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A GENERALIZATION OF THE TORSION FORM

IVAN KOLÁŘ, Brno

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The well-known torsion form of a linear connection on an n-dimensional manifold M is the exterior covariant derivative of the canonical $\mathbf{R}^n$-valued form φ of the bundle of linear frames. As a natural generalization of φ, we have introduced the canonical $(\mathbf{R}^n \oplus \mathfrak{g})$-valued form θ of the first prolongation $W^1(P)$ of an arbitrary principal fibre bundle $P(B, G)$, $n = \dim B$, [5]. In a similar way, we define the torsion form of a connection on $W^1(P)$ to be the exterior covariant derivative of θ. This concept generalizes also the torsion form of a linear connection of higher order in the sense of YUEN, [11]. Using a result by ŠVEC, we find the structure equations of θ. We also deduce that the connections on $W^1(P)$ are in a one-to-one correspondence with certain reductions of the second semi-holonomic prolongation $\overline{W}^2(P)$ of P, and a connection on $W^1(P)$ is without torsion if and only if the corresponding reduction is holonomic. In the special case of a linear connection, these results were established by KOBAYASHI, [3], and LIBERMANN, [8]. In conclusion, we treat the prolongation $p(\Gamma, \Lambda)$ of a connection Γ on P with respect to a linear connection Λ on the base manifold, [7], and we find a necessary and sufficient geometric condition for $p(\Gamma, \Lambda)$ to be without torsion. – Standard terminology and notation of the theory of jets are used throughout the paper, see, e.g., [10]. Our investigations are carried out in the category C^∞.

1. Let G and H be two Lie groups. Assume that every $g \in G$ determines an automorphism $\tilde{g} : H \to H$ such that the mapping $g \mapsto \tilde{g}$ is a right action of G on H. Consider the corresponding semi-direct product $G \mathbin{\overline{\times}} H$, i.e., the multiplication in $G \mathbin{\overline{\times}} H$ is given by

$$(1) \qquad (g_1, h_1)(g_2, h_2) = (g_1 g_2, \tilde{g}_2(h_1)\, h_2), \quad g_1, g_2 \in G, \quad h_1, h_2 \in H.$$

We have natural injections $G \to G \mathbin{\overline{\times}} H$, $g \mapsto (g, e_H)$ and $H \to G \mathbin{\overline{\times}} H$, $h \mapsto (e_G, h)$. In this sense, Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H form two complementary subspaces of the Lie algebra of $G \mathbin{\overline{\times}} H$. One verifies directly that

$$(2) \qquad \operatorname{ad}(g, e_H)(e_G, h) = (e_G, \tilde{g}^{-1}(h)).$$

Consider further a principal fibre bundle $P(B, G, \pi)$. Introduce a projection $P \times H \to B$, $(u, h) \mapsto \pi(u)$, $u \in P$, $h \in H$, and define a right action of $G \overline{\times} H$ on $P \times H$ by

$$(3) \qquad (u, h_1)(g, h_2) = (ug, \tilde{g}(h_1) h_2).$$

Lemma 1. *$P \times H$ with action (3) is a principal fibre bundle $(P \times H)(B, G \overline{\times} H)$.*

Proof is straightforward.

Obviously, $P \approx P \times \{e_H\}$ is a reduction of $P \times H$ to the subgroup $G \subset G \overline{\times} H$. In view of (2), we can apply the result by Švec, [9], p. 572. This proves

Lemma 2. *Let $\omega = \omega_1 \oplus \omega_2$ be a $(\mathfrak{g} \oplus \mathfrak{h})$-valued connection form on $P \times H$ and $\bar{\omega}_1$ or $\bar{\omega}_2$ the restriction of ω_1 or ω_2 to P, respectively. Then $\bar{\omega}_1$ is a connection form and $\bar{\omega}_2$ is an $\mathfrak{h}$-valued tensorial form of type* ad G. *Conversely, if $\bar{\omega}$ is a connection form on P and φ is an $\mathfrak{h}$-valued tensorial form of type* ad G *on P, then there is a unique connection form on $P \times H$ such that its restriction to P is $\bar{\omega} \oplus \varphi$.*

In particular, let ϱ be a representation of G on a finite dimensional vector space V. For $A \in \mathfrak{g}$, $A = j_0^1 \gamma(t)$ and $B \in V$, we set

$$(4) \qquad A \,.\, B = \lim_{t \to 0} \frac{1}{t} \left[\varrho(\gamma(t))(B) - B\right].$$

This defines a bilinear map $\mathfrak{g} \times V \to V$, $(A, B) \mapsto A \,.\, B$. Since V is an Abelian group and $g \mapsto \varrho(g^{-1})$ is a right action of G on V, we can construct the semi-direct product $G \overline{\times} V$. Let ω be a connection form on P and φ a V-valued tensorial 1-form of type ϱ on P. We have the situation of Lemma 2 and one verifies easily that formula (2.23) of [9] is equivalent to the following

Proposition 1. *It is*

$$(5) \qquad \mathrm{d}\varphi = -\omega \,.\, \varphi + D\varphi,$$

where $D\varphi$ is the covariant exterior derivative of φ with respect to ω and $\omega \,.\, \varphi$ means the 2-form on P defined by the extension of bilinear map (4).

2. Consider now the first prolongation $W^1(P)$ of a principal fibre bundle $P(B, G)$, [5]. We recall that $W^1(P) = H^1(B) \oplus J^1 P$ is a principal fibre bundle over B with structure group $G_n^1 = L_n^1 \overline{\times} T_n^1(G)$ (= the semi-direct product with respect to the action $S \mapsto SY$ of L_n^1 on $T_n^1(G)$, $Y \in L_n^1$, $S \in T_n^1(G)$), $n = \dim B$. There are two canonical principal fibre bundle homomorphisms $\beta : W^1(P) \to P$ and $\lambda : W^1(P) \to H^1(B)$. In [5], we have introduced the canonical $(\mathbf{R}^n \oplus \mathfrak{g})$-valued form θ of $W^1(P)$ and we have deduced that θ is a pseudotensorial form of type ϱ, where the representation ϱ of G_n^1 on $\mathbf{R}^n \oplus \mathfrak{g}$ is defined by formula (12) of [5]. If Γ is a connection on $W^1(P)$, then the covariant absolute derivative $D\theta$ of θ will be called the torsion of Γ.

Remark 1. If we consider the trivial one-element group $G = \{e\}$ and the trivial bundle $B \times \{e\}$, then $W^1(B \times \{e\}) = H^1(B)$ and θ coincides with the canonical $\mathbf{R}^n$-valued form of $H^1(B)$. Hence we get really a generalization of the linear case.

Remark 2. Using the identification $\tilde{H}^r(B) \approx W^1(\tilde{H}^{r-1}(B))$ of [5], we obtain the inclusion $\bar{H}^r(M) \subset W^1(\bar{H}^{r-1}(M))$. Further, the restriction of the canonical form of $W^1(\bar{H}^{r-1}(M))$ to $\bar{H}^r(M)$ is the canonical form of $\bar{H}^r(M)$. In this interpretation, our results generalize the investigation of the torsion form of a higher order linear connection by Yuen, [11].

By (4), ϱ determines a bilinear map $\mathfrak{g}_n^1 \times (\mathbf{R}^n \oplus \mathfrak{g}) \to \mathbf{R}^n \oplus \mathfrak{g}$, $(A, B) \mapsto A \,.\, B$. According to [5], we have a decomposition $\mathfrak{g}_n^1 = \mathfrak{g} \oplus (\mathbf{R}^n \otimes \mathbf{R}^{n*}) \oplus (\mathfrak{g} \otimes \mathbf{R}^{n*})$. Hence we can write every $A \in \mathfrak{g}_n^1$ as $A = A_1 + A_2 + A_3$, $A_1 \in \mathfrak{g}$, $A_2 \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$, $A_3 \in \mathfrak{g} \otimes \mathbf{R}^{n*}$, and every $B \in \mathbf{R}^n \oplus \mathfrak{g}$ as $B = B_0 + B_1$, $B_0 \in \mathbf{R}^n$, $B_1 \in \mathfrak{g}$. The same notation will be used for $\mathfrak{g}_n^1$-valued and $(\mathbf{R}^n \oplus \mathfrak{g})$-valued forms. Let $\langle , \rangle : \mathbf{R}^n \times (\mathbf{R}^n \otimes \mathbf{R}^{n*}) \to \mathbf{R}^n$, $\langle , \rangle_G : \mathbf{R}^n \times (\mathfrak{g} \otimes \mathbf{R}^{n*}) \to \mathfrak{g}$ be tensor contractions and $[,]$ the bracket of $\mathfrak{g}$. By direct evaluation, we obtain the formula for $A \,.\, B$

$$(6) \qquad (A \,.\, B)_0 = \langle A_2, B_0 \rangle\,,$$
$$(A \,.\, B)_1 = [A_1, B_1] + \langle A_3, B_0 \rangle_G\,.$$

Proposition 2. (*Structure equations of θ.*) *Let ω be a connection form on $W^1(P)$. Then we have*

$$(7) \qquad \mathrm{d}\theta = -\omega \,.\, \theta + \tfrac{1}{2}[\omega_1, \omega_1] + D\theta\,,$$

where the $\mathfrak{g}$-valued form $[\omega_1, \omega_1]$ is considered an $(\mathbf{R}^n \oplus \mathfrak{g})$-valued form with zero component in $\mathbf{R}^n$.

Proof is based on Proposition 1. However, θ is not horizontal. That is why we shall first consider the tensorial form $\tilde{\theta} = \theta h$, i.e. $\tilde{\theta}(X) = \theta(hX)$, where hX means the horizontal component of the vector $X \in T(W^1(P))$. By Proposition 1,

$$(8) \qquad \mathrm{d}\tilde{\theta}_0 = -\langle \omega_2, \tilde{\theta}_0 \rangle + D\tilde{\theta}_0\,,$$
$$\mathrm{d}\tilde{\theta}_1 = -[\omega_1, \tilde{\theta}_1] - \langle \omega_3, \tilde{\theta}_0 \rangle_G + D\tilde{\theta}_1\,.$$

Further, let Y be a vertical vector on $W^1(P)$, which is the value of the fundamental vector field determined by an element $A \in \mathfrak{g}_n^1$. By the definition of θ, [5], we have $\theta(Y) = A_1$. Hence $\theta = \tilde{\theta} + \omega_1$, where the $\mathfrak{g}$-valued form ω_1 is considered an $(\mathbf{R}^n \oplus \mathfrak{g})$-valued form with zero component in $\mathbf{R}^n$. Substituting it into (8), we obtain

$$(9) \qquad \mathrm{d}\theta_0 = -\langle \omega_2, \theta_0 \rangle + D\theta_0\,,$$
$$\mathrm{d}\theta_1 = -[\omega_1, \theta_1] - \langle \omega_3, \theta_0 \rangle_G + [\omega_1, \omega_1] + \mathrm{d}\omega_1 - D\omega_1 + D\theta_1\,.$$

According to the structure equations of ω, it is

$$d\omega_1 = -\tfrac{1}{2}[\omega_1, \omega_1] + D\omega_1 . \tag{10}$$

Comparing (9) and (10), we deduce (7), QED.

3. We have remarked in [4] that a connection on a principal fibre bundle $P(B, G)$ can be defined as a G-invariant cross section $P \to J^1P$. Consider a connection Γ on $W^1(P)$ in such a form, i.e. $\Gamma : W^1(P) \to J^1\, W^1(P)$. We have $J^1\, W^1(P) = J^1(H^1(B) \oplus J^1P) = J^1\, H^1(B) \oplus \tilde{J}^2P$. There is a standard identification $\varkappa : J^1\, H^1(B) \approx \bar{H}^2(B)$ sending an element $Z = j^1_x\varphi \in J^1\, H^1(B)$, $\varphi(x) = j^1_0\, \psi(y)$ into $\varkappa(Z) = j^1_0[\varphi(\psi(y))\, t_y^{-1}] \in \bar{H}^2(B)$, where $t_y : \mathbf{R}^n \to \mathbf{R}^n$ is the translation $z \mapsto z + y$. On the other hand, the second semi-holonomic prolongation $\bar{W}^2(P)$ of P is equal to $\bar{H}^2(B) \oplus \bar{J}^2P$, [5], so that the jet inclusion $\bar{J}^2P \subset \tilde{J}^2P$ induces the inclusion $\bar{W}^2(P) \subset J^1\, W^1(P)$. We define the reduction $R(\Gamma) \subset \bar{W}^2(P)$ determined by a connection $\Gamma : W^1(P) \to J^1\, W^1(P)$ to be the intersection

$$R(\Gamma) = \Gamma(W^1(P)) \cap \bar{W}^2(P) . \tag{11}$$

Consider the induced connection $\Gamma_0 = \beta_*\Gamma : P \to J^1P$. We recall, [6], that $R(\Gamma_0) := H^1(B) \oplus \Gamma_0(P)$ is a reduction of $W^1(P)$ to the subgroup $L^1_n \times i_1(G) \subset G^1_n$, where $i_1 : G \to T^1_n(G)$ is the canonical injection $g \mapsto j^1_0\hat{g}$, $\hat{g}$ being the constant mapping $x \mapsto g$, $x \in \mathbf{R}^n$.

Lemma 3. *We have $\Gamma(u) \in R(\Gamma)$ if and only if $u \in R(\Gamma_0) \subset W^1(P)$.*

Proof. Let $\Gamma(u) = j^1_x\varphi$, where $\varphi = (\varphi_1, \varphi_2)$ is a local cross section of $H^1(B) \oplus J^1P$. The condition for $j^1_x\varphi_2$ to be semi-holonomic is $\varphi_2(x) = j^1_x(j^0_1\varphi_2) = j^1_x(\beta\varphi) = \Gamma_0(j^0_1u)$, where $j^0_1 : J^1P \to P$ is the jet projection. This is equivalent to $u \in R(\Gamma_0)$, QED.

Consider further the canonical injections $i_2 : G \to T^2_n(G)$, $g \mapsto j^2_0\hat{g}$ and $i : L^1_n \to L^2_n$. The last mapping can be geometrically described as follows. If $Y \in L^1_n$, $Y = j^1_0\, \psi(y)$, then

$$i(Y) = j^1_0[t_{\psi(y)} Y t_y^{-1}] . \tag{12}$$

Our next assertion generalizes a result by Libermann, [8].

Proposition 3. *$R(\Gamma)$ is a reduction of $\bar{W}^2(P)$ to the subgroup $i(L^1_n) \times i_2(G) \subset \bar{L}^2_n \mathbin{\bar{\times}} \bar{T}^2_n(G) = \bar{G}^2_n$. Conversely, every reduction Q of $\bar{W}^2(P)$ to $i(L^1_n) \times i_2(G)$ determines a unique connection $\Gamma(Q)$ on $W^1(P)$ such that $Q = R(\Gamma(Q))$.*

Proof. Put $\Gamma(v, \Gamma_0(u)) = (Z, T) \in J^1\, H^1(B) \oplus \tilde{J}^2P$, $u \in P$, $v \in H^1(B)$, and $Z =$

$= j^1_x\varphi$, $T = j^1_x\sigma$. Starting from the fact that Γ is G^1_n-invariant and using the formula for the action of G^1_n on $W^1(P)$, [5], we find

$$(13) \qquad \Gamma(vY, \Gamma_0(ug)) = j^1_x[\varphi(y)\, Y, \sigma(y) \,.\, (i_1(g)\, Y^{-1}\, \varphi^{-1}(y))] \,,$$

$Y \in L^1_n$, $g \in G$. On the other hand, the formula for the action of $\bar{G}^2_n$ on $\overline{W}^2(P)$ yields

$$(\varkappa(Z), T)\,(i(Y), i_2(g)) = (\varkappa(Z)\, i(Y), T \,.\, (i_2(g)\, i(Y)^{-1}\, \varkappa(Z)^{-1})) \,,$$

see [5]. The relation $\varkappa(Z)\, i(Y) = j^1_x[\varphi(y)\, Y]$ is known from the linear case, [8]. Further, the injection $i_2 : G \to T^2_n(G)$ can be also expressed as $g \mapsto j^1_0[i_1(g)\, t_y^{-1}]$. Then we find easily $i_2(g)\, i(Y)^{-1}\, \varkappa(Z)^{-1} = j^1_x(i_1(g)\, Y^{-1}\, \varphi^{-1}(y))$. Comparing with (13), we conclude that $R(\Gamma)$ is a reduction to the subgroup $i(L^1_n) \times i_2(G)$. The converse assertion can be proved quite similarly, QED.

We shall also need another geometric characterization of $R(\Gamma)$. We recall that a semi-holonomic connection of the second order on P is a G-invariant cross section $P \to \bar{J}^2P$, [4]. For every $(v, u) \in H^1(B) \oplus P$, define $\mu(\Gamma)\,(v, u) = p_2(\Gamma(v, \Gamma_0(u)))$, where $p_2 : J^1\, W^1(P) \to \bar{J}^2P$ is the product projection. According to Lemma 3, the values of $\mu(\Gamma)$ lie in $\bar{J}^2P$.

Lemma 4. *For every $Y \in L^1_n$, it is $\mu(\Gamma)\,(v, u) = \mu(\Gamma)\,(vY, u)$.*

Proof. Let $\Gamma(v, \Gamma_0(u)) = j^1_x(\varphi_1(y), \varphi_2(y))$. Since Γ is invariant, we have $\Gamma(vY, \Gamma_0(u)) = j^1_x(\varphi_1(y)\, Y, \varphi_2(y))$, QED.

Thus, we may consider $\mu(\Gamma)$ to be a cross section $P \to \bar{J}^2P$.

Proposition 4. *$\mu(\Gamma) : P \to \bar{J}^2P$ is a semi-holonomic connection of the second order on P.*

Proof. We have to prove that $\mu(\Gamma)$ is G-invariant. But this is a simple consequence of Proposition 3, QED.

Denote by $\Lambda = \lambda_*\Gamma$ the induced connection on $H^1(B)$ and by $R(\Lambda)$ the corresponding reduction of $\bar{H}^2(B)$. Our previous consideration implies

$$(14) \qquad R(\Gamma) = R(\Lambda) \oplus \mu(\Gamma)\,(P) \,.$$

4. The following assertion generalizes a result by Kobayashi, [3].

Proposition 5. *It is $R(\Gamma) \subset W^2(P)$ if and only if $D\theta = 0$.*

Proof. We first deduce a lemma. Since $\overline{W}^2(P) \subset W^1(W^1(P))$, every $U \in \overline{W}^2(P)$ determines a mapping $\tilde{U}^{-1} : T_u(W^1(P)) \to \mathbf{R}^n \oplus \mathfrak{g}^1_n$, where $u \in W^1(P)$ is the underlying jet of U, [5]. Denote by $q : W^1(P) \to B$ the bundle projection.

Lemma 5. *Let M be a submanifold of $W^1(P)$ such that $q \mid M$ is a submersion.*

Let $\sigma : M \to \overline{W}^2(P)$ be a cross section and θ the $(\mathbf{R}^n \oplus \mathfrak{g}_n^1)$-valued form on M constructed by means of σ, i.e. $\theta \mid T_u(M) = \widetilde{\sigma(u)}^{-1} \mid T_u(M)$, $u \in M$. Then $\sigma(M) \subset W^2(P)$ if and only if

$$(15) \qquad d\theta_0 = -\langle\theta_2, \theta_0\rangle\,, \quad d\theta_1 = -\tfrac{1}{2}[\theta_1, \theta_1] - \langle\theta_3, \theta_0\rangle_G\,.$$

Proof of Lemma 5. For $M = W^1(P)$, the assertion was deduced by direct evaluation by Dekrét, [1]. Using the coordinates of [5] or [1], we have local coordinates a^λ_{ij} on fibred manifold $\overline{W}^2(P) \to W^1(P)$, $i, j, \ldots = 1, \ldots, n$, $\lambda = 1, \ldots, n + \dim G$. The subspace $W^2(P) \subset \overline{W}^2(P)$ is characterized by $a^\lambda_{ij} = a^\lambda_{ji}$. Consider (locally) a cross section $\sigma_1 : W^1(P) \to \overline{W}^2(P)$ extending σ. Let σ_1 be given by some functions $\bar{f}^\lambda_{ij}$, so that σ is given by $f^\lambda_{ij} = \bar{f}^\lambda_{ij} \mid M$. Denote by $\bar{\theta}$ the $(\mathbf{R}^n \oplus \mathfrak{g}_n^1)$-valued form on $W^1(P)$ constructed by means of σ_1. The evaluations by Dekrét imply (in coordinates)

$$(16) \qquad \begin{aligned} d\bar{\theta}^i &= \bar{\theta}^j \wedge \bar{\theta}^i_j + \bar{f}^i_{jk}\bar{\theta}^j \wedge \bar{\theta}^k\,,\\ d\bar{\theta}^\alpha &= -\tfrac{1}{2}c^\alpha_{\beta\gamma}\bar{\theta}^\beta \wedge \bar{\theta}^\gamma + \bar{\theta}^i \wedge \bar{\theta}^\alpha_i + \bar{f}^\alpha_{jk}\bar{\theta}^j \wedge \bar{\theta}^k\,,\end{aligned}$$

$\alpha = n + 1, \ldots, n + \dim G$. Restricting (16) to M, we find that (15) holds if and only if $f^\lambda_{ij} = f^\lambda_{ij}$, thus proving Lemma 5.

We are now in position to prove Proposition 5. Denote by $\tilde{\omega} = \tilde{\omega}_1 \oplus \tilde{\omega}_2 \oplus \tilde{\omega}_3$ or $\tilde{\theta} = \tilde{\theta}_0 \oplus \tilde{\theta}_1$ the restriction of the connection form or the canonical form θ to $R(\Gamma)$, respectively. By the definition of $R(\Gamma)$ and by Lemma 3, it is $\tilde{\omega}_1 = \tilde{\theta}_1$ and $\tilde{\theta}_0 \oplus \tilde{\theta}_1 \oplus \tilde{\omega}_2 \oplus \tilde{\omega}_3$ is the $(\mathbf{R}^n \oplus \mathfrak{g}_n^1)$-valued form constructed by means of the cross section $\Gamma \mid R(\Gamma_0)$. According to (7), we have

$$\begin{aligned} d\tilde{\theta}_0 &= -\langle\tilde{\omega}_2, \tilde{\theta}_0\rangle + D\tilde{\theta}_0\,,\\ d\tilde{\theta}_1 &= -\tfrac{1}{2}[\tilde{\theta}_1, \tilde{\theta}_1] - \langle\tilde{\omega}_3, \tilde{\theta}_0\rangle_G + D\tilde{\theta}_1\,.\end{aligned}$$

Then Proposition 5 follows from Lemma 5, QED.

5. In particular, if Γ is a connection on P and Λ is a linear connection on B, then the prolongation $p(\Gamma, \Lambda)$ of Γ with respect to Λ is an interesting special connection on $W^1(P)$, [7].

Proposition 6. *Connection $p(\Gamma, \Lambda)$ is without torsion if and only if Γ is integrable and Λ is without torsion.*

Proof. As a direct consequence of the definition, we have $\mu(p(\Gamma, \Lambda)) = \Gamma'$, where Γ' means the prolongation of Γ in the sense of Ehresmann, [2]. According to (14), it is $R(p(\Gamma, \Lambda)) = R(\Lambda) \oplus \Gamma'(P)$. By a result by Kobayashi, [3] (or as a special case of Proposition 5), $R(\Lambda) \subset H^2(B)$ if and only if Λ is without torsion. On the other hand, according to Ehresmann, [2], $\Gamma'(P) \subset J^2P$ if and only if Γ is integrable. By Proposition 5 we prove our assertion, QED.