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EXISTENCE OF SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

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In the paper we shall consider the functional-differential equation

$$(1) \qquad y'(t) = f(t, y),$$

where $f : R \times C_n \to R_n$ is a functional continuous with respect to the first variable, R the set of real numbers and C_n the class of continuous functions from R to the n-dimensional Euclidean space R_n. Assume that τ and ϑ are non-negative locally bounded functions $R \to R$. Let $\|\cdot\|$ be the Euclidean norm in R_n. The main result of this paper is the following theorem which is more general then the results recently obtained by JU. A. RJABOV [3], [4] concerning the existence of solutions of linear or weakly non-linear delayed differential equations with small delay; for complete references see a survey paper of R. D. DRIVER [1].

Theorem 1. *Assume that there is a non-negative locally integrable function $h : R \to R$ such that for each $x, y \in C_n$ and each $t \in R$,*

$$(2) \qquad \|f(t, x)\| \leqq h(t) \max \{\|x(t + \xi)\|;\ -\tau(t) \leqq \xi \leqq \vartheta(t)\},$$

$$(3) \qquad \|f(t, x) - f(t, y)\| \leqq h(t) \max \{\|x(t + \xi) - y(t + \xi)\|;\ -\tau(t) \leqq \xi \leqq \vartheta(t)\},$$

and

$$(4) \qquad \max \left(\int_{t-\tau(t)}^{t} h(\xi)\, d\xi,\ \int_{t}^{t+\vartheta(t)} h(\xi)\, d\xi \right) \leqq 1/e\,.$$

Then for each point $(a, b) \in R \times R_n$ there is a solution of (1) defined for all t which passes through (a, b).

Remark. The equation

$$(5) \qquad y'(t) = A(t)\, y(t - \tau(t)) + B(t)\, y(t) + C(t)\, y(t + \vartheta(t))$$

where A, B, C are locally integrable square matrices $R \to R_{n \times n}$, is a particular case of (1). Theorem 1 now asserts that if the function $h(t) = n(\|A(t)\| + \|B(t)\| + \|C(t)\|)$ satisfies (4) for $t \in R$ then a solution of (5) defined for all t passes through each point of $R \times R_n$. Here the norm $\|(a_{ij})\|$ of a matrix is assumed to be $\max_{i,j} |a_{ij}|$.

Proof of Theorem 1. Let Ω be the set of those $x \in C_n$, for which $x(a) = b$ and $\|x(t)\| \leqq \|b\| \exp(e|\int_a^t h(\xi)\, d\xi|)$, for all $t \in R$. Let $\lambda \in (0, 1]$. For $x \in \Omega$ let $F_\lambda(x)$ be the function $R \to R_n$ defined by $F_\lambda(x)(t) = b + \lambda \int_a^t f(\xi, x)\, d\xi$. Using (2) we get

$$\|F_\lambda(x)(t)\| \leqq \|b\| + \lambda \left\| \int_a^t f(\xi, x)\, d\xi \right\| \leqq$$

$$\leqq \|b\| \left(1 + \lambda \left| \int_a^t h(\xi) \exp\left(e \left| \int_a^\xi h(\eta)\, d\eta \right|\right) . \right.\right.$$

$$\left.\left. . \exp\left(\max\left(e \int_{\xi - \tau(\xi)}^{\xi} h(\eta)\, d\eta,\quad e \int_{\xi}^{\xi + \vartheta(\xi)} h(\eta)\, d\eta\right)\right) d\xi \right|\right) \leqq$$

$$\leqq \|b\| \left(1 + e \int_a^t h(\xi) \exp\left(e \left| \int_a^\xi h(\eta)\, d\eta \right|\right) d\xi\right) = \|b\| \exp\left(e \left| \int_a^t h(\xi)\, d\xi \right|\right).$$

Thus $F_\lambda : \Omega \to \Omega$. Now define the following Picard iterations, assuming that λ is fixed, $0 < \lambda < 1$: $x_1(t) \equiv b$ and $x_{k+1} = F_\lambda(x_k)$, for $k = 1, 2, \ldots$. Clearly for each t, $\|x_2(t) - x_1(t)\| \leqq \|b\| \, |\int_a^t h(\xi)\, d\xi| \leqq \|b\| \exp(e|\int_a^t h(\xi)\, d\xi|)$. Assume that, for all t, $\|x_k(t) - x_{k-1}(t)\| \leqq K\|b\| \exp(e|\int_a^t h(\xi)\, d\xi|)$. Then using (3) we obtain

$$\|x_{k+1}(t) - x_k(t)\| \leqq K\|b\|\, \lambda \left| \int_a^t h(\xi) \exp . \right.$$

$$\left. . \left(\max\left(e \left| \int_a^{\xi - \tau(\xi)} h(\eta)\, d\eta \right|,\quad e \left| \int_a^{\xi + \vartheta(\xi)} h(\eta)\, d\eta \right|\right)\right) d\xi \right| \leqq$$

$$\leqq K\lambda\|b\|\, e \left| \int_a^t h(\xi) \exp\left(e \left| \int_a^\xi h(\eta)\, d\eta \right|\right) d\xi \right| = K\lambda\|b\| \exp\left(e \left| \int_a^t h(\xi)\, d\xi \right|\right).$$

Since $0 < \lambda < 1$, the sequence x_n converges almost uniformly to some $x \in \Omega$ such that $F_\lambda(x) = x$.

Let $\{\lambda_n\}$ be a sequence of members of the open interval $(0, 1)$ converging to 1. For every n, let y_n satisfy the equation $y_n = F_{\lambda_n}(y_n)$. All $y_n \in \Omega$ are almost uniformly bounded (i.e. uniformly bounded on each compact). Let $A \subset R$ be a compact. By (2) we have, for each $t \in A$,

$$\|y_n'(t)\| \leqq h(t)\, \|b\| \exp\left(\max\left(e \left| \int_a^u h(\eta)\, d\eta \right|,\quad e \left| \int_a^v h(\eta)\, d\eta \right|\right)\right),$$

where $u = \inf_{\xi \in A} \xi - \tau(\xi)$, $v = \sup_{\xi \in A} \xi + \vartheta(\xi)$. Therefore $\|y_n(t) - y_n(s)\| \leqq \leqq \text{const} \left|\int_t^s h(\xi)\, d\xi\right|$ for $t, s \in A$. Consequently the functions $\{y_n\}$ are equicontinuous on each compact and hence there is a subsequence $\{y_{k(n)}\}$ of y_n which converges almost uniformly to some $y \in \Omega$. Clearly $y(a) = b$. It remains to show that y is a solution of (1) or, which is the same, of the corresponding integral equation.

Let I be a compact subinterval of R. For $t \in I$ we have

$$\left\| y(t) - b - \int_a^t f(\xi, y)\, d\xi \right\| \leqq \|y(t) - y_{n(k)}(t)\| +$$

$$+ \lambda_{n(k)} \|b\| \cdot \left| \int_a^t h(\xi)\, d\xi \right| \max_{\xi \in B} \|y_{n(k)}(\xi) - y(\xi)\| + (1 - \lambda_{n(k)}) \left\| \int_a^t f(\xi, y)\, d\xi \right\|,$$

where $B = [\inf_{\xi \in I} \xi - \tau(\xi), \sup_{\xi \in I} \xi + \vartheta(\xi)]$. Clearly the right-hand side of the inequality tends to 0 whenever $k \to \infty$, q.e.d.

Remark. If the assumptions of Theorem 1 are satisfied with the constant $1/e$ replaced by a positive constant $c < 1/e$ then for each point of $R \times R_n$ there is exactly one solution of (1) which belongs to Ω and passes through the point.

The constant $1/e$ in Theorem 1 is the best possible. To see this we first prove the following

Lemma. *For every sufficiently small $\delta > 0$ there are real numbers a, b with $a < 0$, $0 < b < \pi$ such that $x(t) = e^{at} \cos(bt)$ is a solution of the equation*

$$x'(t) = -e^{\delta - 1} x(t - 1), \tag{6}$$

for all real t.

Proof. For $\xi \leqq 0$ put $\varphi(\xi) = e^{\delta - 1 - \xi} + \xi$. Then $\varphi > 0$. Indeed, if $\varphi(u) = 0$ for some $u < 0$ then we may assume that u is the least root of φ, since $\lim_{\xi \to -\infty} \varphi(\xi) = = +\infty$. In this case we have $\varphi'(u) \leqq 0$, and consequently, $\varphi(u) + \varphi'(u) \leqq 0$, i.e. $u \leqq -1$. On the other hand, φ is a decreasing function in $(-\infty, -1]$, and $\varphi(-1) > > 0$, a contradiction.

Let $\psi(\xi) = \varphi(\xi)(e^{\delta - 1 - \xi} - \xi) = e^{2(\delta - 1 - \xi)} - \xi^2$. Clearly $\psi(\xi) > 0$ for all $\xi \leqq 0$. Let $\omega(\xi) = \xi e^{\xi} e^{1 - \delta} + \cos \sqrt{\psi(\xi)}$. We show that ω has a root in $(-2, 0)$. For sufficiently small δ we have $\psi(0) < \pi^2/4$. Since $\psi(-2) > \pi^2/4$, there is $v \in (-2, 0)$ such that $\psi(v) = \pi^2/4$, i.e. $\omega(v) < 0$. Since $\omega(0) > 0$, there is $a \in (-2, 0)$ such that $\omega(a) = 0$.

The function $x(t) = e^{at} \cos(t \sqrt{\psi(a)})$ is a solution of (6). Indeed, a simple calculation shows that x is a solution of (6) if and only if $ae^{1 + a - \delta} = -\cos \sqrt{\psi(a)}$, and $e^{1 + a - \delta} \sqrt{\psi(a)} = \sin \sqrt{\psi(a)}$. But the first equality is true since it is equivalent to $\omega(a) = 0$. To see that the second equality is also true note that if δ is sufficiently