

Werk

Label: Article

Jahr: 1975

PURL: https://resolver.sub.uni-goettingen.de/purl?31311157X_0100|log57

Kontakt/Contact

[Digizeitschriften e.V.](#)
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

✉ info@digizeitschriften.de

ON ASYMPTOTIC BEHAVIOUR OF CENTRAL DISPERSIONS
OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

MIROSLAV BARTUŠEK, Brno

(Received September 10, 1973)

1.1. Consider a differential equation

$$(q) \quad y'' = q(t)y, \quad q \in C^0[a, b), \quad b \leq \infty$$

where $C^n[a, b)$ (n being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b)$. In all the paper we will deal only with oscillatory ($t \rightarrow b_-$) differential equations (i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$).

Let y be a non-trivial solution of (q) vanishing at $t \in [a, b)$. If $\varphi(t)$ is the first zero of y lying to the right from t , then φ is called the basic central dispersion of the 1-st kind of (q) (briefly, dispersion of (q)). The function φ has the following properties (see [2] § 13):

- (1)
- 1) $\varphi \in C^3[a, b)$,
 - 2) $\varphi'(t) > 0$ on $[a, b)$,
 - 3) $\varphi(t) > t$ on $[a, b)$,
 - 4) $\lim_{t \rightarrow b_-} \varphi(t) = b$,
 - 5) $-\frac{1}{2} \frac{\varphi'''}{\varphi'} + \frac{3}{4} \left(\frac{\varphi''}{\varphi'} \right)^2 + q(\varphi) \varphi'^2 = q(t)$, $t \in [a, b)$,
 - 6) $\varphi'(t) = \frac{q(t_1)}{q(t_2)}$, $t < t_1 < t_2 < \varphi(t)$.

1.2. In our later considerations we will generalize some of the following results which have been proved by the author of [3] (see also [1]):

Theorem 1. Let $q \in C^n[a, \infty)$, $p \in C^n[a, \infty)$ ($n \geq 0$ being an integer) and let $q \in C^1[a, \infty)$ if $n = 0$. Let $\limsup_{t \rightarrow \infty} q(t) < 0$, $\liminf_{t \rightarrow \infty} q(t) > -\infty$, $\lim_{t \rightarrow \infty} (q(t) - p(t)) = 0$, $\lim_{t \rightarrow \infty} q'(t) = 0$ and if $n > 0$ let $\lim_{t \rightarrow \infty} q^{(k)}(t) = 0$, $\lim_{t \rightarrow \infty} p^{(k)}(t) = 0$ for $k = 1, 2, \dots, n$. If $\varphi, \bar{\varphi}$ are the dispersions of (q) and $y'' = p(t)y$, respectively, then

$$\lim_{t \rightarrow \infty} (\varphi(t) - \bar{\varphi}(t)^{(k)}) = 0, \quad k = 1, 2, \dots, n + 3.$$

Lemma 1. Let $q \in C^0[a, b)$, $\limsup_{t \rightarrow b-} q(t) < 0$. Let φ be the dispersion of the differential equation (q) . Then there exists a number $k > 0$ such that

$$\varphi(t) - t \leq k, \quad t \in [a, b).$$

2. Lemma 2. Let φ be the dispersion of an oscillatory $(t \rightarrow b_-)$ differential equation (q) , $q \in C^1[a, b)$, $\limsup_{t \rightarrow b-} q(t) < 0$. Let

$$\lim_{t \rightarrow b-} \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi(t) - t) = 0.$$

Then

$$\lim_{t \rightarrow b-} \varphi'(t) = 1,$$

$$\lim_{t \rightarrow b-} \varphi''(t) = \lim_{t \rightarrow b-} \varphi'''(t) = 0.$$

Proof. It follows from the assumption $\limsup_{t \rightarrow b-} q(t) = c < 0$ and from Lemma 1 that there exist numbers $t_0 \in [a, b)$, $K > 0$ such that we have

$$\varphi(t) - t \leq K, \quad t \in [a, b),$$

$$q(t) \leq \frac{c}{2}, \quad t \in [t_0, b).$$

Then according to (1) 6, we obtain for $t \geq t_0$ that

$$\begin{aligned} |q(\varphi) (1 - \varphi')| &= |q(\varphi)| \cdot \left| 1 - \frac{q(t_1)}{q(t_2)} \right| = \left| \frac{q(\varphi)}{q(t_2)} \right| \cdot |q(t_2) - q(t_1)| \leq \\ &\leq \frac{|q(t_2)| + M_1(t)}{|q(t_2)|} \cdot |q'(\xi)| \cdot (t_2 - t_1) \leq \left(1 + \frac{2}{c} M_2 \right) \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi(t) - t) \end{aligned}$$

holds where $t < t_1 < t_2 < \varphi(t)$, $\xi \in (t_1, t_2)$, $\eta \in (t_2, \varphi(t))$,

$$M_1(t) = |q'(\eta)| (\varphi(t) - t_2) \leq \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi - t) \xrightarrow{t \rightarrow b-} 0,$$

$$M_2 = \max_{t \in [a, b)} M_1(t).$$

Hence it follows

$$(2) \quad \lim_{t \rightarrow b-} |q(\varphi)(1 - \varphi')| = 0$$

and thus $\lim_{t \rightarrow b-} \varphi'(t) = 1$. So the first part of the statement is proved.

According to (2) and (1) 5, we have

$$\begin{aligned} & \lim_{t \rightarrow b-} |-\frac{1}{2}\varphi'''\varphi' + \frac{3}{4}\varphi''^2| = \lim_{t \rightarrow b-} |q(t) - q(\varphi)\varphi'^2| = \\ & = \lim_{t \rightarrow b-} |(q(t) - q(\varphi)) + q(\varphi)(1 - \varphi'^2)| = \lim_{t \rightarrow b-} |q(t) - q(\varphi)| = \\ & = \lim_{t \rightarrow b-} |q'(\xi)(t - \varphi)| \leq \lim_{t \rightarrow b-} \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi - t) = 0 \end{aligned}$$

where $\xi \in (t, \varphi)$. So

$$(3) \quad \lim_{t \rightarrow b-} |-\frac{1}{2}\varphi'''\varphi' + \frac{3}{4}\varphi''^2| = 0.$$

Suppose $\lim_{t \rightarrow b-} \varphi''^2 = c > 0$. Then $\lim_{t \rightarrow b-} \varphi'(t) = \pm \infty$ but this is in contradiction with the proved part of the lemma. Assume that $\lim_{t \rightarrow b-} \varphi''^2$ does not exist. Let $M = \{t \in [a, b), \varphi'''(t) = 0\}$. Then the set M contains every local maximum of the function φ''^2 and the point $t = b$ is an accumulation point of M . According to (3),

$$\lim_{\substack{t \rightarrow b- \\ t \in M}} \varphi''^2(t) = 0$$

holds and hence we have $\lim_{t \rightarrow b-} \varphi''^2(t) = 0$. But this is in contradiction with our assumption.

Thus $\lim_{t \rightarrow b-} \varphi''(t) = 0$ and the rest of the statement follows from (3).

Lemma 3. Let $(q), (\bar{q})$ be oscillatory $(t \rightarrow b_-)$ differential equations such that $q \in C^0[a, b)$, $\bar{q} \in C^1[a, b)$, $\limsup_{t \rightarrow b-} \bar{q}(t) < 0$ and $\lim_{t \rightarrow b-} (q(t) - \bar{q}(t)) = 0$. Let $\varphi(\bar{\varphi})$ be the dispersion of (q) ((\bar{q})) and let

$$\lim_{t \rightarrow b-} \max_{x \in [t, \varphi(t)]} |\bar{q}'(x)| (\bar{\varphi}(t) - t) = 0$$

where $\bar{\varphi}(t) = \max(\varphi(t), \bar{\varphi}(t))$. Then

$$\lim_{t \rightarrow b-} \varphi'(t) = 1,$$

$$\lim_{t \rightarrow b-} \varphi''(t) = \lim_{t \rightarrow b-} \varphi'''(t) = 0.$$

Proof. By virtue of (1) 6, we have:

$$\begin{aligned} \bar{q}(\bar{\varphi}(t)) (\varphi'(t) - \bar{\varphi}'(t)) &= \bar{q}(\bar{\varphi}(t)) \cdot \left(\frac{q(t_1)}{q(t_2)} - \frac{\bar{q}(t_3)}{\bar{q}(t_4)} \right) = \\ &= \frac{\bar{q}(\bar{\varphi}(t))}{q(t_2) \bar{q}(t_4)} (q(t_1) \bar{q}(t_4) - \bar{q}(t_3) q(t_2)) = \\ &= \frac{\bar{q}(\bar{\varphi}(t))}{q(t_2) \bar{q}(t_4)} [\bar{q}(t_4) (q(t_1) - \bar{q}(t_1)) + \bar{q}(t_3) (\bar{q}(t_2) - q(t_2)) + \\ &\quad + \bar{q}(t_1) (\bar{q}(t_4) - \bar{q}(t_2)) - \bar{q}(t_2) (\bar{q}(t_3) - \bar{q}(t_1))], \end{aligned}$$

where

$$t < t_1 < t_2 < \varphi(t), \quad t < t_3 < t_4 < \bar{\varphi}(t).$$

It follows from the relations

$$\limsup_{t \rightarrow b^-} \bar{q}(t) = c < 0, \quad \lim_{t \rightarrow b^-} (q(t) - \bar{q}(t)) = 0$$

that there exists $t_0 \in [a, b)$ such that

$$|q(t)| \geq \frac{1}{2} |\bar{q}(t)| \geq \frac{|c|}{4}, \quad t \in [t_0, b).$$

Then the following inequalities are valid for $t \in [t_0, b)$ and $t_5 \in [t, \bar{\varphi}(t))$ (by the Taylor Theorem):

$$\begin{aligned} \left| \frac{\bar{q}(\bar{\varphi}(t)) \cdot \bar{q}(t_5)}{q(t_2) \cdot \bar{q}(t_4)} \right| &\leq 2 \left| \frac{\bar{q}(\bar{\varphi}(t)) \cdot \bar{q}(t_5)}{\bar{q}(t_2) \cdot \bar{q}(t_4)} \right| \leq 2 \cdot \frac{|\bar{q}(t_2)| + M_1(t)}{|\bar{q}(t_2)|} \cdot \frac{|\bar{q}(t_4)| + M_1(t)}{|\bar{q}(t_4)|} \leq \\ &\leq 2 \cdot \left(1 + \frac{2 \cdot M_1(t)}{|c|} \right) \left(1 + \frac{2 \cdot M_1(t)}{|c|} \right) \leq 2 \cdot \left(1 + \frac{2 \cdot M_2}{|c|} \right)^2 = M_3 < \infty. \end{aligned}$$

Here

$$M_1(t) = \max_{x \in [t, \bar{\varphi}(t)]} |\bar{q}'(x)| (\bar{\varphi}(t) - t) \xrightarrow{t \rightarrow b^-} 0, \quad M_2 = \max_{t \in [t_0, b)} M_1(t).$$

Hence

$$\begin{aligned} |\bar{q}(\bar{\varphi}(t)) (\varphi'(t) - \bar{\varphi}'(t))| &\leq M_3 \cdot [|q(t_1) - \bar{q}(t_1)| + |\bar{q}(t_2) - q(t_2)| + \\ &\quad + |\bar{q}'(\xi_1)| (\bar{\varphi} - t) + |\bar{q}'(\xi_2)| (\bar{\varphi} - t)] \xrightarrow{t \rightarrow b^-} 0 \end{aligned}$$

for $\xi_1 \in (t_2, t_4)$, $\xi_2 \in (t_1, t_3)$ and thus

$$(4) \quad \lim_{t \rightarrow b^-} |\bar{q}(\bar{\varphi}(t)) (\varphi'(t) - \bar{\varphi}'(t))| = 0.$$

This implies $\lim_{t \rightarrow b-} (\varphi'(t) - \bar{\varphi}'(t)) = 0$ and by virtue of Lemma 2 we can see that $\lim_{t \rightarrow b-} \varphi'(t) = 1$ which proves the first part of the lemma.

The dispersions φ and $\bar{\varphi}$ fulfil the non-linear differential equation (1) 5:

$$\begin{aligned} -\frac{1}{2} \frac{\varphi'''}{\varphi'} + \frac{3}{4} \frac{\varphi''^2}{\varphi'^2} &= -q(\varphi) \varphi'^2 + q(t), \\ -\frac{1}{2} \frac{\bar{\varphi}'''}{\bar{\varphi}'} + \frac{3}{4} \frac{\bar{\varphi}''^2}{\bar{\varphi}'^2} &= -\bar{q}(\bar{\varphi}) \bar{\varphi}'^2 + \bar{q}(t). \end{aligned}$$

Subtracting and modifying these equations we get (by (4) and the proved part of Lemma 3):

$$\begin{aligned} A &= \left| -\frac{1}{2} \left(\frac{\varphi'''}{\varphi'} - \frac{\bar{\varphi}'''}{\bar{\varphi}'} \right) + \frac{3}{4} \left(\frac{\varphi''^2}{\varphi'^2} - \frac{\bar{\varphi}''^2}{\bar{\varphi}'^2} \right) \right| = \\ &= |q(t) - \bar{q}(t) - q(\varphi) \varphi'^2 + \bar{q}(\bar{\varphi}) \bar{\varphi}'^2| = \\ &= |(q(t) - \bar{q}(t)) - \bar{q}(\bar{\varphi}) (\varphi' - \bar{\varphi}') (\varphi' + \bar{\varphi}') - \\ &- \varphi'^2 [(q(\varphi) - \bar{q}(\varphi)) + (\bar{q}(\varphi) - \bar{q}(\bar{\varphi}))]| \leq |q(t) - \bar{q}(t)| + \\ &+ |\bar{q}(\bar{\varphi}) (\varphi' - \bar{\varphi}') (\varphi' + \bar{\varphi}')| + \varphi'^2 |q(\varphi) - \bar{q}(\varphi)| + \\ &+ \varphi'^2 \max_{x \in [t, \bar{\varphi}(t)]} |\bar{q}'(x)| (\bar{\varphi}(t) - t) \xrightarrow{t \rightarrow b-} 0. \end{aligned}$$

Taking into account Lemma 2 (for $q \equiv \bar{q}$ we have $\lim_{t \rightarrow b-} \bar{\varphi}''(t) = \lim_{t \rightarrow b-} \bar{\varphi}'''(t) = 0$) we can see from this that

$$(5) \quad \lim_{t \rightarrow b-} A \varphi'^2 = \lim_{t \rightarrow b-} \left| -\frac{1}{2} \varphi'''' \cdot \varphi' + \frac{3}{4} \varphi''^2 \right| = 0$$

holds. The relation (5) is the same as the relation (3) and therefore we can prove in the same way as in Lemma 1 that

$$\lim_{t \rightarrow b-} \varphi''(t) = \lim_{t \rightarrow b-} \varphi'''(t) = 0.$$

So the statement of the lemma is proved.

Theorem 2. Let $(q), (\bar{q})$ be oscillatory $(t \rightarrow \infty)$ differential equations such that $q \in C^0[a, \infty), \bar{q} \in C^1[a, \infty), \limsup_{t \rightarrow \infty} \bar{q}(t) < 0, \lim_{t \rightarrow \infty} (q(t) - \bar{q}(t)) = 0, \lim_{t \rightarrow \infty} \bar{q}'(t) = 0$.

Let φ be the dispersion of (q) . Then

$$\lim_{t \rightarrow \infty} \varphi(t) = 1, \quad \lim_{t \rightarrow \infty} \varphi''(t) = \lim_{t \rightarrow \infty} \varphi'''(t) = 0.$$