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FREDHOLM RADIUS OF A POTENTIAL THEORETIC OPERATOR FOR CONVEX SETS

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Introduction. The use of the method of integral equations for solving boundary value problems for the Laplace equation dates back to the end of the nineteenth century. In that period, however, very strong smoothness restrictions on boundaries of domains in question were imposed. In 1919, J. RADON [10] studied the Dirichlet and the Neumann problems for plane domains bounded by curves of bounded rotation, not necessarily smooth. He investigated properties of the corresponding integral operator and evaluated its Fredholm radius (for the definition see below) in dependence on the character of angular points of the curve. General results for the plane case were obtained by J. KRÁL [2] 1965 (compare also [5] where further references are found). The Neumann problem with a weak characterization of the boundary values was investigated by J. Král [3] 1966 and JU. D. BURAGO, V. G. MAZJA [1] 1967 for general domains in high-dimensional Euclidean spaces.

In order to recall briefly some results of [3] we adopt the following notation. Suppose that M is an arbitrary open set with compact boundary $\partial M \neq \emptyset$ in the Euclidean m-space R^m $(m > 1)$. Given $x \in R^m$, $\theta \in \Gamma = \{z \in R^m; |z| = 1\}$ and $0 < r \leqq +\infty$, let $n_r^M(\theta, x)$ stand for the total number of all points $y \in H_\theta^r(x) = \{x + \varrho\theta; 0 < \varrho < r\}$ such that every neighborhood of y meets both $H_\theta^r(x) \cap M$ and $H_\theta^r(x) - M$ in a set of positive linear measure. (Such a point y is termed a hit of $H_\theta^r(x)$ on M.) The function $\theta \mapsto n_r^M(\theta, x)$ is a Baire function and one may put

$$v_r^M(x) = \int_\Gamma n_r^M(\theta, x)\, \mathrm{d}H_{m-1}(\theta)$$

where H_{m-1} denotes the $(m-1)$-dimensional Hausdorff measure. We shall denote

$$V_0^M = \lim_{r \to 0+} \sup_{y \in \partial M} v_r^M(y)\,.$$

Let us fix now an open set G with compact boundary $B \neq \emptyset$ and denote by $C'(B)$ the Banach space of all finite signed Borel measures with support in B. With each $\mu \in C'(B)$ we associate its potential

$$U\mu(x) = \int_B p(x - y)\, d\mu(y)$$

corresponding to the kernel $p(z) = |z|^{2-m}/(m - 2)$ or $p(z) = \log(1/|z|)$ according to $m > 2$ or $m = 2$ and we form the distribution $N_G U\mu$ (termed the generalized normal derivative of $U\mu$) over the space $\mathscr{D}$ of all infinitely differentiable functions φ with compact support in R^m defining

$$\langle \varphi, N_G U\mu \rangle = \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\mu(x)\, dx \,.$$

It follows from the results of [3], [4] that

$$V_0^G < \infty \tag{1}$$

is a necessary and sufficient condition for the representability of $N_G U\mu$ by means of an element of $C'(B)$ for any $\mu \in C'(B)$. In connection with the applicability of the Riesz-Schauder theory to the operator equation

$$N_G U\mu = \nu \tag{2}$$

over $C'(B)$ (under the hypothesis (1)) it is useful to write (2) in the form

$$[\tfrac{1}{2}AI' + (N_G U - \tfrac{1}{2}AI')]\, \mu = \nu$$

(where $A = H_{m-1}(\Gamma)$ and I' stands for the identity operator on $C'(B)$) and consider the quantity

$$\omega(N_G U - \tfrac{1}{2}AI') = \inf_Q \|N_G U - \tfrac{1}{2}AI' - Q\| \tag{3}$$

where Q varies over the space of all compact operators acting on $C'(B)$. Note that the reciprocal value of the quantity (3) is usually called the Fredholm radius of the operator $N_G U - \frac{1}{2}AI'$ and that the inequality

$$\omega(N_G U - \tfrac{1}{2}AI') < \tfrac{1}{2}A$$

permits one to apply the Fredholm theorems to the equation (2). The quantity (3) is evaluated in [3] in terms of $v_r^G(x)$ and the m-dimensional density $d_G(x)$ of the set G at x and also the relations between analytical properties of the operator $N_G U - \frac{1}{2}AI'$ and geometrical properties of G are studied there. If, in particular, the interior of the closure $\bar{G}$ of G coincides with G, then

$$\omega(N_G U - \tfrac{1}{2}AI') = V_0^G$$

(see [3], Lemma 3.4, Theorem 3.6 and [8], Lemma 30, Theorems 27, 29).

The main objective of this note is to evaluate $\omega(N_G U - \frac{1}{2}AI')$ in terms of d_G for convex G, which shows that boundary value problems for convex sets can always be treated by means of the Fredholm method. This follows from the following

Theorem. *Let $G \subset R^m$ be an open convex set with compact boundary $B \neq \emptyset$. Then there is $y_0 \in B$ such that*

$$\omega(N_G U - \tfrac{1}{2}AI') = \sup_{y \in B} A(\tfrac{1}{2} - d_G(y)) = A(\tfrac{1}{2} - d_G(y_0)) < \tfrac{1}{2}A\,. \tag{4}$$

Corollary. *The operator $N_G U - \frac{1}{2}AI'$ is compact if and only if $d_G(y) = \frac{1}{2}$ for any $y \in B$.*

The formula (4) represents an m-dimensional analog for convex sets of Radon's result on the Fredholm radius of the corresponding operator for sets bounded by curves of bounded rotation (cf. also [11], Chap. V, No. 91).

The relations between convexity of a set and positiveness of an operator of the generalized double layer potential is also studied and sets for which the equality $v_\infty^M = Ad_M$ holds on ∂M are characterized (see Theorems 11 and 14).

1. Notation. For an open set $M \subset R^m$ and $z \in R^m$ we denote

$$Q^M(z) = \{\theta \in \Gamma;\ H_\theta^\infty(z) \cap M \neq \emptyset\}$$

and define

$$a^M(z) = H_{m-1}(Q^m(z))\,.$$

In what follows, G will be a set satisfying the hypotheses of the above theorem. It is clear that it suffices to prove (4) under the additional assumption that G contains the origin. Hence we shall suppose that $0 \in G$ and for $\varrho > 0$ we set

$$G_\varrho = \{\varrho y;\ y \in G\}\,.$$

2. Lemma. *Let $0 < \sigma \leqq \tau \leqq 1$ and $x \in R^m$. Then*

$$a^{G_\tau}(x) \geqq a^{G_\sigma}(x)\,, \quad \lim_{\varrho \to 1-} a^{G_\varrho}(x) = a^G(x)\,.$$

Proof follows easily from the relations $Q^{G_\tau}(x) \supset Q^{G_\sigma}(x)$ and

$$\bigcup_{\varrho \in (0,1)} Q^{G_\varrho}(x) = Q^G(x)\,.$$

3. Lemma. *If $\varrho \in (0, 1)$, then the function a^{G_ϱ} is continuous on $R^m - \overline{G_\varrho}$.*

Proof. Since G_ϱ is convex, we have

$$a^{G_\varrho}(x) = \tfrac{1}{2}v_\infty^{G_\varrho}(x) \leqq \tfrac{1}{2}A\,, \quad x \in R^m - \overline{G_\varrho}\,.$$

Using formula (2.16) of Lemma 2.12 in [3], we obtain

$$a^{G_\varrho}(x) = \frac{1}{2}\int_{\partial G_\varrho} \frac{|n(y).(y - x)|}{|y - x|^m} \, \mathrm{d}H_{m-1}(y), \quad x \in R^m - \overline{G_\varrho},$$

which shows that the function a^{G_ϱ} is continuous on $R^m - \overline{G_\varrho}$. (We have denoted by $n(y)$ the Federer normal of G_ϱ at y.)

4. Proposition. *a^G is a lower semicontinuous function on R^m.*

Proof. It follows from Lemmas 2, 3 that the restriction of a^G to $R^m - G$ is a lower semicontinuous function. Since we have for any $z \in G$ and $x \in R^m$

$$a^G(x) \leqq A = a^G(z),$$

we see that a^G is lower semicontinuous on R^m.

5. Corollary. *d_G is a lower semicontinuous function on B and there is $y_0 \in B$ such that*

$$\inf_{y \in B} d_G(y) = d(y_0) > 0.$$

Proof. Proposition 2.6 and Lemma 2.7 in [3] imply the equality

$$a^G(x) = A\, d_G(x), \quad x \in B. \tag{5}$$

By Proposition 4, d_G is a lower semicontinuous function on B and, consequently, there is $y_0 \in B$ such that $\inf_{y \in B} d_G(y) = d_G(y_0)$. By Lemma 2,

$$d_G(y_0) \geqq A^{-1}\, a^{G_\sigma}(y_0) > 0$$

for any $\sigma \in (0, 1)$.

6. Lemma. *For any $\varepsilon > 0$ there is $\varrho > 1$ such that for each $z \in B$ the inequality*

$$a^G(\varrho z) \geqq A \inf_{y \in B} d_G(y) - \varepsilon \tag{6}$$

holds.

Proof. Suppose that the assertion of the lemma is false. Then there is an $\varepsilon > 0$ and points $y_n \in B$ such that for $z_n = (1 + 1/n)\, y_n$ we have

$$a^G(z_n) < A \inf_{y \in B} d_G(y) - \varepsilon.$$

Since the set B is compact we may suppose $y_n \to y_0 \in B$. Then $z_n \to y_0$ and

$$\liminf_{n \to \infty} a^G(z_n) \leqq A \inf_{y \in B} d_G(y) - \varepsilon \leqq A\, d_G(y_0) - \varepsilon,$$

which contradicts the fact that a^G is lower semicontinuous at y_0 (see also (5)).

7. Proof of the theorem. We know (see Lemma 3.4, Theorem 3.6 in [3] and Theorems 27,29 and Lemma 30 in [8]) that

$$\omega(N_G U - \tfrac{1}{2}AI') = \lim_{r\to 0+} \sup_{y\in B} v_r^G(y) =$$
$$= \lim_{r\to 0+} \sup_{y\in B} [A|\tfrac{1}{2} - d_G(y)| + v_r^G(y)] \geqq \sup_{y\in B} A(\tfrac{1}{2} - d_G(y)) .$$

Hence in order to prove (4) it is sufficient to establish only the inequality

$$\sup_{y\in B} A(\tfrac{1}{2} - d_G(y)) \geqq \lim_{r\to 0+} \sup_{y\in B} v_r^G(y) \tag{7}$$

because the rest follows by Corollary 5.

Fix $\varepsilon > 0$ and choose $\varrho > 1$ by Lemma 6. Since ∂G_ϱ and $\bar{G}$ are disjoint compact sets we can find $r > 0$ such that

$$\operatorname{dist}(y, \bar{G}) \geqq r\varrho$$

for any $y \in \partial G_\varrho$. Consider now $z \in B$. The set G_ϱ being convex, there is a closed half-space P such that $\varrho z \in \partial P$ and $G_\varrho \subset P$. If $\theta \in \Gamma$ is chosen in such a way that $H_\theta^\infty(\varrho z) \cap \cap P = \emptyset$, then $H_\theta^\infty(\varrho z) \cap G_\varrho = \emptyset$. Further, if $\theta \in \Gamma$ and $H_\theta^\infty(\varrho z) \cap G \neq \emptyset$, then $H_\theta^{\varrho r}(\varrho z) \subset G_\varrho$. Hence for these θ's there is no hit of $H_\theta^{\varrho r}(\varrho z)$ on G_ϱ and we conclude that

$$v_{\varrho r}^{G_\varrho}(\varrho z) \leqq \tfrac{1}{2}A - a^G(\varrho z) . \tag{8}$$

Since obviously $v_{\varrho r}^{G_\varrho}(\varrho z) = v_r^G(z)$, (8) and (6) yield

$$v_r^G(z) \leqq \tfrac{1}{2}A - a^G(\varrho z) \leqq \tfrac{1}{2}A - A \inf_{y\in B} d_G(y) + \varepsilon .$$

Consequently,

$$\lim_{r\to 0+} \sup_{y\in B} v_r^G(y) \leqq \sup_{y\in B} v_r^G(y) \leqq \sup_{y\in B} A(\tfrac{1}{2} - d_G(y)) + \varepsilon$$

which proves (7).

The proof of the theorem is complete.

8. Notation. For $x \in R^m$ and $r > 0$ we denote $\Omega_r(x) = \{z \in R^m; |x - z| < r\}$. In the rest of this note we shall suppose that M is a non-empty open set with compact boundary,

$$\partial M = \partial(R^m - \bar{M}) \tag{9}$$

and

$$V_0^M < \infty . \tag{10}$$

Note that by [4], remark on p. 596, (10) implies

$$\sup_{y \in \partial M} v_\infty^M(y) < \infty .$$

$\mathscr{B}$ will stand for the Banach space of all bounded Baire functions on ∂M equipped with the usual supremum norm. Fix $z \in R^m$ and $\theta \in \Gamma$. As in [3] we put for $t > 0$

$$s(t; z, \theta) = \sigma \ (= \pm 1),$$

if there is $\delta > 0$ such that

$$z + (t + \sigma\tau)\,\theta \in R^m - M\,, \quad z + (t - \sigma\tau)\,\theta \in M$$

for a.e. $\tau \in (0, \delta)$; otherwise we set $s(t; z, \theta) = 0$. Of course, if $s(t; z, \theta) \neq 0$, then $z + t\theta$ is a hit of $H_\theta^\infty(z)$ on M. Consequently, we can define as in Lemma 2.5 of [3] for any $f \in \mathscr{B}$

$$Wf(z) = \int_\Gamma \{\sum_{t>0} f(z + t\theta)\, s(t; z, \theta)\}\, \mathrm{d}H_{m-1}(\theta)\,, \quad z \in \partial M\,.$$

It turns out that the function $z \mapsto Wf(z)$ is a bounded Baire function and

$$W : f \mapsto Wf$$

is a bounded linear operator acting on $\mathscr{B}$ (see [3], Sec. 2 and [7], Sec. 7). It should be noted here that Wf is a generalized double layer potential — see Sec. 2 in [3].

Let us finally denote by f_0 the function identically equal to 1 on ∂M and recall that, by Lemma 2.2 in [3],

(11) $$v_\infty^M(z) = \sup \{|Wg(z)|;\ g \in \mathscr{B},\ |g| \leqq 1\}$$

and

(12) $$A\, d_M(z) = Wf_0(z)$$

provided M is bounded (see Proposition 2.6 and Lemma 2.7 in [3]).

9. Lemma. *Let $D \subset R^m$ be an open bounded set, $z \in R^m$ and let H_1 stand for the linear measure. For $\theta \in \Gamma$ define*

$$g(\theta) = H_1(H_\theta^\infty(z) \cap D)\,.$$

Then g is a lower semicontinuous function on Γ.

Proof. Fix $\theta_0 \in \Gamma$ and $c < g(\theta_0)$. There is a finite number of disjoint closed segments contained in $H_{\theta_0}^\infty(z) \cap D$ such that the sum of lengths of these segments exceeds c. Obviously, for any $\theta \in \Gamma$ belonging to a suitably chosen neighborhood of θ_0 we have $g(\theta) > c$.

10. Lemma. *Suppose that the set M is not convex. Then there is $z \in \partial M$ with $d_M(z) = \frac{1}{2}$, a Borel set $\Gamma_z \subset \Gamma$, $H_{m-1}(\Gamma_z) > 0$, and a strictly positive real function φ defined on Γ_z such that*

$$M_z = \{z + \varphi(\theta)\, \theta;\ \theta \in \Gamma_z\}$$

is a Borel subset of ∂M and $s(\varphi(\theta); z, \theta) = -1$ for any $\theta \in \Gamma_z$.

Proof. Since M is not convex, (9) holds and the set

$$\{x \in \partial M;\ d_M(x) = \tfrac{1}{2}\}$$

is dense in ∂M (see Lemma 14 in [8]), we can find $z \in \partial M$ with $d_M(z) = \frac{1}{2}$, $\theta_1 \in \Gamma$, $0 < t^0 < t^1$ and $r > 0$ such that

$$\Omega_r(z + t^0\theta_1) \subset R^m - \overline{M}\,, \quad \Omega_r(z + t^1\theta_1) \subset M\,.$$

According to the hypothesis $v_\infty^M(z) < \infty$ so that

$$n_\infty^M(\theta, z) < \infty \tag{13}$$

for H_{m-1} – a.e. $\theta \in \Gamma$. Let $\Gamma_2 \subset \Gamma$ be a Borel set such that $H_{m-1}(\Gamma_2) = 0$ and that for each $\theta \in \Gamma - \Gamma_2$ the relation (13) holds. Observe that for these θ's the set $H_\theta^\infty(z) \cap \cap M$ is H_1-equivalent to a finite union of disjoint open segments. Denote

$$\Gamma_1 = \{\theta \in \Gamma;\ H_\theta^\infty(z) \cap \Omega_r(z + t^1\theta_1) \neq \emptyset\}$$

and put $\Gamma_z = \Gamma_1 - \Gamma_2$. Obviously, Γ_z is a Borel set and $H_{m-1}(\Gamma_z) > 0$.

Let us now fix $\theta \in \Gamma_z$ and put

$$\varphi(\theta) = \sup\,\{t > t^0;\ \{z + \varrho\theta;\ \varrho \in (t^0, t)\} \cap M = \emptyset\}\,.$$

Of course, $t^0 < \varphi(\theta) < t^1$ and one easily verifies that $z + \varphi(\theta)\,\theta$ is a hit of $H_\theta^\infty(z)$ on M and there is $\delta > 0$ such that

$$H_1(\{z + t\theta;\ \varphi(\theta) < t < t + \delta\} - M) = 0\,.$$

Consequently, $s(\varphi(\theta); z, \theta) = -1$ for each $\theta \in \Gamma_z$.

We intend to show that φ is a Baire function on Γ_z. The proof of this fact is patterned after Mařík's proof of Lemmas 27, 28 in [6].

For $0 < a < b$ we denote $\Omega_{a,b} = \Omega_b(z) - \overline{\Omega_a(z)}$ and

$$N_{a,b} = \{\theta \in \Gamma_z;\ H_1(H_\theta^\infty(z) \cap M \cap \Omega_{a,b}) = b - a\}\,.$$

It follows from Lemma 9 (applied to $D = M \cap \Omega_{a,b}$) that $N_{a,b}$ is a Borel set. One easily verifies that for any $c \in R^1$,

$$\{\theta \in \Gamma_z;\ \varphi(\theta) < c\} = \bigcup_{a,b} N_{a,b}$$

where a, b are rational, $t^0 < a < b < c$. We see that φ is a Baire function and the function ψ on $\Gamma_z \times R^1$ defined by

$$\psi([\theta, t]) = |\varphi(\theta) - t|$$

is consequently a Baire function on $\Gamma_z \times R^1$. Consider now the mapping

$$\Phi : \Gamma \times (0, \infty) \to R^m - \{z\}$$

defined by

$$\Phi([\theta, t]) = z + t\theta\,.$$

Then Φ is a homeomorphism and $\Phi(\psi_{-1}(0))$ is thus a Borel subset of R^m. Putting $M_z = \Phi(\psi_{-1}(0))$, we check easily that z, Γ_z, φ, M_z have the desired properties.

11. Theorem. *The following conditions are equivalent to each other*:

(i) *M is convex.*

(ii) *The operator W is positive.*

Proof. If M is convex, then evidently $s(t; z, \theta) \geqq 0$ for any $z \in \partial M$, $t > 0$, $\theta \in \Gamma$. Consequently, $Wf \geqq 0$ on ∂M provided $f \in \mathscr{B}$, $f \geqq 0$ on ∂M.

Let M be non-convex and let z, M_z, Γ_z, φ have the same meaning as in Lemma 10. Denote by f_1 the function equal to 1 on M_z and zero elsewhere on ∂M. Then, by Lemma 10, $f_1 \in \mathscr{B}$ and since $H_{m-1}(\Gamma_z) > 0$,

$$(14) \quad Wf_1(z) = \int_{\Gamma_z} f_1(z + \varphi(\theta)\,\theta)\, s(\varphi(\theta); z, \theta)\, \mathrm{d}H_{m-1}(\theta) = -H_{m-1}(\Gamma_z) < 0$$

and we conclude that W is not a positive operator.

The proof of the theorem is complete.

12. Remark. Note that if M is not convex and f_1 has the same meaning as above, then Wf_1 is strictly negative on a set of positive H_{m-1} measure. Indeed, observing that $\Omega_{t^0}(z) \cap \overline{M_z} = \emptyset$, we assert that Wf_1 is continuous on $\Omega_{t^0}(z)$ (cf. Lemma 2.12 in [3]) and (14) implies that Wf_1 is strictly negative on a ball Ω with centre z. $H_{m-1}(\Omega \cap \partial M) > 0$ follows from the proof of Lemma 14 in [8] (see inequality (45)).

13. Lemma. *Suppose that M is bounded and non-convex. Then there is $z \in \partial M$ such that*

$$(15) \qquad v_\infty^M(z) > A\, d_M(z) = \tfrac{1}{2}A\,.$$

Proof. Let us take z as in Lemma 10 and let f_1 have the same meaning as above. We have $d_M(z) = \frac{1}{2}$ and $|f_0 - 2f_1| \leqq 1$ (recall that $f_0 \equiv 1$ on ∂M). Since $Wf_1(z) < 0$ (see (14)) we conclude

$$Wf_0(z) < W(f_0 - 2f_1)(z) \leqq \sup\{|Wg(z)|;\ g \in \mathscr{B},\ |g| \leqq 1\}\,.$$

This yields (15) by (11) and (12).

14. Theorem. *The following conditions are equivalent.*

(i) $v_\infty^M(z) = A\, d_M(z)$ *for any* $z \in \partial M$.

(ii) *Either* M *is convex or* $M' = R^m - \overline{M}$ *is convex and* $d_M(z) = \frac{1}{2}$ *for any* $z \in \partial M$.

Proof. Let us start with the following remark. Since both M and M' are non-void, we have $v_\infty^M(z) > 0$ for any $z \in \partial M$. Consequently, if either (i) holds or $d_M = \frac{1}{2}$ on ∂M, then the m-dimensional Lebesgue measure of ∂M is zero. Indeed, in the other case, by the well-known density theorem, there would be at least one $z \in \partial M$ with $d_M(z) = 0$. It follows that

$$d_{M'}(z) = 1 - d_M(z)\,, \quad v_\infty^{M'}(z) = v_\infty^M(z)\,, \quad z \in \partial M\,, \tag{16}$$

by Proposition 1.6 in [3].

First we shall suppose that M is convex. Then, by Theorem 11, W is a positive operator. Consequently,

$$Wf_0(z) = \sup\{|Wg(z)|;\ g \in \mathscr{B},\ |g| \leqq 1\}$$

and (i) follows by (11) and (12).

Let M' be convex and $d_M(z) = \frac{1}{2}$ whenever $z \in \partial M = \partial M'$. We have just proved that

$$v_\infty^{M'}(z) = A\, d_{M'}(z)\,, \quad z \in \partial M'\,,$$

and (i) is a consequence of (16). This completes the proof of the implication (ii) $\Rightarrow$ (i).

Suppose now that (i) is true. If M is bounded, then M is necessarily convex, since otherwise (i) would be violated in virtue of Lemma 13. It remains to consider the case that M is unbounded. In this case M' is bounded and, if non-convex,

$$v_\infty^M(z) = v_\infty^{M'}(z) > A\, d_{M'}(z) = \tfrac{1}{2}A = A\, d_M(z)$$

for a suitable $z \in \partial M' = \partial M$ – a contradiction. We conclude that M' is convex. Since we have already established (ii) $\Rightarrow$ (i) we have by (16)

$$v_\infty^{M'}(z) = A\, d_{M'}(z) = A(1 - d_M(z))\,, \quad z \in \partial M\,,$$

and, by the hypothesis,

$$v_\infty^M(z) = A\, d_M(z)\,, \quad z \in \partial M\,.$$

Using (16) once more, we obtain

$$A\, d_M(z) = A(1 - d_M(z))\,,$$

which yields $d_M(z) = \frac{1}{2}$ for any $z \in \partial M$.

The proof of the theorem is complete.