) \cap \cap \mathrm{Cl}_Y(W) = \emptyset$. Since f is weakly-continuous, there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Cl}_Y(W)$. Therefore, we obtain $f(U) \cap \mathrm{Int}_Y(\mathrm{Cl}_Y(V)) = \emptyset$.

Remark 1. It is obvious that if a function has the property (P), then the graph is closed. The converse is not necessarily true, however, as the following example due to P. Kostyrko [3] shows.

Example 1. Let X and Y be the sets of positive integers. Let X have the discrete topology, Y have the cofinite topology and $f : X \to Y$ be the identity mapping. Then, although $G(f)$ is closed, f does not hold the property (P).

Corollary 1. *If $f : X \to Y$ is weakly-continuous and Y is Hausdorff, then $G(f)$ is closed.*

R. V. Fuller showed that if $f : X \to Y$ has the closed graph, then the inverse image $f^{-1}(K)$ of each compact set K of Y is closed in X [2, Theorem 3.6]. We shall obtain an analogous result to this theorem.

Theorem 2. *If $f : X \to Y$ has the property (P), then the inverse image $f^{-1}(K)$ of each N-closed set K of Y is closed in X.*

Proof. Assume that there exists a N-closed set $K \subset Y$ such that $f^{-1}(K)$ is not closed in X. Then, there exists a point $x \in \mathrm{Cl}_X(f^{-1}(K)) - f^{-1}(K)$. Since $f(x) \notin K$, for each $y \in K$ we have $(x, y) \notin G(f)$. Therefore, there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x)) \cap \mathrm{Int}_Y(\mathrm{Cl}_Y(V(y))) = \emptyset$. The family $\{V(y) \mid y \in K\}$ is a cover of K by open sets of Y. Since K is N-closed, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in K such that $K \subset \bigcup_{j=1}^{n} \mathrm{Int}_Y(\mathrm{Cl}_Y(V(y_j)))$. Now, put $U = \bigcap_{j=1}^{n} U_{y_j}(x)$. Then we obtain $f(U) \cap K = \emptyset$. On the other hand, since $x \in \mathrm{Cl}_X(f^{-1}(K))$, we have $f(U) \cap K \neq \emptyset$ because U is an open set containing x. This is a contradiction.

Remark 2. The converse to Theorem 2 is not always true, as Example 1 shows.

Corollary 2. *Let Y be a Hausdorff space such that every closed set is N-closed. If $f : X \to Y$ is weakly-continuous, then it is continuous.*

Proof. This is an immediate consequence of Theorem 1 and Theorem 2.

In [6, Theorem 7], it is shown that if Y is a rim-compact space and $f : X \to Y$ is an almost-continuous function with the closed graph, then f is continuous. We shall show that "almost-continuous" in this theorem can be replaced by "weakly-continuous".

Theorem 3. *If Y is a rim-compact space and $f : X \to Y$ is a weakly-continuous function with the closed graph, then f is continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since Y is rim-compact, there exists an open set $W \subset Y$ such that $f(x) \in W \subset V$ and the frontier $\mathrm{Fr}(W)$ is compact. It is obvious that $f(x) \notin \mathrm{Fr}(W)$. Thus, for each $y \in \mathrm{Fr}(W)$, we have $(x, y) \notin G(f)$. Since $G(f)$ is closed, there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x)) \cap V(y) = \emptyset$. The family $\{V(y) \mid y \in \mathrm{Fr}(W)\}$ is a cover of $\mathrm{Fr}(W)$ by open sets of Y. Since $\mathrm{Fr}(W)$ is compact, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in $\mathrm{Fr}(W)$ such that $\bigcup_{j=1}^{n} V(y_j) \supset \mathrm{Fr}(W)$. Now, since f is weakly-continuous, there exists an open set $U_0 \subset X$ containing x such that $f(U_0) \subset \mathrm{Cl}_Y(W)$. Put $U = U_0 \cap [\bigcap_{j=1}^{n} U_{y_j}(x)]$, then U is an open set containing x such that

$$f(U) \cap (Y - W) = f(U) \cap \mathrm{Fr}(W) \subset \bigcup_{j=1}^{n} f(U) \cap V(y_j) \subset \bigcup_{j=1}^{n} f(U_{y_j}(x)) \cap V(y_j) = \emptyset .$$

This shows that $f(U) \subset V$ and hence f is continuous.

Theorem 4. *Every rim-compact Hausdorff space is regular.*

Proof. This proof is similar to that of Theorem 3.

Corollary 3. *If Y is rim-compact Hausdorff and $f : X \to Y$ is weakly-continuous, then f is continuous.*

Proof. This follows immediately from [4, Theorem 2].

In [6, Theorem 8], it is shown that if f is an almost-continuous function of a first countable space into a countably compact Hausdorff space, then f is continuous. The following theorem shows that "almost-continuous" in this result can be replaced by "weakly-continuous".

Theorem 5. *Let X be a first countable space and Y a countably compact Hausdorff space. If $f : X \to Y$ is weakly-continuous, then f is continuous.*

Proof. This is an immediate consequence of Corollary 1 and [5, Theorem 2].