Summary

ORTHOGONALITY ON SETS

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The article deals with a certain natural generalization of the concept of orthogonality, as is known e.g. from the Hilbert space theory. The article is divided into two parts. The first part is of preparatory character, the second one is devoted to the problem itself. Henceforth the terminology of lattice theory will be used.

1. If $\mathscr{P} = (P, \leqq, 1, \perp)$ stands for an orthocomplemented lattice, then $\mathscr{P}$ is orthomodular if and only if

$$p, q, r \in P\,, \quad r \leqq q\,, \quad p \leqq q^{\perp} \text{ implies } (r \vee p) \wedge q = r\,.$$

Moreover, in this case if $p, q, r, s, t \in P$, $r \leqq q$, $t \leqq q$, $p \leqq q^{\perp}$, $s \leqq q^{\perp}$ and $r \vee p = t \vee s$, then $r = t$ and $p = s$.

Let $\mathscr{P} = (P, \leqq, 1, \perp)$ be an orthomodular lattice. Let $p \in P$, $0 \neq p \neq 1$ and r, s be atoms of $\mathscr{P}$ such that $r \leqq p$, $s \leqq p^{\perp}$. If $q \in P$ such that $q \neq 0$, $q \wedge p = q \wedge p^{\perp} = 0$ and $q \leqq r \vee s$, then $r \vee q = r \vee s = q \vee s$. The above assumptions concerning r and s determine the atoms r, s uniquely.

An orthocomplemented lattice $\mathscr{P} = (P, \leqq, 1, \perp)$ is called a V-lattice if it satisfies the following condition: For every atom q of $\mathscr{P}$ and for every p of $\mathscr{P}$ such that $0 \neq p \neq 1$, $q \wedge p = q \wedge p^{\perp} = 0$ there exist atoms r and s of $\mathscr{P}$ such that $r \leqq p$, $s \leqq p^{\perp}$, and $q \leqq r \vee s$.

If the V-lattice is orthomodular, then the atoms r and s are uniquely determined.

If $\mathscr{P} = (P, \leqq, 1, \perp)$ is an orthomodular V-lattice and if $r \neq s$ are atoms of $\mathscr{P}$, then there exists an atom t of $\mathscr{P}$ such that $r \perp t$ and $r \vee s = r \vee t$. Moreover, if r and s are atoms of $\mathscr{P}$ such that $r \perp s$ and t is an atom of $\mathscr{P}$ such that $r \neq t \neq s$ and $t \leqq r \vee s$, then there exists an atom u of $\mathscr{P}$ such that $t \perp u$ and the equality $r \vee s = t \vee u$ holds.

In a complete orthomodular lattice $\mathscr{P} = (P, \leqq, 1, \perp)$, assume that q is an atom such that $q \nleqq p$ and $q \nleqq p^{\perp}$. If $\{(r_i, s_i) : i \in I\}$ is a system of all r_i and s_i of $\mathscr{P}$ such that $r_i \leqq p$, $s_i \leqq p^{\perp}$ and $q \leqq r_i \vee s_i$ for all $i \in I$, then there is a least element r_0 or s_0 among r_i, $i \in I$, or among s_i, $i \in I$, respectively. Moreover, we have $r_0 = (q \vee p^{\perp}) \wedge p$ and $s_0 = (q \vee p) \wedge p^{\perp}$.

A complete orthomodular lattice $\mathscr{P}$ is a V-lattice if and only if the following statement is true: For an arbitrary element p of $\mathscr{P}$, $0 \neq p \neq 1$ and for an arbitrary atom q of $\mathscr{P}$ such that $q \nleqq p$ and $q \nleqq p^{\perp}$, the elements $(q \vee p^{\perp}) \wedge p$ and $(q \vee p) \wedge p^{\perp}$ are atoms of $\mathscr{P}$.

A complete orthomodular lattice $\mathscr{P} = (P, \leqq, 1, \perp)$ is a V-lattice if and only if it has the covering property.

2. Let a binary relation $\perp$ be defined on a nonempty set Ω satisfying the following axioms:

(A I) $\perp$ is symmetric.

(A II) There is an element o of Ω such that $o \perp x$ for all x of Ω.

(A III) If $x \perp x$ for x of Ω then $x = o$.

In this case we say that on Ω orthogonality $\perp$ is defined, and we write $(\Omega, \perp)$. Two elements x and y of Ω are called orthogonal if $x \perp y$. For an element x of Ω and for any nonempty subset A of Ω the symbol $x \perp A$ means that $x \perp y$ for all y of A. Put $A^{\perp} = \{x \in \Omega : x \perp A\}$.

The following statements are valid:

(i) $\{o\}^{\perp} = \Omega$ and $\Omega^{\perp} = \{o\}$;

(ii) If $A \subset \Omega$ is nonempty then $o \in A^{\perp}$;

(iii) If $A \subset \Omega$ is nonempty then $A \subset A^{\perp\perp}$ (here the symbol $A^{\perp\perp}$ is used for $(A^{\perp})^{\perp}$; similarly $A^{\perp\perp\perp}$ for $(A^{\perp\perp})^{\perp}$);

(iv) If $A \subset \Omega$ is nonempty and $A \subset B \subset \Omega$, then $B^{\perp} \subset A^{\perp}$;

(v) If $A \subset \Omega$ is nonempty, then $A^{\perp} = A^{\perp\perp\perp}$;

Putting in the sequel $S = \{A \subset \Omega : \emptyset \neq A = A^{\perp\perp}\}$ and $T = \{A^{\perp} : \emptyset \neq A \subset \Omega\}$, we have

(vi) $S = T$;

(vii) $\mathscr{S} = (S, \subset, \Omega, \perp)$ is a complete orthocomplemented lattice, where

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i\,, \quad \bigvee_{i \in I} A_i = \Big(\bigcap_{i \in I} A_i^{\perp}\Big)^{\perp}\,, \quad 0 = \{o\}\,,$$

$1 = \Omega$ for $A_i \in S$, $i \in I$.

Let $(\Omega, \perp)$ and $\mathscr{S} = (S, \subset, \Omega, \perp)$ be given. For any nonempty $A \subset \Omega$ there is a least element B of $\mathscr{S}$ such that $A \subset B$. Moreover, $B = A^{\perp\perp}$.

If $A, B \in S$, then $A \perp B$ (i.e. $A \subset B^{\perp}$) if and only if $x \perp y$ for all $x \in A$ and $y \in B$.

Given a set Ω and an element $o \in \Omega$, let $\mathscr{S} = (S, \subset, \Omega, \perp)$ be a complete orthocomplemented lattice of subsets of the set Ω such that

(i) $\{o\}$ is a least element of $\mathscr{S}$;

(ii) If I is any nonempty set and $A_i \in S$ for $i \in I$, then $\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$.

We define a binary relation $\top$ on Ω by: $x \top y$ if and only if there is $A \in S$ such that $x \in A$ and $y \in A^{\perp}$. The relation $\top$ defined above is an orthogonality on Ω and the complete orthocomplemented lattice $(T, \subset, \Omega, \top)$ coincides with the lattice $\mathscr{S}$.

Given $(\Omega, \perp)$, we define for $x, y \in \Omega$

$$R(x, y) = 0 \quad \text{for} \quad x \perp y\,, \quad R(x, y) = 1 \quad \text{otherwise}\,.$$

The mapping $R : \Omega \times \Omega \to G$ (here G stands for the set of all complex numbers) has the following basic properties:

(a) $R(x, y) = \overline{R(y, x)}$ for every $x, y \in \Omega$ (here bar denotes the complex conjugate);

(b) $R(o, o) = 0$ and $R(x, x) > 0$ for every $x \in \Omega$, $x \neq o$;

(c) $|R(x, y)|^2 \leqq R(x, x) \,.\, R(y, y)$ for all $x, y \in \Omega$.

It is evident that $R(o, x) = R(x, o) = 0$ for all $x \in \Omega$.

A mapping $R : \Omega \times \Omega \to G$ is said to be a quasiscalar product if it satisfies conditions (a)–(c). A pair (Ω, R) is called a space with quasiscalar product.

Conversely: given a space with quasiscalar product (Ω, R), we can define a binary relation $\perp$ on Ω by: $x \perp y$ if and only if $R(x, y) = 0$. Then $\perp$ is an orthogonality on Ω. Another orthogonality on Ω is defined by: $x \perp y$ if and only if $\operatorname{Re}(R(x, y)) \leqq 0$ (here Re denotes the real part of a complex number).

If Ω is a Hilbert space and R is a scalar (also quasiscalar) product, then the first definition leads to the well-known case. The second definition produces the lattice $\mathscr{S} = (S, \subset, \Omega, \perp)$, where S the system of all closed convex cones with the vertices situated at the origin; this is an example of a complete atomic V-lattice which is not orthomodular and also not a C-lattice (cf. [1]).

If $(\Omega, \perp)$ and $\mathscr{S} = (S, \subset, \Omega, \perp)$ are given and A is an atom of $\mathscr{S}$, then $A = \{x\}^{\perp\perp}$ for every $x \in A$, $x \neq o$.

The following statements are equivalent:

1) $\mathscr{S}$ is an atomic lattice.

2) For every nonempty subset $B \subset \Omega$, $B^\perp \neq \{o\}$ there is $x \in B^\perp$, $x \neq o$ such that $\{y\}^\perp \subset \{x\}^\perp$ for all $y \in \{x\}^{\perp\perp}$, $y \neq o$.

Axiom (A). Given $(\Omega, \perp)$ and $\mathscr{S} = (S, \subset, \Omega, \perp)$, then $\{x\}^{\perp\perp}$ is an atom of $\mathscr{S}$ for every $x \in \Omega$, $x \neq o$.

If (A) takes place for $\mathscr{S}$, then $\mathscr{S}$ is an atomic lattice. Moreover, if $A \in S$, $x \in \Omega$, $x \neq o$, then $A \cap \{x\}^{\perp\perp} = \{o\}$ if and only if $x \notin A$.

For a nonempty set Ω and $o \in \Omega$ let $\mathscr{S} = (S, \subset, \Omega, \{o\})$ be a complete atomic and antiatomic lattice of subsets of Ω in which the meet is given by the set theoretical intersection having Ω and $\{o\}$ for the greatest and the least element, respectively. Put $\mathscr{A}$ and $\mathscr{B}$ for the system of all atoms and antiatoms, respectively. An orthogonality in $\mathscr{S}$ can be introduced if the following conditions are satisfied.

1) Let $A \in S$, $A \neq \{o\}$. If $\{A_i\}$, $i \in I$ is the system of all atoms of $\mathscr{S}$ such that $A_i \subset A$ for all $i \in I$, then $A = \bigvee_{i \in I} A_i$.

2) Let $A \in S$, $A \neq \Omega$. If $\{B_j\}$, $j \in J$ is the system of all antiatoms of $\mathscr{S}$ such that $A \subset B_j$ for all $j \in J$, then $A = \bigcap_{j \in J} B_j$.

3) There is a bijection f of $\mathscr{A}$ onto $\mathscr{B}$ such that

$$A_1, A_2 \in \mathscr{A}\,, \quad A_1 \subset f(A_2) \quad \text{imply} \quad A_2 \subset f(A_1)\,. \tag{3.1}$$

$$A \in \mathscr{A} \quad \text{implies} \quad A \cap f(A) = \{o\}\,. \tag{3.2}$$

Axiom (V). Let $(\Omega, \perp)$ and $\mathscr{S} = (S, \subset, \Omega, \perp)$ be given. Let $A \in S$, $\{o\} \neq A \neq \Omega$ and $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$. Then there are atoms A_1 and A_2 of $\mathscr{S}$ such that $A_1 \subset A$, $A_2 \subset A^{\perp}$, and $x \in A_1 \vee A_2$.

Numerous results of the first part of the article can be used when assuming that the lattice $\mathscr{S}$ satisfies the axiom (A) or (V).