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ON TREE-COMPLETE GRAPHS

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If G_0 is a graph, then we denote by $V(G_0)$, $E(G_0)$, and $\Delta(G_0)$ the vertex set of G_0, the edge set of G_0, and the maximum degree of G_0, respectively; the number of vertices of G_0 is called the order of G_0. For the notions not defined here, see BEHZAD and CHARTRAND [2].

We shall say that a graph G of order p is *tree-complete* if for every tree T of order p there is a spanning subgraph T' of G such that the graphs T and T' are isomorphic. Obviously, every complete graph is tree-complete. In the present paper, we shall construct tree-complete graphs. First, we shall prove three lemmas.

Let F be a forest. A vertex u of F is said to be semi-terminal if either u is an end-vertex or there is an end-vertex v such that the vertices u and v lie in the same component and the maximum degree among the vertices lying on the $u - v$ path in F is two.

Lemma 1. *Let F be a forest. Then either $\Delta(F) \leqq 2$ or F contains a vertex u of degree $d \geqq 3$ such that u is adjacent to at least $d - 1$ semi-terminal vertices.*

Proof. Assume $\Delta(F) \geqq 3$. Then there is a component T of F such that $\Delta(T) \geqq 3$. This means that T contains a vertex u of degree $d \geqq 3$ such that for every vertex $v \in V(T)$ of degree $d' \geqq 3$, $e(u) \geqq e(v)$, where is the eccentricity of the vertex w in the tree T. Clearly, u is adjacent to at least $d - 1$ semi-terminal vertices of F.

Lemma 2. *Let T be a tree of order $p \geqq 4$. Then there are distinct vertices $v_1, \ldots$ $\ldots, v_{[p/4]}$ such that*

$$\Delta(T - v_1 - \ldots - v_{[p/4]}) \leqq 2 .$$

Proof. Let F be a forest. Assume that F contains a vertex v of degree $d \geqq 3$ such that at least $d - 1$ vertices adjacent to v are semi-terminal. If at least three semi-terminal vertices are adjacent to v, then v is referred to as an auxiliary vertex. If precisely two vertices adjacent to v are semi-terminal, then $d = 3$ and the only non-semi-terminal vertex adjacent to v is said to be auxiliary. If $\Delta(F) \leqq 2$, then an arbitrary vertex is said to be auxiliary.

Let v_1 be an auxiliary vertex of T. For every integer i, $1 \leqq i < [p/4]$, let v_{i+1} be an auxiliary vertex of the forest $T - v_1 - \ldots - v_i$. The inequality of the lemma follows.

Lemma 3. *Let $p \geqq 8$, p be an integer. Then there is a tree T of order p such that*

(1) *for every sequence of distinct vertices $u_1, \ldots, u_{[p/4]-1}$, $\Delta(T - u_1 - \ldots - u_{[p/4]-1}) \geqq 3$.*

Proof. Let $p = 4m + k$, where $k \in \{0, 1, 2, 3\}$. We denote by T the tree in Fig. 1 (if $m \geqq 3$, then each of the vertices $s_3, \ldots, s_m$ has degree 4). It is easy to prove that T fulfils (1). Hence the lemma follows.

Let G be a graph. We denote by $\mathscr{H}_p(G)$ the graph with the vertex set $V(G) \cup V(G')$ and with the edge set

$$E(G) \cup E(G') \cup \{uv \mid u \in V(G),\ v \in V(G')\},$$

where G' is the path of order p, and $V(G) \cap V(G') = \emptyset$.

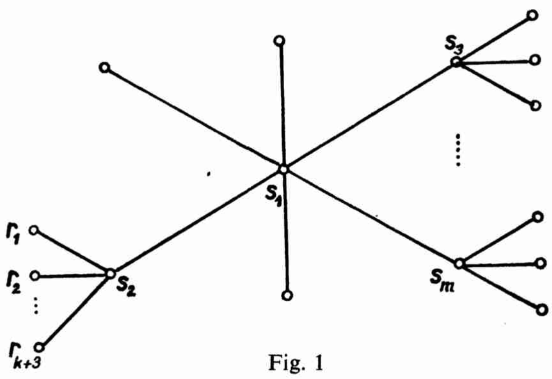

Fig. 1

Theorem 1. *Let p be an integer, $p \geqq 4$, and let G be a tree-complete graph of order n. Then the graph $\mathscr{H}_p(G)$ is tree-complete if and only if $n \geqq [(p - 1)/3]$.*

Proof. It is routine to prove that $n \geqq [(p - 1)/3]$ if and only if $n \geqq [(p + n)/4]$.

Let $n \geqq [(p + n)/4]$, and let G' be the same as in the definition of $\mathscr{H}_p(G)$. Consider a tree T of order $p + n$. Then there are distinct vertices $v_1, \ldots, v_n$ of T such that the forest $T - v_1 - \ldots - v_n$ is isomorphic to a spanning subgraph of G'. The subgraph of T induced by $\{v_1, \ldots, v_n\}$ is isomorphic to a spanning subgraph of G. Hence T is isomorphic to a spanning subgraph of $\mathscr{H}_p(G)$.

Let $n < [(p + n)/4]$. Then $p + n \geqq 8$. If the tree T in Fig. 1 has order $p + n$, then Lemma 3 implies that T is isomorphic to no spanning subgraph of $\mathscr{H}_p(G)$. Hence the theorem follows.

Obviously, every tree-complete graph is connected. Since a tree-complete graph contains both a spanning path and a spanning star, we get the following

Proposition. *Every tree-complete graph has at most two blocks.*

In the remainder of the paper we shall discuss tree-complete graphs with a cut-vertex.

Theorem 2. *Let G be a tree-complete graph of order p, and let B be a block of G having order n, where $n \leqq (p + 1)/2$. If $p \neq 8, 11$, then $n \leqq 3$. If $p = 8$, then $n \leqq 4$. If $p = 11$, then $n \leqq 5$ and $n \neq 4$.*

Proof. Let $n \geqq 4$. Obviously, $p \geqq 2n - 1 \geqq 7$. If $2n - 1 \leqq p \leqq 2n + 1$, then we denote by $T_{p,n}$ the tree in Fig. 2 (r_{p-2n+2}, t_n, and u_n are all the end-vertices). If $p \geqq 2n + 2$, then we denote by $T_{p,n}$ the tree in Fig. 3 (v_0, w_0, v_n, and w_n are all the end-vertices). It is not difficult to see that $T_{p,n}$ is isomorphic to no spanning subgraph of G, except the following cases: $p = 8$ and $n = 4$; $p = 9$ and $n = 4$; $p = 11$ and $n = 5$. If $p = 9$ and $n = 4$, then the subdivision graph of the star $K(1, 4)$ is isomorphic to no spanning subgraph of G. Hence the theorem follows.

Note that there is a tree-complete graph of order 8 which contains a block of order 4, and that there is a tree-complete graph or order 11 which contains a block of order 5.

Let G be a graph. We denote by $\mathscr{Y}_1(G)$ the graph G_1 with $V(G_1) = V(G) \cup \{u, v\}$ and with $E(G_1) = \{tu \mid t \in V(G)\} \cup \{uv\}$, where u and v are distinct vertices not belonging to G. We denote by $\mathscr{Y}_2(G)$ the graph G_2 with $V(G_2) = V(G_1) \cup \{w\}$ and with $E(G_2) = E(G_1) \cup \{uw, vw\}$, where $w \notin V(G_1)$.

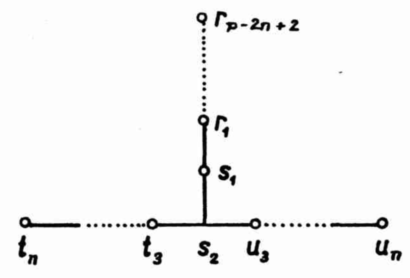

Fig. 2

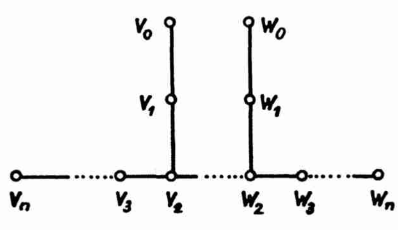

Fig. 3

Theorem 3. *Let $i \in \{1, 2\}$, and let G be a graph of order p such that every tree T_0 of order p with $\Delta(T_0) \leqq [(p + i)/2]$ is isomorphic to a spanning subgraph of G. Then $\mathscr{Y}_i(G)$ is tree-complete.*

Proof. Let T be a tree of order $p + i + 1$. A vertex of T adjacent to an end-vertex will be referred to as an e_1-vertex. A vertex of T adjacent either to at least two end-vertices or to an e_1-vertex of degree 2 will be referred to as an e_2-vertex. We denote by d_i the maximum degree among the e_i-vertices.

Let $i = 1$. The case $p \leqq 2$ is obvious. Assume that $p \geqq 3$. Consider an e_1-vertex r_1 of degree d_1 and an end-vertex s_1 adjacent to r_1. We have $\Delta(T - r_1 - s_1) \leqq \leqq [(p + 1)/2]$. As $T - r_1 - s_1$ is a forest, it is a spanning subgraph of a tree T_1 with $\Delta(T_1) = \max(2, \Delta(T - r_1 - s_1)) \leqq [(p + 1)/2]$. As T_1 is isomorphic to a spanning subgraph of G, $T - r_1 - s_1$ is also isomorphic to a spanning subgraph of G. Hence T is isomorphic to a spanning subgraph of $\mathcal{Y}_1(G)$.

Let $i = 2$. Consider an e_2-vertex r_2 of degree d_2, and distinct vertices s_2 and t_2 such that s_2 is adjacent to r_2, t_2 is an end-vertex, and either (a) s_2 is an end-vertex and t_2 is adjacent to r_2 or (b) s_2 is an e_1-vertex of degree 2 and t_2 is adjacent to s_2. We have $\Delta(T - r_2 - s_2 - t_2) \leqq [(p + 2)/2]$. Clearly, $T - r_2 - s_2 - t_2$ is a spanning subgraph of a tree T_2 with $\Delta(T_2) \leqq [(p + 2)/2]$. This means that $T - r_2 - - s_2 - t_2$ is isomorphic to a spanning subgraph of G. Hence T is isomorphic to a spanning subgraph of $\mathcal{Y}_2(G)$ and the proof is complete.

Note that – in a certain sense – the value $[(p + i)/2]$ in Theorem 3 is the best possible. This follows from Fig. 4 (for even $p + i + 1$) and from Fig. 5 (for odd $p + i + 1$).

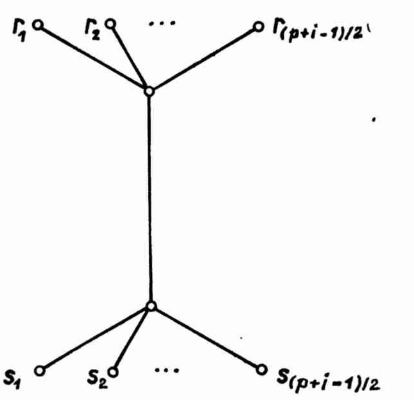

Fig. 4

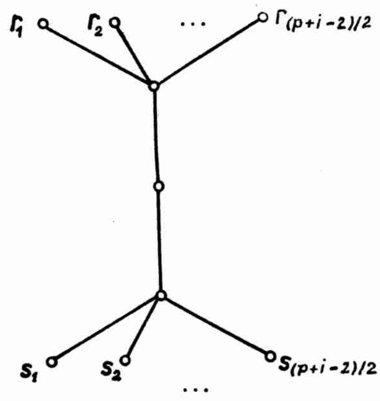

Fig. 5

Corollary 1. *Let G be a tree-complete graph. Then both $\mathcal{Y}_1(G)$ and $\mathcal{Y}_2(G)$ are tree-complete.*

We denote by D_1 and D_2 the trivial graph and the connected graph with exactly one edge. If p is a positive integer, then we denote by D_{p+2} the graph $\mathcal{Y}_1(D_p)$. As has been shown by Behzad and Chartrand [1], the graph D_p, $p \geqq 2$, is (up to iso-