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A SIMPLE PROOF OF CAUCHY THEOREM

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Our aim is to prove the following assertion:

Cauchy Theorem. *If Γ is a cycle homologic with* 0 *in a region Ω and if a function F is holomoprhic in Ω, then $\int_\Gamma F = 0$.*

Let us first explain the necessary notions and notation: $\mathbf{C}$ and $\mathbf{R}$ denote respectively the sets of all (finite) complex and real numbers. Re z and Im z denote respectively the real and the imaginary parts of a number $z \in \mathbf{C}$. A continuous mapping $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ (where $\langle \alpha, \beta \rangle \subset \mathbf{R}$) for which the supremum $l(\varphi)$ of numbers $\sum_{k=1}^{n} |\varphi(t_k) - \varphi(t_{k-1})|$, $\alpha = t_0 < t_1 < \ldots < t_n = \beta$ is finite, is called a curve. If $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ is a curve, then we denote $\langle \varphi \rangle = \varphi(\langle \alpha, \beta \rangle)$. The curvilinear integral $\int_\varphi F$ of a continuous function $F : \langle \varphi \rangle \to \mathbf{C}$ over a curve $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ is defined to be the Stieltjes integral $\int_\alpha^\beta (F \circ \varphi)\, d\varphi$. The index of a point $\zeta \in \mathbf{C} - \langle \varphi \rangle$ with respect to a closed curve φ is denoted by $\operatorname{ind}_\varphi \zeta$ (a closed curve is a curve $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ for which $\varphi(\alpha) = \varphi(\beta)$). A region is an open connected set $\Omega \subset \mathbf{C}$. Any finite system $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ of closed curves satisfying $\langle \Gamma \rangle = \bigcup_{k=1}^{n} \langle \varphi_k \rangle \subset \Omega$ is called a cycle in Ω. The index of a point $\zeta \in \mathbf{C} - \langle \Gamma \rangle$ with respect to a cycle Γ is then defined by the relation $\operatorname{ind}_\Gamma \zeta = \sum_{k=1}^{n} \operatorname{ind}_{\varphi_k} \zeta$, the curvilinear integral of a continuous function $F : \langle \Gamma \rangle \to \mathbf{C}$ over a cycle Γ is defined by $\int_\Gamma F = \sum_{k=1}^{n} \int_{\varphi_k} F$; we set similarly $l(\Gamma) = \sum_{k=1}^{n} l(\varphi_k)$. If Γ is a cycle in Ω and $\operatorname{ind}_\Gamma \zeta = 0$ for every $\zeta \in \mathbf{C} - \Omega$, then the cycle Γ is said to be homologic with 0 in Ω. A set

$$Q = \{z;\ \alpha \leqq \operatorname{Re} z \leqq \beta,\ \gamma \leqq \operatorname{Im} z \leqq \delta\} \tag{1}$$

with $\alpha < \beta$, $\gamma < \delta$ is called a rectangle. Oriented sides of a rectangle Q as well as the oriented boundary (Q) of a rectangle Q are curves defined as usual (see [1], Introduction, § 8); the fact that

(2) $$\operatorname{ind}_{(Q)} \zeta = \begin{cases} 1 & \text{if } \zeta \in \operatorname{int} Q \text{ [1]} \\ 0 & \text{if } \zeta \in \mathbf{C} - Q \end{cases}$$

will be needed in the sequel (see [1], Chap. II, (4.7)), as well as the assertion

(3) if λ and λ^* are oriented sides of rectangles Q and Q^* respectively, satisfying $\langle\lambda\rangle = \langle\lambda^*\rangle$, $\operatorname{int} Q \cap \operatorname{int} Q^* = \emptyset$, then $\int_\lambda f = -\int_{\lambda^*} f$ for any continuous function $f : \langle\lambda\rangle \to \mathbf{C}$.

(This follows immediately from the definition of the oriented side of a rectangle.)

If $M, N \subset \mathbf{C}$ and $\Phi : M \times N \to \mathbf{C}$, then $\Phi(\cdot, \zeta)$ means for every $\zeta \in N$ the mapping of the set M into $\mathbf{C}$ assuming at a point $z \in M$ the value $\Phi(z, \zeta)$; the mapping $\Phi(z, \cdot)$ of the set N for $z \in M$ is defined similarly.

Proof of the Cauchy Theorem. In what follows Ω stands always for a fixed region, F is a fixed function holomorphic in Ω. For the sake of brevity we shall write

(4) $$\Phi(z, \zeta) = \frac{F(z)}{z - \zeta};$$

the definition domain of the function will be always evident from the context.

We shall prove the Cauchy Theorem in the above form under the assumption that the theorem as well as the Cauchy Formula were already proved for a rectangle $Q \subset \Omega$:

(5) If $Q \subset \Omega$ is a rectangle, then $\int_{(Q)} F = 0$ and $\int_{(Q)} \Phi(\cdot, \zeta) = 2\pi i\, F(\zeta) \operatorname{ind}_{(Q)} \zeta$ for $\zeta \in \Omega - \partial Q$.

(For a proof of the Cauchy Theorem for a rectangle as well as a proof of the identity $\int_{(Q)} \Phi(\cdot, \zeta) = 2\pi i\, F(\zeta)$ for $\zeta \in \operatorname{int} Q$ see [1], Chap. II, §§ 4 and 5; the identity $\int_{(Q)} \Phi(\cdot, \zeta) = 2\pi i\, F(\zeta) \operatorname{ind}_{(Q)} \zeta = 0$ for $\zeta \in \Omega - Q$ follows by virtue of the Cauchy Theorem applied to the function $\Phi(\cdot, \zeta)$ and the region $\Omega - \{\zeta\}$ in which this function is holomorphic.)

Let Γ be a cycle homologic with 0 in Ω and denote

(6) $$N = \{z; \operatorname{ind}_\Gamma z \neq 0\} \cup \langle\Gamma\rangle .$$

Since the set $\mathbf{C} - N$ is the union of all components of the set $\mathbf{C} - \langle\Gamma\rangle$ with $\operatorname{ind}_\Gamma = 0$, it is open. Since $\operatorname{ind}_\Gamma z = 0$ for sufficiently large $|z|$, the set N is bounded. Hence N is a compact subset of the region Ω and there exists a $\delta > 0$ so that

(7) $$\operatorname{dist}(z, N)\text{ [2]} < 2\delta \Rightarrow z \in \Omega .$$

[1]) $\operatorname{int} M$ denotes the interior of a set $M \subset \mathbf{C}$ while ∂M denotes its boundary.

[2]) If $z \in \mathbf{C}$ and $\emptyset \neq M \subset \mathbf{C}$, then $\operatorname{dist}(z, M) = \inf_{w \in M} |z - w|$.

Let $\mathcal{S}_0$ be the system of all squares of the form

$$(8) \qquad \{z; (m-1)\,\delta \leqq \operatorname{Re} z \leqq m\delta,\ (n-1)\,\delta \leqq \operatorname{Im} z \leqq n\delta\}$$

where m, n are integers. Let $\mathcal{S}$ be the system of all squares $Q \in \mathcal{S}_0$ satisfying $Q \cap N \neq \emptyset$. With regard to $\bigcup_{Q \in \mathcal{S}_0} Q = \mathbf{C}$ and to (7) we have

$$(9) \qquad N \subset \bigcup_{Q \in \mathcal{S}} Q \subset \Omega\,.$$

Since the set N is bounded, the system $\mathcal{S}$ is finite.

Let us divide the system $\mathcal{T}$ of all oriented sides of all squares $Q \in \mathcal{S}$ into two subsystems: The system $\mathcal{T}_1$ let consist of all curves λ for which the segment $\langle\lambda\rangle$ is a side of precisely one square from $\mathcal{S}$, and let $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$. According to (3) obviously

$$(10) \qquad \sum_{Q \in \mathcal{S}} \int_{(Q)} f = \sum_{\lambda \in \mathcal{T}} \int_\lambda f = \sum_{\lambda \in \mathcal{T}_1} \int_\lambda f \quad \text{for any continuous function } f : \bigcup_{Q \in \mathcal{S}} \partial Q \to \mathbf{C};$$

by virtue of (5) we have

$$(11) \qquad F(\zeta) = \frac{1}{2\pi i} \sum_{Q \in \mathcal{S}} \int_{(Q)} \Phi(\cdot, \zeta) \quad \text{for every} \quad \zeta \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q\,.$$

Since the function $\Phi(\cdot, \zeta)$ is continuous in $\bigcup_{Q \in \mathcal{S}} \partial Q$ for $\xi \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q$, (10) and (11) imply

$$(12) \qquad F(\zeta) = \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{T}_1} \int_\lambda \Phi(\cdot, \xi) \quad \text{for every} \quad \zeta \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q\,.$$

However, the function on the right hand side of the equality (12) is continuous in $\mathbf{C} - \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle$ by the well-known theorems; since the function F is continuous in Ω, (12) implies that

$$(13) \qquad F(\zeta) = \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{T}_1} \int_\lambda \Phi(\cdot, \zeta) \quad \text{for every} \quad \zeta \in \bigcup_{Q \in \mathcal{S}} Q - \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle\,.$$

If λ is an oriented side of a square $Q \in \mathcal{S}$ and if $\langle\lambda\rangle \cap N \neq \emptyset$, then both squares from $\mathcal{S}_0$ whose side is the segment $\langle\lambda\rangle$ belong to $\mathcal{S}$ and consequently, the curve λ belongs to $\mathcal{T}_2$. This implies in virtue of (9) that

$$(14) \qquad \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle \subset \Omega - N$$

so that according to the definition of the set N .

$$(15) \qquad z \in \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle \Rightarrow \operatorname{ind}_\Gamma z = 0\,.$$