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VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES

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1. Definitions, basic theorems. In the whole paper, meromorphic functions are understood to be meromorphic in $\mathbf{C}$.

Let f be a meromorphic function, $n(r, f)$ let denote the number of poles of the function f that lie in the disc $|z| \leqq r$ and $n(r, a)$ let denote the number of roots of the equation $f(z) = a$ in the disc $|z| \leqq r$, each point counted with regard to its multiplicity. Usually it has been put $n(r, f) = n(r, \infty)$.

Let us set

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t}\,\mathrm{d}t + n(0, f) \ln r\,,$$

$$N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t}\,\mathrm{d}t + n(0, a) \ln r\,,$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\varphi})|\,\mathrm{d}\varphi\,,$$

$$m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \frac{1}{|f(re^{i\varphi}) - a|}\,\mathrm{d}\varphi\,.$$

The function $T(r, f) = m(r, f) + N(r, f)$ is called *Nevanlinna characteristic function of f*. Further, let us denote by $\bar{n}(r, f)$ ($\bar{n}(r, a)$, $a \in \mathbf{C}$) the number of different poles (different roots of the equation $f(z) = a$, respectively) that lie in the disc $|z| \leqq r$.

Analogously we define

$$\bar{N}(r, f) = \int_0^r \frac{\bar{n}(t, f) - \bar{n}(0, f)}{t}\,\mathrm{d}t + \bar{n}(0, f) \ln r\,,$$

$$\bar{N}(r, a) = \int_0^r \frac{\bar{n}(t, a) - \bar{n}(0, a)}{t}\,\mathrm{d}t + \bar{n}(0, a) \ln r\,.$$

1.1 Theorem. (First Main Theorem of the value distribution theory.) *For any meromorphic function f, the equation*

$$m(r, a) + N(r, a) = T(r, f) + \varepsilon(r, a)$$

holds for each $a \in \mathbf{C}$, *where* $\varepsilon(r, a) = O(1)$ *for* $r \to \infty$.

1.2 Theorem. (Second Main Theorem of the value distribution theory.) *Let f be a nonconstant meromorphic function. If* $a_1, a_2, \ldots, a_q$, $q \geqq 1$, *are mutually distinct finite or infinite complex numbers, then*

$$\sum_{\nu=1}^{q} m(r, a_\nu) \leqq 2\, T(r, f) - N_1(r) + S(r, f),$$

where $N_1(r) = N(r, 1/f') + 2\, N(r, f) - N(r, f')$ *and the remainder* $S(r, f)$ *satisfies the following conditions:* $S(r, f) = o\{T(r, f)\}$ *with at most the exception of a set* $\mathbf{E}$ *of values* (r) *of finite Lebesgue measure. If f is of finite order, then* $S(r, f) = o\{T(r, f)\}$ *without exceptional intervals.*

1.3 Definition. Let f be a meromorphic function, $a \in \mathbf{C} \cup \{\infty\}$. Let us set

$$\delta(a) = \delta(a, f) = \varliminf_{r \to \infty} \frac{m(r, a)}{T(r, f)} = 1 - \varlimsup_{r \to \infty} \frac{N(r, a)}{T(r, f)},$$

$$\Theta(a) = \Theta(a, f) = 1 - \varlimsup_{r \to \infty} \frac{\overline{N}(r, a)}{T(r, f)},$$

$$\vartheta(a) = \vartheta(a, f) = \varliminf_{r \to \infty} \frac{N(r, a) - \overline{N}(r, a)}{T(r, f)}.$$

Recall that $\Theta(a) \geqq \delta(a) + \vartheta(a)$. The quantity $\delta(a)$ is called the *deficiency* of the value a, $\vartheta(a)$ is called the *ramification index* of the point a. The value a is called *deficient value* (or *Nevanlinna exceptional value*) if $\delta(a) > 0$.

If the equation $f(z) = a$, $a \in \mathbf{C}$, has only a finite number of roots, then the value a is called *Picard exceptional value.* The function f must be transcendental. It is clear that every *Picard exceptional value* is *Nevanlinna exceptional value*, but the contrary is not true.

In the following we shal need the following theorems ($S(r, f)$ has the same meaning as in Theorem 1.2):

1.4 Theorem. (Milloux, see [5] or [2].) *Let f be a meromorphic function,* $k \in \mathbf{N}$ *arbitrary. Then*

$$T(r, f^{(k)}) \leqq (k + 1)\, T(r, f) + S(r, f). \tag{1}$$

1.5 Theorem. (See [5].) *Let f be a entire function,* $k \in \mathbf{N}$ *arbitrary. Then*

$$T(r, f^{(k)}) \leqq T(r, f) + S(r, f). \tag{2}$$

1.6 Theorem. (See [8] or [4].) *Let f be a meromorphic function for which $S(r,f) = o\{T(r,f)\}$. Then*

$$\frac{1}{n+1} \sum_{a \neq \infty} \delta(a,f) \leqq \delta(0, f^{(n)}) \tag{3}$$

for arbitrary $n \in \mathbf{N}$.

1.7 Note. The relation $S(r,f) = o\{T(r,f)\}$ is valid for every function of finite order, but it need not be fulfilled for functions of infinite order.

1.8 Theorem. (Hayman, see [2].) *Let f be a transcendental meromorphic function, $k \in \mathbf{N}$ arbitrary. Then*

$$T(r,f) \leqq \left(2 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{k}\right) \overline{N}\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r,f). \tag{4}$$

Corollary. *Either the function f assumes every finite value infinitely many times, or $f^{(k)}$ ($k \in \mathbf{N}$) assumes every nonzero finite value infinitely many times.*

1.9 Theorem. (Milloux, see [3], p. 132.) *Let f be a transcendental meromorphic function, $k \in \mathbf{N}$, $a \in \mathbf{C}$, $b \neq 0$. Then*

$$\begin{aligned} T(r,f) &\leqq \overline{N}(r,f) + \\ &+ N\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) - N\left(r, \frac{f^{(k)} - b}{f^{(k+1)}}\right) + S(r,f). \end{aligned} \tag{5}$$

1.10 Theorem. (See [1], [2].) *Let f be a meromorphic function. Then the quantity $\Theta(a)$ vanishes for all except at most a countable set of values a. Furthermore,*

$$\sum_a \{\delta(a) + \vartheta(a)\} \leqq \sum_a \Theta(a) \leqq 2. \tag{6}$$

1.11 Note. In the proof of Theorem 2.2 we shall need the following inequality which was obtained when proving Theorem 1.2 (see [1], [4], [8]):

$$N\left(r, \frac{1}{f'}\right) + \sum_{\nu=1}^{q} m(r, a_\nu) + S(r,f) \leqq T(r,f') \leqq N(r,f') + m(r,f) + S(r,f). \tag{7}$$

Here $a_1, a_2, \ldots, a_q$ are arbitrary finite complex numbers.

2. Some generalizations of Polya-Saxer theorem. The following theorem was proved by Polya and Saxer (1923).

2.1 Theorem. (See [6].) *If an entire transcendental function has finite Picard exceptional value, then every its derivative assumes all finite nonzero values infinitely many times.*

The Nevanlinna theory is a tool for finer investigation in this direction, and with its help it is possible to prove theorems that generalize, in different ways, the Polya-Saxer theorem.

The following generalization of the Polya-Saxer theorem is a consequence of the inequality (3).

2.2 Theorem. *Let f be a meromorphic function for which $S(r,f) = o\{T(r,f)\}$ and $\delta(\infty,f) = 1$. If the function f has finite Nevanlinna exceptional value a (that means $\delta(a) > 0$), then every derivative of f assumes all finite nonzero values infinitely many times.*

Proof. First we shall prove (under our suppositions) that the relation $\delta(\infty,f) = 1$ implies the relation $\delta(\infty,f^{(k)}) = 1$, for arbitrary $k \in \mathbf{N}$. From the evident relation

$$\frac{N(r,f')}{T(r,f')} = \frac{N(r,f) + \bar{N}(r,f)}{T(r,f)} \cdot \frac{T(r,f)}{T(r,f')}$$

we get

$$(8) \quad 1 - \delta(\infty,f') = \varlimsup_{r\to\infty} \frac{N(r,f')}{T(r,f')} \leqq [1 - \delta(\infty,f) + 1 - \Theta(\infty,f)] \varlimsup_{r\to\infty} \frac{T(r,f)}{T(r,f')}.$$

The inequality (7) yields a lower estimate for $\varliminf_{r\to\infty} T(r,f')/T(r,f)$ (and thereby also an upper estimate for $\varlimsup_{r\to\infty} T(r,f)/T(r,f')$). From the inequality (7) we obtain easily

$$\varliminf_{r\to\infty} \frac{T(r,f')}{T(r,f)} \geq \varliminf_{r\to\infty} \frac{N\left(r,\dfrac{1}{f'}\right) + \sum_{1}^{q} m(r,a_\nu)}{T(r,f)} +$$

$$+ \varliminf_{r\to\infty} \frac{S(r,f)}{T(r,f)} \geq \varliminf_{r\to\infty} \frac{N\left(r,\dfrac{1}{f'}\right)}{T(r,f)} + \sum_{\nu=1}^{q} \delta(a_\nu,f).$$

According to the suppositions of our theorem it is $\delta(a) > 0$. If we choose $a_\nu = a$ for any ν, then $\varliminf_{r\to\infty} T(r,f')/T(r,f) \geqq \delta(a) > 0$. Hence also

$$\varlimsup_{r\to\infty} \frac{T(r,f)}{T(r,f')} = \frac{1}{\varliminf_{r\to\infty} \dfrac{T(r,f')}{T(r,f)}} < +\infty.$$

If $\delta(\infty,f) = 1$, then $\Theta(\infty,f) = 1$, and from the inequality (8) we get $\delta(\infty,f') = 1$. The validity of the relation $\delta(\infty,f^{(k)}) = 1$, for arbitrary $k \in \mathbf{N}$ is obtained by simple induction.

The relation (6) from Theorem 1.10 applied to $f^{(k)}$ gives, with respect to $\delta(\infty, f^{(k)}) = 1$,

$$\sum_{b \neq 0,\infty} \delta(b, f^{(k)}) + \delta(0, f^{(k)}) \leqq 1 .$$

From (3) we get

$$\sum_{b \neq 0,\infty} \delta(b, f^{(k)}) \leqq 1 - \frac{1}{k+1} \sum_{a \neq \infty} \delta(a, f) .$$

According to the suppositions, there exist $a \in \mathbf{C}$ that $\delta(a, f) > 0$. Then $\sum_{b \neq 0,\infty} \delta(b, f^{(k)}) < 1$. Thus $\delta(b, f^{(k)}) < 1$ for every finite nonzero complex value b. This implies that the function f assumes the value b infinitely many times, for the function f is transcendental.

The corollary of Theorem 1.8 yields a further generalization of the Polya-Saxer theorem.

2.3 Theorem. *Let f be a transcendental meromorphic function. If the function f has finite Picard exceptional value, then every derivative of f assumes all finite nonzero values infinitely many times.*

3. In this section some further generalizations of the Polya-Saxer theorem will be proved.

3.1 Theorem. *Let f be a transcendental meromorphic function for which $\overline{N}(r, f) = o\{T(r, f)\}$ (that means $\Theta(\infty, f) = 1$). If the function f has finite Nevanlinna exceptional value, then every derivative of f assumes all finite nonzero value infinitely many times.*

3.2 Theorem. *Let f be a transcendental meromorphic function for which $S(r, f) = o\{T(r, f)\}$ and $\Theta(\infty, f) = 1$. If $\delta(a, f) > 0$, $a \in \mathbf{C}$, then for every finite nonzero complex number b and arbitrary $k \in \mathbf{N}$, the inequality*

$$\delta(b, f^{(k)}) \leqq \Theta(b, f^{(k)}) \leqq 1 - \frac{\delta(a, f)}{k+1} \tag{9}$$

holds.

3.3 Theorem. *Let f be an entire transcendental function for which $S(r, f) = o\{T(r, f)\}$. If $\delta(a, f) > 0$, $a \in \mathbf{C}$, then for every finite nonzero complex number b and arbitrary $k \in \mathbf{N}$, the inequality*

$$\delta(b, f^{(k)}) \leqq \Theta(b, f^{(k)}) \leqq 1 - \delta(a, f) \tag{10}$$

holds.

Proof of Theorem 3.1. We use the inequality (5), choosing the notation so that $\delta(a,f) > 0$. Let us suppose that the equation $f^{(k)}(z) = b$ has only a finite number of roots.

Then

$$\bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) = o\{T(r,f)\}.$$

Let us divide the inequality (5) by $T(r,f)$. We obtain the inequality

$$(11)\quad 1 \leqq \frac{\bar{N}(r,f)}{T(r,f)} + \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} + \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-b}\right)}{T(r,f)} - \frac{\bar{N}\left(r, \frac{f^{(k)}-b}{f^{(k+1)}}\right)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}.$$

In (11) we let $r \to \infty$, $r \notin \mathbf{E}$, where the set $\mathbf{E}$ has finite Lebesgue measure. The set $\mathbf{E}$ is "the exceptional set" from the Nevanlinna Second Main Theorem. Recall that $S(r,f) = o\{T(r,f)\}$ for $r \to \infty$, $r \notin \mathbf{E}$.

Since

$$\delta(a,f) > 0,$$

it is

$$(12)\quad \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} = A < 1.$$

According to our notation $\delta(a,f) = 1 - A$. From (11), (12) we obtain the inequality

$$1 \leqq \overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} \leqq \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} = A.$$

This contradicts the inequality $A < 1$. Therefore the supposition that there is only a finite number of roots of the equation $f^{(k)}(z) = b$ is not correct, hence the function $f^{(k)}$ assumes the value b infinitely many times, QED.

Proof of Theorem 3.2. Again we use the inequality (5). Now we consider functions for which $S(r,f) = o\{T(r,f)\}$! From (11) we obtain the inequality

$$(13)\quad \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-b}\right)}{T(r,f)} \geqq 1 - A.$$

The definition of the deficiency and the inequality (1) imply the following inequalities:

$$1 - A \leqq \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r,f)} = (k+1) \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{(k+1)\,T(r,f) + S(r,f)} \leqq$$

$$\leqq (k+1) \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r, f^{(k)})},$$

$$(k+1) - (k+1) \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r, f^{(k)})} \leqq A + k = 1 - \delta(a,f) + k\,,$$

$$1 - \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r, f^{(k)})} \leqq \frac{k + 1 - \delta(a,f)}{k+1} = 1 - \frac{\delta(a,f)}{k+1}.$$

The last inequality may be rewritten in the form

$$\delta(b, f^{(k)}) \leqq \Theta(b, f^{(k)}) \leqq 1 - \frac{\delta(a,f)}{k+1} \quad \text{QED}\,.$$

P r o o f o f T h e o r e m 3.3. The proof of the inequality (10) is analogous to that of the inequality (9), we only use the stricter inequality (2) instead of (1).

From the inequalities (13) and (2) we get

$$1 - A \leqq \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r,f)} = \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r,f) + S(r,f)} \leqq \varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f^{(k)} - b}\right)}{T(r, f^{(k)})}.$$

Then

$$\Theta(b, f^{(k)}) \leqq A = 1 - \delta(a,f)\,,$$

which together with the well-known inequality $\delta(a) \leqq \Theta(a)$ yields (10), QED.

4. Remarks.

4.1. The supposition in Theorem 3.1 about the existence of finite Nevanlinna exceptional value is essential. It will be seen in the following

Assertion. *Let $f(z) = e^z + z$. Then $\delta(a,f) = 0$ is valid for every $a \in \mathbf{C}$, that means, f has not finite Nevanlinna exceptional value.*

The function f is entire, hence the second supposition of Theorem 3.1 is evidently fulfilled. Its derivative $f'(z) = e^z + 1$ never assumes the nonzero value 1.

P r o o f of the assertion. We use the theorem which supplies in the terms of covering sufficient conditions for the validity of $\delta(a, f) = 0$. First let us recall some concepts. Let $w = f(z)$ be an entire function. Let us denote by $\mathscr{F}$ the Riemann surface of the analytic function f^{-1}. Its ramification points lie just over the points $w_k \in \mathbf{C}$, $w_k = f(z_k)$, for which $f'(z_k) = 0$. Further let us denote by π the natural projection $\mathscr{F}$ on $\mathbf{C}$ ($\pi(\mathfrak{Z}) = w$, where $\mathfrak{Z}$ is the algebraic element of the analytic function f^{-1}, with the centre at w).

Now, the following theorem (see [3], p. 431) is valid:

Theorem. *Let $w = f(z)$ be an entire function, $\mathscr{F}$ the Riemann surface of f^{-1}, $a \in \mathbf{C}$ arbitrary. Let $\Lambda > 0$ and an η-neighbourhood $U(a, \eta)$ exist with the following properties: If $\mathscr{F}_\nu \subset \mathscr{F}$ is an arbitrary domain over $U(a, \eta)$ $(\pi(\mathscr{F}_\nu) = U(a, \eta))$, then over every point $w \in U(a, \eta)$ there lie just λ_ν points and $1 \leqq \lambda_\nu \leqq \Lambda$ (every ramification point of the order m is counted $(m - 1)$-times). Then $\delta(a, f) = 0$.*

Now let us construct the Rieman surface $\mathscr{F}$ of the analytic function f^{-1}, where $f(z) = e^z + z$. It is $f'(z) = e^z + 1 = 0$ for $z_k = (2k + 1)\pi i$, $k = 0, \pm 1, \pm 2, \ldots$. The ramification points lie over $w_k = e^{z_k} + z_k = -1 + (2k + 1)\pi i$. These ramification points are of the first order, for $f''(z) = e^z \neq 0$. The function $e^z + z$ maps conformally the strip (see [7], p. 481, example 7)

$$\Omega_k = \{z \in \mathbf{C},\ (2k - 1)\pi < \operatorname{Im} z < (2k + 1)\pi\}$$

onto $\mathbf{C} \setminus (p_1^{(k)} \cup p_2^{(k)})$, where

$$p_1^{(k)} = \{z \in \mathbf{C},\ z = x + (2k + 1)\pi i,\ x \in (-\infty, -1\rangle\},$$
$$p_2^{(k)} = \{z \in \mathbf{C},\ z = x + (2k - 1)\pi i,\ x \in (-\infty, -1\rangle\}.$$

Let $\mathscr{F}$ be constructed so that the k-st sheet $\mathscr{P}_k$ of the plane $\mathbf{C}$, which is cut along the rays $p_1^{(k)}$ a $p_2^{(k)}$, is connected in the usual way with the $(k + 1)$-st sheet $\mathscr{P}_{k+1}$ along $p_2^{(k)}$ and with the $(k - 1)$-st sheet $\mathscr{P}_{k-1}$ along $p_1^{(k)}$, $k = 0, \pm 1, \pm 2, \ldots$. We obtain infinitely-many-sheeted surface. All domains lying over an arbitrary disc D with the centre at a point $w \neq w_k$, which contains none of the points w_k, are discs. All domains over an arbitrary disc D_k with the centre at w_k are discs except a single one which is a two-sheeted disc. In this domain, exactly two points lie over each point $w \in \mathsf{D}_k$. We can choose $\Lambda = 2$ or $\Lambda = 1$, hence the function $w = e^z + z$ has no Nevanlinna exceptional value.

4.2. Let us compare the suppositions of Theorems 2.2 and 3.1.

a. Theorem 2.2 applies only to functions with $S(r, f) = o\{T(r, f)\}$, while in Theorem 3.1 this condition does not appear.