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COMPATIBLE RELATIONS ON ALGEBRAS

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The concept of tolerance relation compatible with a given algebra is studied in [3], [4], [5]. A tolerance relation is (according to [1], [2]) a reflexive and symmetric binary relation. Here we shall extend the definition of compatibility onto relations which are not tolerances in general.

Let an algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ with finitary operations be given. (Here A denotes the set of elements of $\mathfrak{A}$ and $\mathscr{F}$ denotes the set of operations.) Let ϱ be a binary relation on A. We say that ϱ is compatible with the algebra $\mathfrak{A}$, if and only if the following condition is satisfied: If $f \in \mathscr{F}$ is an n-ary operation (n is a positive integer), $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are elements of A, $(x_i, y_i) \in \varrho$ for $i = 1, \ldots, n$, then $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \varrho$.

We shall prove several theorems; some of them are generalizations of the results from [3] and [4]. When we speak about an algebra, we always mean an algebra in which all operations are finitary.

Even an empty relation on A can be considered a relation compatible with $\mathfrak{A}$. If ϱ is a binary relation on a set A, then by ϱ^* we denote the relation $\{(y, x) \mid x \in A, y \in A, (x, y) \in \varrho\}$.

Theorem 1. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ϱ_1, ϱ_2 be two relations on A compatible with $\mathfrak{A}$. Then $\varrho_1 \cap \varrho_2$, ϱ^* are relations compatible with $\mathfrak{A}$.*

Proof. Let $f \in \mathscr{F}$ be an n-ary operation, let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be elements of A such that $(x_i, y_i) \in \varrho_1 \cap \varrho_2$ for $i = 1, \ldots, n$. As $(x_i, y_i) \in \varrho_1$ for $i = 1, \ldots, n$, we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \varrho_1$. As $(x_i, y_i) \in \varrho_2$ for $i = 1, \ldots, n$, we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \varrho_2$. Thus $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \varrho_1 \cap \varrho_2$ and $\varrho_1 \cap \varrho_2$ is a relation compatible with $\mathfrak{A}$. The assertion for ϱ^* is evident.

Theorem 2. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ϱ be a reflexive relation on A compatible with $\mathfrak{A}$. Then $\varrho \cap \varrho^*$ is a tolerance compatible with $\mathfrak{A}$.*

Proof. The reflexivity and the symmetry of $\varrho \cap \varrho^*$ is evident. Its compatibility with $\mathfrak{A}$ follows from Theorem 1.

Theorem 3. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ϱ be a reflexive and transitive relation (i.e. a quasi-ordering) on A compatible with $\mathfrak{A}$. Then $\varrho \cap \varrho^*$ is a congruence on $\mathfrak{A}$.*

Proof is analogous to that of Theorem 2.

Let the product $\varrho_1 \varrho_2$ of two binary relations ϱ_1, ϱ_2 on the same set A be defined so that $(x, y) \in \varrho_1 \varrho_2$ for $x \in A$, $y \in A$, if and only if there exists $z \in A$ such that $(x, z) \in \varrho_1$, $(z, y) \in \varrho_2$. We can define also the n-th power of a binary relation ϱ so that $\varrho^n = \varrho$ for $n = 1$ and $\varrho^n = \varrho\varrho^{n-1}$ for $n \geqq 2$.

It is easy to prove the following

Theorem 4. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ϱ_1, ϱ_2 be two relations on A compatible with $\mathfrak{A}$. Then their product $\varrho_1 \varrho_2$ is compatible with $\mathfrak{A}$.*

Now we shall prove

Theorem 5. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let $\{\varrho_j\}_{j=1}^{\infty}$ be a sequence of compatible relations on $\mathfrak{A}$ such that $\varrho_j \subseteq \varrho_{j+1}$ for every positive integer j. Then $\bigcup_{j=1}^{\infty} \varrho_j = \varrho$ is compatible relation on $\mathfrak{A}$.*

Proof. Let $f \in F$ be an n-ary operation, let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be elements of A such that $(x_i, y_i) \in \varrho$ for each $i = 1, \ldots, n$. Then for each $i = 1, \ldots, n$ we have $(x_i, y_i) \in \varrho_{j(i)}$ for a positive integer $j(i)$. Let $j = \max_{1 \leqq i \leqq n} j(i)$. Then $(x_i, y_i) \in \varrho_j$ for each $i = 1, \ldots, n$ and thus $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \varrho_j \subseteq \varrho$.

Theorem 6. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ϱ be a reflexive relation on A compatible with $\mathfrak{A}$. Then the transitive hull ϱ_T of ϱ is compatible with $\mathfrak{A}$.*

Proof. We have $\varrho_T = \bigcup_{j=1}^{\infty} \varrho^j$. According to Theorem 4 the relation ϱ^j is compatible with A for every positive integer j. As ϱ is reflexive, we have $\varrho^j \subseteq \varrho^{j+1}$ for every positive integer j. Thus according to Theorem 5 the relation $\varrho_T = \bigcup_{i=1}^{\infty} \varrho^j$ is compatible with $\mathfrak{A}$.

Example 1. This example will show us that:

1) the reflexive hull and the symmetric hull of a relation compatible with $\mathfrak{A}$ need not be compatible with $\mathfrak{A}$;

2) the union of two relations compatible with $\mathfrak{A}$ need not be compatible with $\mathfrak{A}$.

Let $\mathfrak{A}$ be the semigroup with elements a, b, c, d, e, f, g, h given by the following Cayley table:

	a	b	c	d	e	f	g	h
a	a	e	h	h	e	h	h	h
b	e	b	f	h	e	f	h	h
c	h	f	c	g	h	f	g	h
d	h	h	g	d	h	h	g	h
e	e	e	h	h	e	h	h	h
f	h	f	f	h	h	f	h	h
g	h	h	g	g	h	h	g	h
h	h	h	h	h	h	h	h	h

Let $\varrho = \{(a, c), (b, d), (e, g)\}$. This is a compatible relation on $\mathfrak{A}$. The reflexive hull ϱ_R of ϱ is not compatible with $\mathfrak{A}$; we have $(a, c) \in \varrho_R$, $(c, c) \in \varrho_R$, $ac = h$, $cc = c$, but $(h, c) \notin \varrho_R$. This is also an example that the union of two compatible relations on $\mathfrak{A}$ need not be a compatible relation on $\mathfrak{A}$, because the reflexive hull of ϱ is the union of ϱ and of the relation of equality on A which is evidently also compatible with $\mathfrak{A}$. Also the symmetric hull $\varrho \cup \varrho^* = \{(a, c), (c, a), (b, d), (d, b), (e, g), (g, e)\}$ is not compatible with $\mathfrak{A}$. We have $(a, c) \in \varrho \cup \varrho^*$, $(d, b) \in \varrho \cup \varrho^*$, $ad = h$, $cb = f$, but $(h, f) \notin \varrho \cup \varrho^*$.

Example 2. This example will show that the reflexivity of ϱ in Theorem 6 is substantial.

Let $\mathfrak{A}$ be the semigroup with elements a, b, c, d, e, f given by the following Cayley table:

	a	b	c	d	e	f
a	a	d	f	d	f	f
b	d	b	e	d	e	f
c	f	e	c	f	e	f
d	d	d	f	d	f	f
e	f	e	e	f	e	f
f	f	f	f	f	f	f

Let $\varrho = \{(a, b), (b, c), (d, e)\}$. This is a compatible relation on A, evidently not reflexive. The transitive hull of ϱ is $\varrho_T = \{(a, b), (b, c), (a, c), (d, e)\}$. We have $(a, b) \in \varrho_T$, $(a, c) \in \varrho_T$, $aa = a$, $bc = e$, but $(a, e) \notin \varrho_T$. Thus ϱ_T is not compatible with $\mathfrak{A}$.

Theorem 7. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ϱ be a relation on A compatible with $\mathfrak{A}$. Let e be an idempotent element of $\mathfrak{A}$ (i.e. such an element that $f(e, e, \ldots, e) =$*

$= e$ *for each* $f \in \mathcal{F}$). *The set* A_e *of all elements* $x \in A$ *such that* $(e, x) \in \varrho$ *forms a subalgebra of* $\mathfrak{A}$.

Proof. For $i = 1, \ldots, n$ let $x_i \in A_e$, this means $(e, x_i) \in \varrho$. If $f \in F$ is an n-ary operation, then $(e, f(x_1, \ldots, x_n)) = (f(e, \ldots, e), f(x_1, \ldots, x_n)) \in \varrho$, because ϱ is compatible with $\mathfrak{A}$. This means $f(x_1, \ldots, x_n) \in A_e$. As the elements $x_1, \ldots, x_n$ and the operation f were chosen arbitrarily, A_e forms a subalgebra of $\mathfrak{A}$.

Corollary 1. *Let L be a lattice (or semilattice), let* ϱ *be a compatible relation on L. Then for each* $x \in L$ *the set* L_x *of all elements* $y \in L$ *such that* $(x, y) \in \varrho$ *forms a sublattice (or subsemilattice respectively) of L.*

Remark. Theorem 7 implies immediately Theorem 11 from [3].

Theorem 8. *Let G be a group, let* ϱ *be a compatible relation on G. Let* ϱ *be reflexive. The set N of all* $x \in G$ *satisfying* $(e, x) \in \varrho$ *is a normal subgroup of G. (The symbol e denotes the unit of G.)*

Proof. From Theorem 7 it follows that set N is a subgroup of G. Let $x \in N$, i.e. $(e, x) \in \varrho$. From the reflexivity of ϱ we obtain $(z, z) \in \varrho$ and $(z^{-1}, z^{-1}) \in \varrho$ for arbitrary $z \in G$. From the compatibility of ϱ we obtain finally $(e, z^{-1}xz) = (z^{-1}ez, z^{-1}xz) \in \varrho$, thus $z^{-1}xz \in N$. Therefore N is a normal subgroup of G.

Remark. In [4] it is proved that each compatible relation on a group which is reflexive and symmetric is also transitive, i.e., it is a congruence.

Theorem 9. *Let G be an involutory group (i.e.* $x^2 = e$ *for each* $x \in G$, *where e is the unit of G), let* ϱ *be a reflexive compatible relation on G. Then* ϱ *is a congruence relation on G.*

Proof. Let $(x, y) \in \varrho$ for $x \in G$, $y \in G$. From the reflexivity of ϱ we have $(x^{-1}, x^{-1}) \in \varrho$, $(y^{-1}, y^{-1}) \in \varrho$ and from the compatibility of ϱ we have $(e, x^{-1}y) = (x^{-1}x, x^{-1}y) \in \varrho$ and thus $(y^{-1}, x^{-1}) = (ey^{-1}, x^{-1}yy^{-1}) \in \varrho$. But G is an involutory group; this means $y^{-1} = y$, $x^{-1} = x$, thus $(x, y) \in \varrho$ implies $(y, x) \in \varrho$. By the theorem in [4] quoted in the above remark ϱ is a congruence on G.

Theorem 10. *Let* $L(\vee)$ *be a complete* $\vee$*-semilattice, let* ϱ *be a compatible relation on* $L(\vee)$. *Denote* $M(x) = \bigvee_{(x,y) \leqq \varrho} y$ *for* $x \in L(\vee)$. *The mapping M which assigns the element* $M(x)$ *to any* $x \in L(\vee)$ *is an isotone mapping of* $L(\vee)$ *into itself.*

Proof. Let $x \in L(\vee)$, let ϱ be a compatible relation on $L(\vee)$. The existence of $M(x)$ for each $x \in L(\vee)$ follows from the completeness of $L(\vee)$. Let $x \leqq y$, i.e. $x \vee y = y$. From the definition of $M(x)$ we have $(x, M(x)) \in \varrho$, $(y, M(y)) \in \varrho$ (because $L(\vee)$ is complete) and from the compatibility of ϱ we obtain $(x \vee y,$

$M(x) \vee M(y)) \in \varrho$ therefore $M(x) \vee M(y)$ is one factor in the join $\bigvee_{(x \vee y, z) \in \varrho} z =$ $= M(x \vee y)$. This means $M(x) \vee M(y) \leqq M(x \vee y)$. But $x \vee y = y$ and thus $M(x) \vee M(y) \leqq M(y)$, which means $M(x) \leqq M(y)$.

Corollary 2. *Let $L(\wedge)$ be a complete $\wedge$-semilattice, let ϱ be a compatible relation on $L(\wedge)$. Denote $m(x) = \bigwedge_{(x,y) \in \varrho} y$ for $x \in L(\wedge)$. The mapping m which assigns the element $m(x)$ to any $x \in L(\wedge)$ is an isotone mapping of $L(\wedge)$ into itself.*

Proof of Corollary 2 is dual to that of Theorem 10.

Corollary 3. *Let L be a complete lattice, let ϱ be a compatible relation on L. Let $M(x)$ and $m(x)$ be defined as in Theorem 11 and Corollary 2. The mappings $M : x \to M(x)$, $m : x \to m(x)$ are isotone mappings of L into itself.*

Theorem 11. *Let S be a semigroup, let ϱ be a compatible relation on S, let T be a subsemigroup of S. The set ϱT of all elements $x \in S$ such that $(x, x') \in \varrho$ for some $x' \in T$ is a subsemigroup of S.*

Proof. Let $x \in \varrho T$, $y \in \varrho T$. Then there exist elements $x' \in T$, $y' \in T$ such that $(x, x') \in \varrho$, $(y, y') \in \varrho$. From the compatibility of ϱ we have $(xy, x'y') \in \varrho$. But $x'y' \in T$, because T is a subsemigroup of S, thus $xy \in \varrho T$ and ϱT is a subsemigroup of S.

Theorem 12. *Let S be a semigroup, let ϱ be a compatible relation on S. Let ϱ be reflexive. Let T be an ideal of S (right or left or two-sided). The set ϱT defined in Theorem 11 is an ideal of the semigroup S (right or left or two-sided, respectively).*

Proof. Let $x \in \varrho T$, let T be a left ideal of S. There exists $x' \in T$ such that $(x, x') \in \varrho$. Let $y \in S$; from the reflexivity of ϱ we have $(y, y) \in \varrho$. From $(x, x') \in \varrho$ and $(y, y) \in \varrho$ we obtain $(xy, x'y) \in \varrho$. But $x'y \in T$, because T is a left ideal of S. Therefore $xy \in \varrho T$ and ϱT is a left ideal of S. Analogously for right and two-sided ideals.

Theorem 13. *Let R be a ring, let ϱ be a compatible relation on R, let O be the zero element of R. Let ϱ be reflexive. The set R_0 of all $x \in R$ such that $(O, x) \in \varrho$ (or $(x, O) \in \varrho$) is an ideal of R.*

Proof follows immediately from Theorems 12, 8 and 1.

For a ring whose additive group is involutory, the assumption that ϱ is reflexive is unnecessary. We obtain

Corollary 4. *Let R be a ring whose additive group is involutory, let ϱ be a compatible relation on R. The set R_0 of all $x \in R$ for which $(O, x) \in \varrho$ (or $(x, O) \in \varrho$) holds (where O is the zero element of R) is a subring of the ring R.*