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ON THE TIMOSHENKO TYPE EQUATIONS

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In this paper, we shall investigate the correctness of the following initial value problem:

$$\text{(T)} \qquad u''''(t) + (\alpha A^{1/2} + \beta)\, u''(t) + (aA + bA^{1/2} + c)\, u(t) = 0\,,$$
$$u(0_+) = u'(0_+) = u''(0_+) = 0\,, \quad u'''(0_+) = x\,,$$

where t is positive time, x and $u(t)$ are elements of a Hilbert space H, A is a nonzero, nonnegative, selfadjoint operator in H and α, β, a, b, c are real constants.

PRELIMINARIES

Let R denote the real and C the complex number fields.

We shall write in the sequel: $\xi^{1/2}$ is the nonnegative root of $\xi \geqq 0$ and $\xi^{1/2} = \mathrm{i}|\xi|^{1/2}$ for $\xi < 0$.

Further, let sign $\xi = 1$ for $\xi \geqq 0$ and sign $\xi = -1$ for $\xi < 0$.

Lemma 1. *Let $\varphi \in (0, \infty) \to R$. If*

(1) *the function φ is continuous on $(0, \infty)$ and bounded on $(0, 1)$,*

(2) *there exists a nondecreasing function $\psi \in (0, \infty) \to R$ such that for every $t \in (0, \infty)$,*

$$|\varphi(t)| \leqq \psi(t) \int_0^t |\varphi(\tau)|\, \mathrm{d}\tau\,,$$

then φ is identically zero.

BASIC NOTIONS

We shall denote by H a fixed Hilbert space over C.

We suppose that there are given

(A) a nonzero nonnegative selfadjoint operator A from H into H,
(B) constants $a, b, c, \alpha, \beta \in R$.

All the following definitions and results are related to these fixed data.

Lemma 2. *If P is an orthogonal projector such that AP is bounded and $PA \subseteq AP$, then*

(1) *$AP^{1/2}$ is a nonnegative selfadjoint operator,*
(2) *$A^{1/2}P$ is bounded,*
(3) *$A^{1/2}P$ is a nonnegative selfadjoint operator,*
(4) *$A^{1/2}P \supseteq PA^{1/2}$,*
(5) *$A^{1/2}P = (AP)^{1/2}$.*

Lemma 3. *If $\mathscr{E}$ is the spectral resolution of the operator A, then for every $\alpha \geqq 0$:*

(1) *$\mathscr{E}(\langle 0, \alpha\rangle)$ is an orthogonal projector,*
(2) *$A\mathscr{E}(\langle 0, \alpha\rangle)$ is a bounded operator,*
(3) *$\mathscr{E}(\langle 0, \alpha\rangle)\, A \subseteq A\mathscr{E}(\langle 0, \alpha\rangle)$.*

Lemma 4. *If $\mathscr{E}$ is the spectral resolution of the operator A, then for every $x \in H$,*

$$\mathscr{E}(\langle 0, \alpha\rangle)\, x \underset{\alpha \to \infty}{\to} x\,.$$

A function $u \in (0, \infty) \to H$ will be called a solution of the Timoshenko type equation if

(I) u is four-times continuously differentiable on $(0, \infty)$,
(II) $u^{(4)}$ is bounded on $(0, 1)$,
(III) $u(t) \in D(A)$, $u''(t) \in D(A^{1/2})$ for every $t \in (0, \infty)$,
(IV) $u''''(t) + (\alpha A^{1/2} + \beta)\, u''(t) + (aA + bA^{1/2} + c)\, u(t) = 0$ for every $t \in (0, \infty)$.

Remark 1. More precisely, we should speak about "Timoshenko type equation, generated by the operator A". Nonetheless, since the operator A is supposed to be fixed, we shall omit the latter part of this term.

Remark 2. It is not difficult to establish that the above notion of the Timoshenko type equation includes some operator variants of the equation of transverse vibrations of the beam, discovered in 1916 by S. P. Timoshenko (cf. [1], p. 338):

$$u_{tttt} - \alpha u_{tt\xi\xi} + \beta u_{tt} + a u_{\xi\xi\xi\xi} - b u_{\xi\xi} + cu = 0$$

with appropriate boundary conditions at the end of the beam.

The coefficients a, b, c, α, β are interpreted in terms of certain physical constants of the material of beam.

The analogous equation for transverse vibrations of plates was discovered by UFLJAND (cf. [2], [3]) and its operator variants may be also included in our abstract model.

The same is true for the Mindlin-Hermann equation of longitudinal vibrations of a rod (see [4], p. 488 and 481).

All these equations, which are of order four in the time variable, take into account some mechanical phenomena (e.g. rotary inertia and shear deflection in the case of transverse vibrations) which were neglected in classical models.

Proposition 1. *If u is a solution of the Timoshenko type equation, then $u(0_+)$, $u'(0_+)$, $u''(0_+)$ and $u'''(0_+)$ exist.*

Proposition 2. *If u is a solution of the Timoshenko type equation and P an orthogonal projector such that AP is bounded and $PA \subseteq AP$, then the function Pu is also a solution of the Timoshenko type equation.*

Proof. It is easy to verify the properties (I)–(IV) from the definition of solution.

A solution u of the Timoshenko type equation will be called Duhamel solution of this equation if $u(0_+) = u'(0_+) = u''(0_+) = 0$.

A Duhamel solution of the Timoshenko type equation will be called a null solution of this equation if $u'''(0_+) = 0$.

Proposition 3. *If u is a Duhamel (null) solution of the Timoshenko type equation and P an orthogonal projector such that AP is bounded and $PA \subseteq AP$, then the function Pu is also a Duhamel (null) solution of the Timoshenko type equation.*

Proof. Immediate consequence of Proposition 2.

UNIQUENESS FOR TIMOSHENKO TYPE EQUATION

Lemma 5. *Let $z \in (0, \infty) \to H$. If the function z is continuously differentiable on $(0, \infty)$, z' is bounded on $(0, 1)$ and $z(0_+) = 0$, then for every $t \in (0, \infty)$*

$$\operatorname{Re} \int_0^t \langle z(\tau), z'(\tau) \rangle \, d\tau = \tfrac{1}{2}\|z(t)\|^2 , \tag{1}$$

$$\int_0^t \|z(\tau)\| \, \|z'(\tau)\| \, d\tau \leqq \frac{t}{2} \int_0^t \|z'(\tau)\|^2 \, d\tau . \tag{2}$$

Proof. To prove (1), we write

$$\operatorname{Re} \int_0^t \langle z(\tau), z'(\tau) \rangle \, d\tau = \tfrac{1}{2} \int_0^t \frac{d}{d\tau} \|z(\tau)\|^2 \, d\tau = \tfrac{1}{2}\|z(t)\|^2 .$$

Using Lemma 4, we obtain (2) in the following way

$$\int_0^t \|z(\tau)\| \, \|z'(\tau)\| \, d\tau = \int_0^t \left\| \int_0^\tau z'(\sigma) \, d\sigma \right\| \|z'(\tau)\| \, d\tau \leqq$$

$$\leqq \int_0^t \left(\int_0^\tau \|z'(\sigma)\| \, d\sigma \right) \|z'(\tau)\| \, d\tau =$$

$$= \tfrac{1}{2} \int_0^t \frac{d}{d\tau} \left(\int_0^\tau \|z'(\sigma)\| \, d\sigma \right)^2 d\tau = \tfrac{1}{2} \left[\int_0^t \|z'(\tau)\| \, d\tau \right]^2 \leqq$$

$$\leqq \frac{t}{2} \int_0^t \|z'(\tau)\|^2 \, d\tau .$$

We shall say that the Timoshenko type equation is definite if every null solution of the Timoshenko type equation is identically zero.

Theorem 1. *The Timoshenko type equation is always definite.*

Proof. We shall proceed indirectly.

Hence we suppose that

(1) there exists a null solution u of the Timoshenko type equation and a point $t_0 \in (0, \infty)$ so that $u(t_0) \neq 0$.

Let $\mathscr{E}$ be the spectral resolution of the operator A.

Using (1), we obtain from Lemma 4 that

(2) there exists an $\alpha > 0$ so that $\mathscr{E}(\langle 0, \alpha \rangle) \, u(t_0) \neq 0$.

For the sake of simplicity we shall write

(3) $$P = \mathscr{E}(\langle 0, \alpha \rangle) .$$

Then by (2) and (3)

(4) $$P \, u(t_0) \neq 0 .$$

Let us now denote

(5) $$u_0(t) = P \, u(t) \quad \text{for} \quad t \in (0, \infty) .$$

By (4) and (5), we have

(6) $$u_0(t_0) \neq 0 .$$

Using (3)–(5), we obtain by means of Lemmas 2 and 3 that

(7) $$u_0 \text{ is four times continuously differentiable on } (0, \infty) ,$$

(8) $$u_0'''' \text{ is bounded on } (0, 1) ,$$

$$(9) \qquad u_0''''(t) + (\alpha(AP)^{1/2} + \beta)\, u_0''(t) + (a(AP) + b(AP)^{1/2} + c)\, u_0(t) = 0\,,$$

$$(10) \qquad u_0(0_+) = u_0'(0_+) = u_0''(0_+) = u_0'''(0_+) = 0\,.$$

Multiplying (9) scalarly by u_0''', we obtain

$$(11) \qquad \langle u_0''''(t), u_0'''(t)\rangle + \langle(\alpha(AP)^{1/2} + \beta)\, u_0''(t), u_0'''(t)\rangle + \\ + \langle(a(AP) + b(AP)^{1/2} + c)\, u_0(t), u_0'''(t)\rangle = 0 \quad \text{for every} \quad t \in (0, \infty)\,.$$

In virtue of (3) and of Lemmas 2 and 3 we can define

$$(12) \qquad K = \max\left(\|a(AP) + b(AP)^{1/2} + c\|,\ \|\alpha(AP)^{1/2} + \beta\|\right).$$

Using (11) and (12), we obtain

$$(13) \qquad \left|\int_0^t \langle u_0''''(\tau), u_0'''(\tau)\rangle\, d\tau\right| \leqq K\left[\int_0^t \|u_0''(\tau)\|\ \|u_0'''(\tau)\|\, d\tau + \right. \\ \left. + \int_0^t \|u_0(\tau)\|\ \|u_0'''(\tau)\|\, d\tau\right] \quad \text{for every} \quad t \in (0, \infty)\,.$$

On the other hand, by Lemma 5, we have for every $t \in (0, \infty)$

$$(14) \qquad \operatorname{Re}\int_0^t \langle u_0''''(\tau), u_0'''(\tau)\rangle\, d\tau = \tfrac{1}{2}\|u_0'''(t)\|^2\,,$$

$$(15) \qquad \int_0^t \|u_0''(\tau)\|\ \|u_0'''(\tau)\|\, d\tau \leqq \frac{t}{2}\int_0^t \|u_0'''(\tau)\|^2\, d\tau\,,$$

$$(16) \qquad \int_0^t \|u_0(\tau)\|\ \|u_0'''(\tau)\|\, d\tau = \\ = \int_0^t \left\|\int_0^\tau \int_0^\sigma u_0''(\varrho)\, d\varrho\, d\sigma\right\|\ \|u_0'''(\tau)\|\, d\tau \leqq \\ \leqq t^2 \int_0^t \|u_0''(\tau)\|\ \|u_0'''(\tau)\|\, d\tau \leqq \frac{t^3}{2}\int_0^t \|u_0'''(\tau)\|^2\, d\tau\,.$$

It follows from (13)–(16) that

$$(17) \qquad \|u_0'''(t)\|^2 \leqq K(t + t^3)\int_0^t \|u_0'''(\tau)\|^2\, d\tau\,.$$

By Lemma 1, it follows from (17) that

$$(18) \qquad u_0'''(t) = 0 \quad \text{for every} \quad t \in (0, \infty)\,.$$

But (10) and (18) give immediately

$$(19) \qquad u_0(t) = 0 \quad \text{for every} \quad t \in (0, \infty)\,.$$

The contradiction between (6) and (19) implies that (1) cannot hold and hence the proof is complete.

FOURIER FUNCTION OF TIMOSHENKO TYPE EQUATION

A function $m \in (0, \infty) \times \langle 0, \infty) \to C$ will be called the Fourier function of the Timoshenko type equation if

(I) $\quad m(\cdot, s)$ is four times differentiable on $(0, \infty)$ for every $s \in \langle 0, \infty)$,

(II) $$m_{tttt}(t, s) + (\alpha s^{1/2} + \beta)\, m_{tt}(t, s) + (as + bs^{1/2} + c)\, m(t, s) = 0$$
$$\text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty)\,,$$

(III) $$m(0_+, s) = m_t(0_+, s) = m_{tt}(0_+, s) = 0\,, \quad m_{ttt}(0_+, s) = 1$$
$$\text{for every} \quad s \in \langle 0, \infty)\,.$$

Proposition 4. *The Fourier function of the Timoshenko type equation always exists and is unique.*

Proof follows from the theory of ordinary differential equations.

Let us denote for $s \in \langle 0, \infty)$

$$\mu_{1,2}(s) = -\frac{\alpha s^{1/2} + \beta}{2} \pm \left[\left(\frac{\alpha s^{1/2} + \beta}{2}\right)^2 - (as + bs^{1/2} + c)\right]^{1/2}.$$

Lemma 6. *For every $s \in \langle 0, \infty)$, $\mu_{1,2}(s)$ are all possible roots of the polynomial*

$$\mu^2 + (\alpha s^{1/2} + \beta)\, \mu + (as + bs^{1/2} + c) = 0\,.$$

Lemma 7. *For every $s \in \langle 0, \infty)$*

(1) $$\mu_1(s) + \mu_2(s) = -(\alpha s^{1/2} + \beta)\,,$$

(2) $$\mu_1(s) - \mu_2(s) = [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2}\,.$$

Lemma 8. *For $s \in \langle 0, \infty)$*

$$|\mu_{1,2}(s)| \leqq (1 + |a| + |b| + |c| + |\alpha| + |\beta|)\,(1 + s^{1/2})\,.$$

Proof. We have for $s \in \langle 0, \infty)$

$$|\mu_{1,2}(s)| \leqq \frac{|\alpha|\, s^{1/2} + |\beta|}{2} + \left[\left(\frac{|\alpha|\, s^{1/2} + |\beta|}{2}\right)^2 + |a|\, s + |b|\, s^{1/2} + |c|\right]^{1/2} \leqq$$

$$\leqq \frac{|\alpha|\, s^{1/2} + |\beta|}{2} + \left[\frac{|\alpha|\, s^{1/2} + |\beta|}{2} + (|a|\, s + |b|\, s^{1/2} + |c|)^{1/2}\right] \leqq$$

$$\leqq |\alpha|\, s^{1/2} + |\beta| + (|a|\, s + |b|\, s^{1/2} + |c|)^{1/2} \leqq$$
$$\leqq (|\alpha| + |\beta|)(1 + s^{1/2}) + (|a| + |b| + |c|)^{1/2}(1 + s)^{1/2} \leqq$$
$$\leqq (|\alpha| + |\beta|)(1 + s^{1/2})(1 + |a| + |b| + |c|)(1 + s^{1/2}).$$

Let us denote for $s \in \langle 0, \infty)$

$$\lambda_{1,2}(s) = \frac{1}{2^{1/2}} \{[(\mathrm{Re}\, \mu_{1,2}(s))^2 + (\mathrm{Im}\, \mu_{1,2}(s))^2]^{1/2} + \mathrm{Re}\, \mu_{1,2}(s)\}^{1/2} +$$
$$+ \frac{i \,\mathrm{sign}\,(\mathrm{Im}\, \mu_{1,2}(s))}{2^{1/2}} \{[(\mathrm{Re}\, \mu_{1,2}(s))^2 + (\mathrm{Im}\, \mu_{1,2}(s))^2]^{1/2} - \mathrm{Re}\, \mu_{1,2}(s)\}^{1/2},$$
$$\lambda_{3,4}(s) = -\lambda_{1,2}(s).$$

Lemma 9. *For every $s \in \langle 0, \infty)$, $\lambda_{1,2,3,4}(s)$ are all possible roots of the polynomia*

$$\lambda^4 + (\alpha s^{1/2} + \beta)\, \lambda^2 + (as + bs^{1/2} + c) = 0.$$

Proof. A consequence of Lemma 6 by virtue of the formula

$$a + ib = \left(\pm \left\{\frac{1}{2^{1/2}} [(a^2 + b^2)^{1/2} + a]^{1/2} + \frac{i \,\mathrm{sign}\, b}{2^{1/2}} [(a^2 + b^2)^{1/2} - a]^{1/2}\right\}\right)^2.$$

Lemma 10. *For every $s \in \langle 0, \infty)$,* $\mathrm{Re}\, \lambda_{1,2}(s) \geqq 0$.

Lemma 11. *For every $s \in \langle 0, \infty)$,* $\lambda_{1,2}(s)^2 = \mu_{1,2}(s)$.

Lemma 12. *For every $s \in \langle 0, \infty)$,*

$$|\lambda_{1,2}(s)| \leqq (1 + |a| + |b| + |c| + |\alpha| + |\beta|)(1 + s^{1/4}).$$

Proof. From Lemma 11, we have $|\lambda_{1,2}(s)| = |\mu_{1,2}(s)|^{1/2}$ and it suffices to apply Lemma 8.

Lemma 13. *If m is the Fourier function of the Timoshenko type equation, then for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$*

(1) $$m(t, s) = \int_0^t \left(\int_0^{t-\tau} \cosh(\lambda_1(s)\, \sigma)\, \mathrm{d}\sigma \int_0^\tau \cosh(\lambda_2(s)\, \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}\tau,$$

(2) $$m_t(t, s) = \int_0^t \cosh(\lambda_1(s)(t - \tau)) \left(\int_0^\tau \cosh(\lambda_2(s)\, \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}\tau =$$
$$= \int_0^t \left(\int_0^{t-\tau} \cosh(\lambda_2(s)\, \sigma)\, \mathrm{d}\sigma\right) \cosh(\lambda_1(s)\, \tau)\, \mathrm{d}\tau,$$

(3) $$m_{tt}(t, s) = \int_0^t \cosh(\lambda_2(s)(t - \tau)) \cosh(\lambda_1(s)\, \tau)\, \mathrm{d}\tau,$$

(4) $$m_{ttt}(t, s) = \lambda_1(s) \int_0^t \sinh(\lambda_1(s)(t - \tau)) \cosh(\lambda_2(s)\tau) \, d\tau + \cosh(\lambda_2(s) t),$$

(5) $$m_{tttt}(t, s) = \lambda_1(s)^2 \int_0^t \cosh(\lambda_1(s)(t - \tau)) \cosh(\lambda_2(s)\tau) \, d\tau + \\ + \lambda_2(s) \sinh(\lambda_2(s) t),$$

(6) $$m_{tt}(t, s) - 1 = \lambda_1(s) \int_0^t \sinh(\lambda_1(s)(t - \tau)) \cosh(\lambda_2(s)\tau) \, d\tau + \\ + \lambda_2(s) \int_0^t \sinh(\lambda_2(s)\tau) \, d\tau.$$

Proof. Let us denote for $t \in (0, \infty)$ and $s \in \langle 0, \infty)$

(7) $$n(t, s) = \int_0^t \left(\int_0^{t-\tau} \cosh(\lambda_1(s)\sigma) \, d\sigma \int_0^\tau \cosh(\lambda_2(s)\sigma) \, d\sigma \right) d\tau.$$

We obtain easily that for $t \in (0, \infty)$ and $s \in \langle 0, \infty)$

(8) $$n_t(t, s) = \int_0^t \left[\cosh(\lambda_1(s)(t - \tau)) \int_0^\tau \cosh(\lambda_2(s)\sigma) \, d\sigma \right] d\tau,$$

(9) $$n_{tt}(t, s) = \lambda_1(s) \int_0^t \left[\sinh(\lambda_1(s)(t - \tau)) \int_0^{t-\tau} \cosh(\lambda_2(s)\sigma) \, d\sigma \right] d\tau + \\ + \int_0^t \cosh(\lambda_2(s)\tau) \, d\tau =$$

$$= \lambda_1(s)^2 \int_0^t \left(\int_0^{t-\tau} \cosh(\lambda_1(s)\sigma) \, d\sigma \int_0^\tau \cosh(\lambda_2(s)\sigma) \, d\sigma \right) d\tau + \int_0^t \cosh(\lambda_2(s)\tau) \, d\tau =$$

$$= \lambda_1(s)^2 \, n(t, s) + \int_0^t \cosh(\lambda_2(s)\tau) \, d\tau,$$

(10) $$n_{ttt}(t, s) = \lambda_1(s)^2 \, n_t(t, s) + \cosh(\lambda_2(s) t),$$

(11) $$n_{tttt}(t, s) = \lambda_1(s)^2 \, n_{tt}(t, s) + \lambda_2(s) \sinh(\lambda_2(s) t) =$$

$$= \lambda_1(s)^4 \, n(t, s) + \lambda_1(s)^2 \int_0^t \cosh(\lambda_2(s)\tau) \, d\tau + \lambda_2(s) \sinh(\lambda_2(s) t) =$$

$$= \lambda_1(s)^4 \, n(t, s) + (\lambda_1(s)^2 + \lambda_2(s)^2) \int_0^t \cosh(\lambda_2(s)\tau) \, d\tau.$$

It follows immediately from (1)–(5) that

(12) $n(\cdot, s)$ is four times differentiable on $(0, \infty)$ for every $s \in \langle 0, \infty)$.

By Lemma 9 we have

(13) $$\lambda_1(s)^4 + (\alpha s^{1/2} + \beta)\,\lambda_1(s)^2 + (as + bs^{1/2} + c) = 0 \text{ for every } s \in \langle 0, \infty) .$$

By Lemmas 7 and 11 we have

(14) $$\lambda_1(s)^2 + \lambda_2(s)^2 + \alpha s^{1/2} + \beta = 0 \quad \text{for every} \quad s \in \langle 0, \infty) .$$

Now it follows from (7), (9), (11), (13) and (14) that

(15) $$\begin{aligned} n_{tttt}(t, s) + (\alpha s^{1/2} + \beta)\, n_{tt}(t, s) + (as + bs^{1/2} + c)\, n(t, s) = \\ = [\lambda_1(s)^4 + (\alpha s^{1/2} + \beta)\,\lambda_1(a)^2 + (as + bs^{1/2} + c)]\, n(t, s) + \\ + [\lambda_1(s)^2 + \lambda_2(s)^2 + \alpha s^{1/2} + \beta] \int_0^t \cosh(\lambda_2(s)\,\tau)\,\mathrm{d}\tau = 0 \end{aligned}$$
$$\text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty) .$$

Finally, we see from (7)–(10) that

(16) $$\begin{gathered} n(0_+, s) = n_t(0_+, s) = n_{tt}(0_+, s) = 0 , \\ n_{ttt}(0_+, s) = 1 \quad \text{for every} \quad s \in \langle 0, \infty) . \end{gathered}$$

We obtain easily from (12), (13) and (15), (16) by means of Proposition 4 that $m = n$ and then the statement of our Lemma follows from (7)–(11).

Proposition 5. *If m is the Fourier function of the Timoshenko type equation, then the functions m, m_t, m_{tt}, m_{ttt} and m_{tttt} are continuous and bounded on $(0, T) \times \langle 0, S\rangle$ for every $T \in (0, \infty)$ and $S \in \langle 0, \infty)$.*

Proof. Immediately from Lemma 13 in view of the definition of functions $\lambda_{1,2,3,4}$.

Proposition 6. *Let m be the Fourier function of the Timoshenko type equation, $s \in \langle 0, \infty)$ and let $\varkappa$ be a nonnegative constant. If $\operatorname{Re} \lambda_{1,2}(s) \leqq \varkappa$, then for every $t \in (0, \infty)$*

(1) $$|m(t, s)| \leqq \frac{t^3}{6}\, e^{\varkappa t} ,$$

(2) $$|m_t(t, s)| \leqq \frac{t^2}{2}\, e^{\varkappa t} ,$$

(3) $$|m_{tt}(t, s)| \leqq t e^{\varkappa t} ,$$

(4) $$|m_{ttt}(t, s)| \leqq (1 + |a| + |b| + |c| + |\alpha| + |\beta|)\,(1 + s^{1/4})\, t e^{\varkappa t} + e^{\varkappa t} ,$$

(5) $$\begin{aligned} |m_{tttt}(t, s)| \leqq (1 + |a| + |b| + |c| + |\alpha| + |\beta|)^2\,(1 + s^{1/4})^2\, t e^{\varkappa t} + \\ + (1 + |a| + |b| + |c| + |\alpha| + |\beta|)\,(1 + s^{1/4})\, e^{\varkappa t} , \end{aligned}$$

(6) $$|m_{ttt}(t, s) - 1| \leqq 2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)(1 + s^{1/4})\, te^{\varkappa t}\,.$$

Proof. By means of Lemmas 11–13.

Lemma 14. *If m is the Fourier function of the Timoshenko type equation, then for every* $t \in (0, \infty)$ *and* $s \in \langle 0, \infty)$

$$(\lambda_1(s)^2 - \lambda_2(s)^2)\, m(t, s) =$$
$$= \int_0^t \cosh(\lambda_1(s)\,\tau)\,d\tau - \int_0^t \cosh(\lambda_2(s)\,\tau)\,d\tau\,.$$

Proof. Let us denote for $t \in (0, \infty)$ and $s \in \langle 0, \infty)$

(1) $$n(t, s) = \int_0^t \cosh(\lambda_1(s)\,\tau)\,d\tau - \int_0^t \cosh(\lambda_2(s)\,\tau)\,d\tau\,.$$

We obtain easily that for $t \in (0, \infty)$ and $s \in \langle 0, \infty)$

(2) $$n_t(t, s) = \cosh(\lambda_1(s)\,t) - \cosh(\lambda_2(s)\,t)\,,$$

(3) $$n_{tt}(t, s) = \lambda_1(s)\sinh(\lambda_1(s)\,t) - \lambda_2(s)\sinh(\lambda_2(s)\,t) =$$
$$= \lambda_1(s)^2 \int_0^t \cosh(\lambda_1(s)\,\tau)\,d\tau - \lambda_2(s)^2 \int_0^t \cosh(\lambda_2(s)\,\tau)\,d\tau\,,$$

(4) $$n_{ttt}(t, s) = \lambda_1(s)^2\cosh(\lambda_1(s)\,t) - \lambda_2(s)^2\cosh(\lambda_2(s)\,t)\,,$$

(5) $$n_{tttt}(t, s) = \lambda_1(s)^3\sinh(\lambda_1(s)\,t) - \lambda_2(s)^3\sinh(\lambda_2(s)\,t) =$$
$$= \lambda_1(s)^4 \int_0^t \cosh(\lambda_1(s)\,\tau)\,d\tau - \lambda_2(s)^4 \int_0^t \cosh(\lambda_2(s)\,\tau)\,d\tau\,.$$

It follows immediately from (1)–(5) that

(6) $n(\cdot, s)$ is four times differentiable on $(0, \infty)$ for every $s \in \langle 0, \infty)$.

By Lemma 9 we have

(7) $$\lambda_{12}(s)^4 + (\alpha s^{1/2} + \beta)\,\lambda_{12}(s)^2 + (as + bs^{1/2} + c) = 0$$
for every $s \in \langle 0, \infty)$.

Now it follows from (1), (3), (5) and (7) that

(8) $$n_{tttt}(t, s) + (\alpha s^{1/2} + \beta)\, n_{tt}(t, s) + (as + bs^{1/2} + c)\, n(t, s) =$$
$$= [(\lambda_1(s)^4 + (\alpha s^{1/2} + \beta)\,\lambda_1(s)^2 + (as + bs^{1/2} + c)] \int_0^t \cosh\lambda_1(s)\,\tau\,d\tau +$$

$$+ [\lambda_2(s)^4 + (\alpha s^{1/2} + \beta)\,\lambda_2(s)^2 + (as + bs^{1/2} + c)] \int_0^t \cosh \lambda_2(s)\,\tau\, d\tau = 0$$

for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$.

Finally, we see from (1)–(4) that

$$(9) \qquad n(0_+, s) = n_t(0_+, s) = n_{tt}(0_+, s) = 0\,, \qquad n_{ttt}(0_+, s) = \lambda_1(s)^2 - \lambda_2(s)^2 \quad \text{for every} \quad s \in \langle 0, \infty)\,.$$

We obtain easily from (6), (7) and (8), (9) that $n(t, s) = (\lambda_1(s)^2 - \lambda_2(s)^2)\, m(t, s)$ for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$ and the statement of our Lemma follows from (1).

Proposition 7. *Let m be the Fourier function of the Timoshenko type equation, $s \in (0, \infty)$, $\varkappa$ a nonnegative and q a positive constant. If*

$$\operatorname{Re} \lambda_{12}(s) \leqq \varkappa\,, \quad |\lambda_1(s)^2 - \lambda_2(s)^2| \geqq q s^{1/2}\,,$$

then for every $t \in (0, \infty)$

$$(1) \quad |m(t, s)| \leqq \frac{2}{q} \frac{1}{s^{1/2}}\, t e^{\varkappa t}\,,$$

$$(2) \quad |m_t(t, s)| \leqq 2 e^{\varkappa t}\,,$$

$$(3) \quad |m_{tt}(t, s)| \leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)}{q} \frac{1}{s^{1/4}} \left(1 + \frac{1}{s^{1/4}}\right) e^{\varkappa t}\,,$$

$$(4) \quad |m_{ttt}(t, s)| \leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^2}{q} \left(1 + \frac{1}{s^{1/4}}\right)^2 e^{\varkappa t}\,,$$

$$(5) \quad |m_{tttt}(t, s)| \leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^3}{q}\, s^{1/4} \left(1 + \frac{1}{s^{1/4}}\right)^3 e^{\varkappa t}\,,$$

$$(6) \quad |m_{ttt}(t, s) - 1| \leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^3}{q}\, s^{1/4} \left(1 + \frac{1}{s^{1/4}}\right)^3 t e^{\varkappa t}\,.$$

Proof. By Lemma 14, we obtain for $t \in (0, \infty)$

$$(7) \quad m(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} \int_0^t \cosh \lambda_1(s)\,\tau\, d\tau - \int_0^t \cosh \lambda_2(s)\,\tau\, d\tau\,,$$

$$(8) \quad m_t(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} \cosh \lambda_1(s)\, t - \cosh \lambda_2(s)\, t\,,$$

$$(9) \quad m_{tt}(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s) \sinh \lambda_1(s)\, t - \lambda_2(s) \sinh \lambda_2(s)\, t]\,,$$

$$(10) \quad m_{ttt}(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^2 \cosh(\lambda_1(s)\, t) - \lambda_2(s)^2 \cosh(\lambda_2(s)\, t)]\,,$$

$$(11) \quad m_{tttt}(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^3 \sinh(\lambda_1(s)\, t) - \lambda_2(s)^3 \sinh(\lambda_2(s)\, t)]\,,$$

$$(12)\quad m_{ttt}(t, s) - 1 = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^2 \cosh(\lambda_1(s)\, t) - \lambda_1(s)^2 - \lambda_2(s)^2 \cosh(\lambda_2(s)\, t) + \lambda_2(s)^2] =$$

$$= [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} \left[\lambda_1(s)^3 \int_0^t \sinh(\lambda_1(s)\,\tau)\, d\tau - \lambda_2(s)^3 \int_0^t \sinh(\lambda_2(s)\,\tau)\, d\tau\right].$$

By means of Lemmas 10 and 12, we obtain from the identities (7)–(12) that for every $t \in (0, \infty)$

$$(13)\quad |m(t, s)| \leqq \frac{2}{\lambda_1(s)^2 - \lambda_2(s)^2}\, te^{\varkappa t} \leqq \frac{2}{q} \frac{1}{s^{1/2}}\, te^{\varkappa t},$$

$$(14)\quad |m_t(t, s)| \leqq 2e^{\varkappa t},$$

$$(15)\quad |m_{tt}(t,s)| \leqq 2\, \frac{|\lambda_1(s)| + |\lambda_2(s)|}{|\lambda_1(s)^2 - \lambda_2(s)^2|}\, e^{\varkappa t} \leqq$$

$$\leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)}{q} \frac{1 + s^{1/4}}{s^{1/2}}\, e^{\varkappa t} =$$

$$= \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)}{q} \frac{1}{s^{1/4}} \left(1 + \frac{1}{s^{1/4}}\right) e^{\varkappa t},$$

$$(16)\quad |m_{tttt}(t, s)| \leqq \frac{|\lambda_1(s)|^2 + |\lambda_2(s)|^2}{|\lambda_1(s)^2 - \lambda_2(s)^2|}\, e^{\varkappa t} \leqq$$

$$\leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^2 (1 + s^{1/4})^2}{qs^{1/2}}\, e^{\varkappa t} =$$

$$= \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^2}{q} \left(1 + \frac{1}{s^{1/4}}\right)^2 e^{\varkappa t},$$

$$(17)\quad |m_{tttt}(t, s)| \leqq \frac{|\lambda_1(s)|^3 + |\lambda_2(s)|^3}{|\lambda_1(s)^2 - \lambda_2(s)^2|}\, e^{\varkappa t} \leqq$$

$$\leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^3 (1 + s^{1/4})^3}{qs^{1/2}}\, e^{\varkappa t} =$$

$$= \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^3}{q}\, s^{1/4} \left(1 + \frac{1}{s^{1/4}}\right)^3 e^{\varkappa t},$$

$$(18)\quad |m_{ttt}(t, s) - 1| \leqq \frac{|\lambda_1(s)|^3 + |\lambda_2(s)|^3}{|\lambda_1(s)^2 - \lambda_2(s)^2|}\, te^{\varkappa t} \leqq$$

$$\leqq \frac{2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)^3}{q}\, s^{1/4} \left(1 + \frac{1}{s^{1/4}}\right)^3 te^{\varkappa t}.$$

But (13)–(18) are in fact the desired estimates (1)–(6).

Proposition 8. *Let m be the Fourier function of the Timoshenko type equation and $\mathscr{E}$ the spectral resolution of the operator A. For every $x \in H$ and $K \in \langle 0, \infty)$, the function $u \in R^+ \to H$ defined for $t \in (0, \infty)$ by*

$$(1) \qquad u(t) = \int_0^K m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, x$$

is a Duhamel solution of the Timoshenko type equation such that

$$(2) \qquad u'''(0_+) = \mathscr{E}(\langle 0, K \rangle)\, x\,.$$

Proof. An immediate consequence of Proposition 5 by means of operational calculus.

Proposition 9. *Let m be the Fourier function of the Timoshenko type equation and $\mathscr{E}$ the spectral resolution of the operator A. If u is a Duhamel solution of the Timoshenko type equation, then*

$$u^{(j)}(t) = \lim_{K \to \infty} \int_0^K m_t^{(j)}(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, u'''(0_+)$$

for every $t \in (0, \infty)$ and $j \in \{0, 1, 2, 3, 4\}$.

Proof. Let u be an arbitrary Duhamel solution of the Timoshenko type equation. We define

$$(1) \qquad u_K(t) = \mathscr{E}(0, K)\, u(t) \quad \text{for every} \quad t \in (0, \infty) \quad \text{and} \quad K \geqq 0\,.$$

It follows from Lemma 3 and Proposition 3 that

(2) for every $K \geqq 0$, u_K is a Duhamel solution of the Timoshenko type equation such that $u_K'''(0_+) = \mathscr{E}(\langle 0, K \rangle)\, u'''(0_+)$.

Further, by Lemma 4

$$(3) \qquad u_K^{(j)}(t) \xrightarrow[K \to \infty]{} u^{(j)}(t) \quad \text{for every} \quad t \in (0, \infty) \quad \text{and} \quad j \in \{0, 1, 2, 3, 4\}\,.$$

In virtue of Proposition 5 we can define

$$(4) \qquad v_K(t) = \int_0^K m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, u'''(0_+) \quad \text{for every} \quad t \in (0, \infty) \quad \text{and} \quad K \geqq 0\,.$$

By Proposition 8

(5) for every $K \geqq 0$, v_K is a Duhamel solution of the Timoshenko type equation such that $v_K'''(0_+) = \mathscr{E}(\langle 0, K \rangle)\, u'''(0_+)$.

By Theorem 1, we obtain from (2) and (5)

$$(6) \qquad u_K = v_K \quad \text{for every} \quad K \geqq 0\,.$$

Our statement follows from (3), (4) and (6).

We shall say that the Timoshenko type equation is extensive if there exists a dense subset $Z \subseteq H$ such that for every $x \in Z$ there exists a Duhamel solution u of the Timoshenko type equation for which $u'''(0_+) = x$.

Theorem 2. *The Timoshenko type equation is always extensive.*

Proof. Let us denote by $\mathscr{E}$ the spectral resolution of the operator A.

Now we have $Z_k = \mathscr{E}(\langle 0, k\rangle)(H)$ for every $k \in \{1, 2, \dots\}$ and $Z = \bigcup_{k=1}^{\infty} Z_k$.

It is easy to see from the properties of the function $\mathscr{E}$ that

(1) Z is a dense subset of H.

Let now $x \in Z$. Then there exists a $k \in \{1, 2, \dots\}$ so that $x \in Z_k$. Now let us define for $t \in R^+$

$$u(t) = \int_0^k m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, x .$$

Since $\mathscr{E}(\langle 0, k\rangle)\, x = x$, it follows from Proposition 8 that

(2) u is a Duhamel solution of the Timoshenko type equation such that $u'''(0_+) = x$.

Our Theorem follows from (1) and (2).

We shall say that the Timoshenko type equation is exponential if there exist two nonnegative constants M, ω such that for every Duhamel solution u of the Timoshenko type equation, there is for every $t \in (0, \infty)$

$$\|u(t)\| \leqq M e^{\omega t} \|u'''(0_+)\| .$$

Theorem 3. *If the operator A is bounded, then the Timoshenko type equation is always exponential.*

Proof. Immediate consequence of Proposition 6.

Lemma 15. *If m is the Fourier function of the Timoshenko type equation, then for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$*

$$m(t, s) + (\alpha s^{1/2} + \beta) \int_0^t \int_0^\tau m(\sigma, s)\, d\sigma\, \mathrm{d}\tau +$$

$$+ (as + bs^{1/2} + c) \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau = \frac{t^3}{6} .$$

Proof follows immediately from the properties of the Fourier function by integrating four times the determining equation.

Lemma 16. *Let m be the Fourier function of the Timoshenko type equation, $s \in \langle 0, \infty)$ and let ω be a nonnegative constant. If the function $e^{-\omega t} m(t, s)$ is bounded on $(0, \infty)$, then* $\operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \omega$.

Proof. We have by our hypothesis:

(1) there exists a constant c such that for every $t \in (0, \infty)$ and $\lambda \in C$

$$|e^{-\lambda t} m(t, s)| \leqq c e^{(\omega - \operatorname{Re} \lambda) t}.$$

Using (1), we obtain by integrating by parts in the equation of Lemma 15

$$\lambda^4 \int_0^\infty e^{-\lambda \tau} m(\tau, s)\, d\tau + (\alpha s^{1/2} + \beta)\, \lambda^2 \int_0^\infty e^{-\lambda \tau} m(\tau, s)\, d\tau + \tag{2}$$

$$+ (a s + b s^{1/2} + c) \int_0^\infty e^{-\lambda \tau} m(\tau, s)\, d\tau = 1$$

for every $\operatorname{Re} \lambda > \omega$.

We can rewrite (2) as follows

$$[\lambda^4 + (\alpha s^{1/2} + \beta)\, \lambda^2 + (a s + b s^{1/2} + c)] \int_0^\infty e^{-\lambda \tau} m(\tau, s)\, d\tau = 1 \tag{3}$$

for every $\operatorname{Re} \lambda > \omega$.

Now the statement of our Lemma follows immediately from (3) by means of Lemma 9.

Proposition 10. *Let m be the Fourier function of the Timoshenko type equation. If there exist an unbounded subset $S \subseteq \langle 0, \infty)$ and a nonnegative constant ω so that the function $e^{-\omega t} m(t, s)$ is bounded for every $s \in S$, then $a \geqq 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$.*

Proof. Let us denote

$$q_{1,2} = -\frac{\alpha}{2} \pm \frac{(\alpha^2 - 4a)^{1/2}}{2}. \tag{1}$$

We see from the definition of $\mu_{1,2}$ that

$$\frac{\mu_{1,2}(s)}{s^{1/2}} \to q_{1,2} \quad (s \to \infty). \tag{2}$$

On the other hand, from the formulas for $\lambda_{1,2}$ we have

$$\text{(3)} \qquad \frac{\operatorname{Re} \lambda_{1,2}(s)}{s^{1/4}} = \frac{1}{2^{1/2}} \left\{ \left[\left(\frac{\operatorname{Re} \mu_{1,2}(s)}{s^{1/2}} \right)^2 + \left(\frac{\operatorname{Im} \mu_{1,2}(s)}{s^{1/2}} \right)^2 \right]^{1/2} + \frac{\operatorname{Re} \mu_{1,2}(s)}{s^{1/2}} \right\}^{1/2}$$

for every $s \in (0, \infty)$.

Consequently, (2) and (3) imply

$$\text{(4)} \qquad \frac{\operatorname{Re} \lambda_{1,2}(s)}{s^{1/4}} \to \frac{1}{2^{1/2}} \{[(\operatorname{Re} q_{1,2})^2 + (\operatorname{Im} q_{1,2})^2]^{1/2} + \operatorname{Re} q_{1,2}\}^{1/2} \, (s \to \infty) .$$

On the other hand, we have by Lemma 16

$$\text{(5)} \qquad \operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \omega \quad \text{for every} \quad s \in S .$$

It follows from (4) and (5) that

$$\text{(6)} \qquad [(\operatorname{Re} q_{1,2})^2 + (\operatorname{Im} q_{1,2})^2]^{1/2} + \operatorname{Re} q_{1,2} \leqq 0 .$$

Now it is clear from (6) that

$$\text{(7)} \qquad \operatorname{Im} q_{1,2} = 0 .$$

But (1) and (7) imply

$$\text{(8)} \qquad \alpha^2 \geqq 4a .$$

On the other hand, (6) and (7) give $|\operatorname{Re} q_{1,2}| + \operatorname{Re} q_{1,2} \leqq 0$ which is possible if and only if

$$\text{(9)} \qquad \operatorname{Re} q_{1,2} \leqq 0 .$$

By (1) and (8), $\operatorname{Re} q_{1,2} = q_{1,2}$ and hence (9) implies

$$\text{(10)} \qquad q_{1,2} \leqq 0 .$$

But from (1) and (10) we obtain

$$\text{(11)} \qquad \frac{(\alpha^2 - 4a)^{1/2}}{2} \leqq \frac{\alpha}{2} .$$

It is immediate from (8) and (11) that

$$\text{(12)} \qquad \alpha \geqq 0 .$$

However, (8) and (11) give $\frac{1}{4}(\alpha^2 - 4a) \leqq \frac{1}{4}\alpha^2$ which implies

$$\text{(13)} \qquad a \geqq 0 .$$

The Proposition is proved by (8), (12) and (13).

Lemma 17. *If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$, then there exists a nonnegative constant $\varkappa$ such that* $\operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \varkappa$ *for every* $s \in \langle 0, \infty)$.

Proof. Since by hypothesis $a > 0$, $\alpha > 0$, find an $\bar{s} \in \langle 0, \infty)$ so that

$$\alpha s^{1/2} + \beta > 0 \quad \text{for} \quad s \geqq \bar{s}\,, \tag{1}$$

$$as + bs^{1/2} + c \geqq 0 \quad \text{for} \quad s \geqq \bar{s}\,. \tag{2}$$

Let us now suppose that one of the following conditions holds:

$$\alpha^2 > 4a\,, \tag{3}$$

$$\alpha^2 = 4a\,, \quad \alpha\beta > 2b\,, \tag{4}$$

$$\alpha^2 = 4a\,, \quad \alpha\beta = 2b\,, \quad \beta^2 \geqq 4c\,. \tag{5}$$

It follows easily from (3)–(5) that there exists an $s_0 \geqq \bar{s}$ so that

$$(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + \beta^2 - 4c \geqq 0 \quad \text{for} \quad s \geqq s_0\,. \tag{6}$$

Now we obtain from (2) and (6) that

$$0 \leqq \left(\frac{\alpha s^{1/2} + \beta}{2}\right)^2 - \left(as + bs^{1/2} + c\right) \leqq \left(\frac{\alpha s^{1/2} + \beta}{2}\right)^2 \tag{7}$$

for every $s \geqq s_0$.

Using (1) and (7) we have

$$\mu_{1,2}(s) \leqq 0 \quad \text{for} \quad s \geqq s_0\,. \tag{8}$$

But (8) gives immediately

$$\operatorname{Re} \lambda_{1,2,3,4}(s) = 0 \quad \text{for} \quad s \geqq s_0\,. \tag{9}$$

Now by (3), (4), (5) and (9)

(10) if one of the conditions (3), (4), (5) is fulfilled, then there exists an $s_0 \in \langle 0, \infty)$ so that $\operatorname{Re} \lambda_{1,2,3,4}(s) = 0$ for $s \geqq s_0$.

Further let

$$\alpha^2 = 4a\,, \quad \alpha\beta = 2b\,, \quad \beta^2 < 4c\,. \tag{11}$$

In this case obviously

$$\left(\frac{\alpha s^{1/2} + \beta}{2}\right)^2 - \left(as + bs^{1/2} + c\right) = \frac{\beta^2 - 4c}{4} < 0 \quad \text{for} \quad s \geqq 0\,. \tag{12}$$

We obtain easily from (12) that

$$\text{(13)}\quad \operatorname{Re}\mu_{1,2}(s) = -\frac{\alpha s^{1/2}+\beta}{2}, \quad \operatorname{Im}\mu_{1,2}(s) = \pm(4c-\beta)^{1/2} \quad \text{for} \quad s \geqq 0\,.$$

Using (1), (2), (13) we obtain that for $s \geqq \bar{s}$,

$$\text{(14)}\quad \operatorname{Re}\lambda_{1,2}(s) = \frac{1}{2^{1/2}}\left\{\left[\left(\frac{\alpha s^{1/2}+\beta}{2}\right)^2 + (4c-\beta^2)\right]^{1/2} - \frac{\alpha s^{1/2}+\beta}{2}\right\}^{1/2} =$$

$$= \frac{1}{2^{1/2}}\left\{\frac{\left(\frac{\alpha s^{1/2}+\beta}{2}\right)^2 + (4c-\beta^2) - \left(\frac{\alpha s^{1/2}+\beta}{2}\right)^2}{\left[\left(\frac{\alpha s^{1/2}+\beta}{2}\right)^2 + (4c-\beta^2)\right]^{1/2} + \frac{\alpha s^{1/2}+\beta}{2}}\right\}^{1/2} =$$

$$= \frac{1}{2^{1/2}}\left\{\frac{4c-\beta^2}{\left[\left(\frac{\alpha s^{1/2}+\beta}{2}\right)^2 + (4c-\beta^2)\right]^{1/2} + \frac{\alpha s^{1/2}+\beta}{2}}\right\}^{1/2}.$$

It follows easily from (11) and (14) that

$$\text{(15)}\qquad 0 \leqq \operatorname{Re}\lambda_{1,2}(s) \leqq \frac{(4c-\beta^2)^{1/4}}{2^{1/2}} \quad \text{for} \quad s \geqq \bar{s}\,.$$

Further we obtain from (15)

$$\text{(16)}\qquad \operatorname{Re}\lambda_{1,2,3,4}(s) \leqq \frac{(4c-\beta^2)^{1/2}}{2^{1/2}} \quad \text{for} \quad s \geqq \bar{s}\,.$$

Using (11) and (16) we can state that

(17) if the condition (11) si fulfilled, then there exists an $s_0 \in \langle 0, \infty)$ so that

$$\operatorname{Re}\lambda_{1,2,3,4}(s) \leqq \frac{(4c-\beta^2)^{1/4}}{2^{1/2}} \quad \text{for} \quad s \geqq s_0\,.$$

Finally, let

$$\text{(18)}\qquad \alpha^2 = 4a\,, \quad \alpha\beta < 2b\,.$$

In the case (18) we can choose an $s_{00} \geqq \bar{s}$ so that

$$\text{(19)}\qquad 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c) < 0 \quad \text{for} \quad s \geqq s_{00}\,.$$

It follows from (19) that

$$\text{(20)} \qquad \operatorname{Re}\mu_{1,2}(s) = -\frac{\alpha s^{1/2}+\beta}{2}, \quad \operatorname{Im}\mu_{1,2}(s) = \\ = \pm\frac{[2(2b-\alpha\beta)\,s^{1/2}+(4c-\beta^2)]^{1/2}}{2} \quad \text{for} \quad s \geqq s_{00}\,.$$

We obtain from (20)

$$\text{(21)} \quad \operatorname{Re}\lambda_{1,2}(s) = \frac{1}{2^{1/2}}\left\{\left[\left(\frac{\alpha s^{1/2}+\beta}{2}\right)^2 + \frac{2(2b-\alpha\beta)\,s^{1/2}+(4c-\beta^2)}{4}\right]^{1/2} - \right. \\ \left. - \frac{\alpha s^{1/2}+\beta}{2}\right\}^{1/2} = \\ = \frac{1}{2^{1/2}}\left\{\frac{(\alpha^2 s+4bs^{1/2}+4c)^{1/2}}{2} - \frac{\alpha s^{1/2}+\beta}{2}\right\}^{1/2} = \\ = \tfrac{1}{2}\{(\alpha^2 s+4bs^{1/2}+4c)^{1/2} - (\alpha s^{1/2}+\beta)\}^{1/2} \quad \text{for} \quad s \geqq s_{00}\,.$$

We have on the one hand

$$\text{(22)} \qquad \alpha s^{1/2}+\beta > 0 \quad \text{for} \quad s \geqq s_{00}\,.$$

On the other hand, writing

$$\alpha^2 s + 4bs^{1/2} + 4c = \alpha^2 s + 2\alpha\beta s^{1/2} + \beta^2 + [2(2b-\alpha\beta)\,s^{1/2} + (4c-\beta^2)] = \\ = (\alpha s^{1/2}+\beta)^2 - [2(\alpha\beta-2b)\,s^{1/2} + (\beta^2-4c)]$$

we obtain from (19) that

$$\text{(23)} \qquad \alpha^2 s + 4bs^{1/2} + 4c > 0 \quad \text{for} \quad s \geqq s_{00}\,.$$

Using (21), (22) and (23) we obtain

$$\text{(24)} \qquad \operatorname{Re}\lambda_{1,2}(s) = \frac{1}{2}\left\{\frac{2(2b-\alpha\beta)\,s^{1/2}+(4c-\beta^2)}{(\alpha^2 s+4bs^{1/2}+4c)^{1/2}+\alpha s^{1/2}+\beta}\right\}^{1/2} \\ \text{for} \quad s \geqq s_{00}\,.$$

It follows from (24) that

$$\text{(25)} \qquad \operatorname{Re}\lambda_{1,2}(s) \to \frac{1}{2}\left(\frac{(2b-\alpha\beta)}{\alpha}\right)^{1/2} \quad (s\to\infty)\,.$$

It is clear from (18) and (25) that there exists an $s_0 \in \langle 0, \infty)$ so that

$$\text{(26)} \qquad 0 \leqq \operatorname{Re}\lambda_{1,2}(s) \leqq \left(\frac{(2b-\alpha\beta)}{\alpha}\right)^{1/2} \quad \text{for} \quad s \geqq s_0\,.$$

With regard to (26) we can write

$$(27) \qquad \operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \left(\frac{(2b - \alpha\beta)}{2}\right)^{1/2} \quad \text{for} \quad s \geqq s_0 .$$

It follows from (18) and (27) that

(28) if the condition (18) is fulfilled, then there exists an $s_0 \in \langle 0, \infty)$ so that

$$\operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \left(\frac{(2b - \alpha\beta)}{\alpha}\right)^{1/2} \quad \text{for} \quad s \geqq s_0 .$$

Since the hypothesis of our proposition is evidently fulfilled if and only if one of the conditions (3), (4), (5), (11) and (18) holds we can resume (10), (17) and (28) as

(29) there exists a $\varkappa_0 \geqq 0$ and $s_0 \in \langle 0, \infty)$ so that $\operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \varkappa_0$ for $s \geqq s_0$.

On the other hand, it is easy to see from Lemma 12 that

(30) there exists a $\varkappa_{00} \geqq 0$ so that $\operatorname{Re} \lambda_{1,2,3,4}(s) \leqq \varkappa_{00}$ for $0 \leqq s \leqq s_0$.

Taking $\varkappa = \max(\varkappa_0, \varkappa_{00})$, we obtain the statement of our Lemma from (29) and (30).

Proposition 11. *Let m be the Fourier function of the Timoshenko type equation. If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$, then there exists a nonnegative constant $\varkappa$ so that*

$$|m(t, s)| \leqq \frac{t^3}{6} e^{\varkappa t} \quad \textit{for every} \quad t \in (0, \infty)$$

and $s \in \langle 0, \infty)$.

Proof. An immediate consequence of Lemma 17 and Proposition 6.

Theorem 4. *If the operator A is unbounded and the coefficient a is nonzero, then the Timoshenko type equation is exponential if and only if*

$$(\mathrm{C}) \qquad a > 0\,, \quad \alpha \geqq 0\,, \quad \alpha^2 \geqq 4a\,.$$

Proof. "Only if" part.

First, by Theorem 2, we can choose a fixed dense subset $Z \subseteq H$ from the definition of extensivity.

Further, by our hypothesis, we can choose fixed nonnegative constants M, ω from the definition of exponentiality.

Let us have

(1) $\qquad S = \sigma(A) \setminus \{0\}$, where $\sigma(A)$ is the spectrum of A.

Since the operator A is supposed to be nonnegative and unbounded, we have

(2) $$S \subseteq \langle 0, \infty) \text{ is unbounded}.$$

Finally, let m be the Fourier function of the Timoshenko type equation existing in virtue of Proposition 4.

We shall now prove that

(3) $$|m(t, s)| \leqq Me^{\omega t} \quad \text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in S.$$

Indeed, suppose that (3) is not true. Then there exist $t_0 \in (0, \infty)$ and $s_0 \in S$ so that $|m(t_0, s_0)| > Me^{\omega t_0}$. Since $s_0 > 0$ and m is continuous in both variables we can find $0 < r \leqq s_0$ and $\delta > 0$ so that $|m(t_0, s)| > Me^{\omega t_0} + \delta$ for every $|s - s_0| \leqq r$. Summarizing these considerations we can state that

(4) $$\text{there exist } t_0 \in \langle 0, \infty),\ s_0 \in S,\ 0 < r \leqq s_0 \text{ and } \delta > 0 \text{ so that}$$
$$|m(t_0, s)| > Me^{\omega t_0} + \delta \quad \text{for every} \quad |s - s_0| \leqq r.$$

Now let $\mathscr{E}$ be the spectral resolution of the operator A.

Let

(5) $$P = \mathscr{E}(\langle s_0 - r, s_0 + r \rangle).$$

Since, by (1) and (4), $s_0 \in \sigma(A)$, we obtain easily from the operational calculus

(6) $$P \neq 0.$$

Since the subset $Z \subseteq H$ is supposed to be dense in H, we see easily from (6) that

(7) $$\text{there exists } z \in Z \text{ so that } \|Pz\| = 1.$$

Since $z \in Z$ by (7), we can find a function $u \in (0, \infty) \to H$ such that

(8) u is the Duhamel solution of the Timoshenko type equation such that

$$u'''(0_+) = z.$$

Let us now take

(9) $$u_1(t) = P\,u(t) \quad \text{for} \quad t \in (0, \infty).$$

It follows from Lemma 3 and Proposition 3 and from (5), (8), (9) that

(10) u_1 is the Duhamel solution of the Timoshenko type equation such that $u_1'''(0_+) = Pz$.

In virtue of exponentiality, we obtain from (7) and (10)

(11) $$\|u_1(t)\| \leqq Me^{\omega t} \quad \text{for} \quad t \in (0, \infty).$$

Let us now define

(12) $$u_2(t) = \int_{s_0-r}^{s_0+r} m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, z \quad \text{for} \quad t \in (0, \infty)\,.$$

We obtain from the properties of the Fourier function m by means of the operational calculus that

(13) u_2 is the Duhamel solution of the Timoshenko type equation such that $u_2'''(0_+) = Pz$.

Now, by (4), (5) and (7), we obtain from (12) that

$$\|u_2(t_0)\|^2 = \left\| \int_{s_0-r}^{s_0+r} m(t_0, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, z \right\|^2 = \int_{s_0-r}^{s_0+r} m(t_0, \sigma)^2\, \|\mathscr{E}(\mathrm{d}\sigma)\, z\|^2 \geqq$$

$$\geqq (Me^{\omega t_0} + \delta)^2 \int_{s_0-r}^{s_0+r} \|\mathscr{E}(\mathrm{d}\sigma)\, z\|^2 = (Me^{\omega t_0} + \delta)^2\, \|Pz\|^2 = (Me^{\omega t_0} + \delta)^2\,.$$

Hence $\|u_2(t_0)\| \geqq Me^{\omega t_0} + \delta$ which implies

(14) $$\|u_2(t_0)\| > Me^{\omega t_0}\,.$$

On the other hand, using Theorem 1 we obtain from (10) and (13) that

(15) $$u_1(t) = u_2(t) \quad \text{for every} \quad t \in (0, \infty)\,.$$

However, (14) and (15) contradict (11) and hence (3) holds.

Now the "only if" part follows from (3) by means of Proposition 10.

"If" part.

Let u be an arbitrary Duhamel solution for the Timoshenko type equation.

If m is the Fourier function of the Timoshenko type equation and $\mathscr{E}$ the spectral resolution of the operator A, then by Proposition 9

(16) $$u(t) = \lim_{k\to\infty} \int_0^k m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, u'''(0_+) \quad \text{for every} \quad t \in R^+\,.$$

Using Proposition 11 we obtain from (16) that there exists a nonnegative constant $\varkappa$ for which

(17) $$\begin{aligned} \|u(t)\| &= \lim_{k\to\infty} \left\| \int_0^k m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, u'''(0_+) \right\| \leqq \\ &\leqq \lim_{k\to\infty} \frac{t^3}{6}\, e^{\varkappa t} \|\mathscr{E}(\langle 0, k\rangle)\, u'''(0_+)\| = \\ &= \frac{t^3}{6}\, e^{\varkappa t} \|u'''(0_+)\| \leqq e^{(\varkappa+1)t} \|u'''(0_+)\| \quad \text{for every} \quad t \in R^+\,. \end{aligned}$$

Now (17) proves the exponentiality with $M = 1$, $\omega = \varkappa + 1$.

Remark 3. From Theorem 4 we see the interesting (but well known) fact due to FATTORINI [9] that the simplest Timoshenko type equation with $\alpha = 0$ is never exponential for A unbounded, even when A is positive. Hence the role of the "mixed" term $\alpha A^{1/2} u''(t)$ is decisive for the exponentiality of Timoshenko type equations.

Remark 4. The situation described by Theorems 1, 2, 3 for Timoshenko type equation is analogous to the well known results of Hadamard for wave equation.

In abstract form, if we replace the Laplacian by our selfadjoint nonnegative operator A, Hadamard considered the Duhamel problem for the equation

$$u''(t) + aA\,u(t) = 0\,.$$

As in Timoshenko case, this equation is always definite and extensive, but exponentiality fails for $a < 0$.

Proposition 12. *Let m be the Fourier function of the Timoshenko type equation. If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$, then there exists a nonnegative constant $\varkappa$ such that for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$ the estimates (1)–(6) of Proposition 6 hold.*

Proof. An immediate consequence of Lemma 17 and Proposition 6.

Theorem 5. *If the operator A is unbounded and $a > 0$, $\alpha \geqq 0$, $\alpha^2 \geqq 4a$, then there exists a nonnegative constant $\varkappa$ so that for every $x \in \mathsf{D}(A)$ there exists a Duhamel solution u of the Timoshenko type equation for which*

$$u'''(0_+) = x\,, \tag{1}$$

$$\|u^{(j)}(t)\| \leqq \frac{t^{3-j}}{(3-j)!}\, e^{\varkappa t}\|u'''(0_+)\| \tag{2}$$

for every $t \in (0, \infty)$ and $j \in \{0, 1, 2\}$.

Proof. An immediate consequence of Propositions 5 and 12 by means of operational calculus if we seek the desired solution u in the form

$$u(t) = \int_0^\infty m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, x$$

where $\mathscr{E}$ is the spectral resolution of A.

Proposition 13. *Let m be the Fourier function of the Timoshenko type equation. If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 > 4a$, then there exist two nonnegative constants N, $\varkappa$ so that for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$*

$$|m(t, s)| \leqq N \frac{1}{1 + s^{1/2}}\, e^{\varkappa t}\,, \tag{1}$$

$$(2) \qquad |m_t(t, s)| \leqq N e^{\varkappa t},$$

$$(3) \qquad |m_{tt}(t, s)| \leqq N \frac{1}{1 + s^{1/4}} e^{\varkappa t},$$

$$(4) \qquad |m_{ttt}(t, s)| \leqq N e^{\varkappa t},$$

$$(5) \qquad |m_{tttt}(t, s)| \leqq N(1 + s^{1/4}) e^{\varkappa t},$$

$$(6) \qquad |m_{ttt}(t, s) - 1| \leqq N(1 + s^{1/4}) t e^{\varkappa t}.$$

Proof. By Lemmas 7 and 11, we have for $s \in \langle 0, \infty)$

$$(7) \qquad \lambda_1(s)^2 - \lambda_2(s)^2 = [(\alpha^2 - 4a) s + 2(\alpha\beta - 2b) s^{1/2} + (\beta^2 - 4c)]^{1/2}.$$

By our assumption $\alpha^2 > 4a$ it is clear form (7) that there exists an $s_0 > 0$ so that

$$(8) \qquad |\lambda_1(s)^2 - \lambda_2(s)^2| \geqq \frac{(\alpha^2 - 4a)^{1/2}}{2} s^{1/2} \quad \text{for} \quad s \geqq s_0.$$

Using (8), we obtain from Lemma 17 and Proposition 7 that there exist two nonnegative constants N_1, $\varkappa_1$ such that for every $t \in (0, \infty)$ and $s \geqq s_0$

$$(9) \qquad |m(t, s)| \leqq N_1 \frac{1}{s^{1/2}} e^{\varkappa_1 t},$$

$$(10) \qquad |m_t(t, s)| \leqq N_1 e^{\varkappa_1 t},$$

$$(11) \qquad |m_{tt}(t, s)| \leqq N_1 \frac{1}{s^{1/4}} e^{\varkappa_1 t},$$

$$(12) \qquad |m_{ttt}(t, s)| \leqq N_1 e^{\varkappa_1 t},$$

$$(13) \qquad |m_{tttt}(t, s)| \leqq N_1 s^{1/4} e^{\varkappa_1 t},$$

$$(14) \qquad |m_{ttt}(t, s) - 1| \leqq N_1 s^{1/4} t e^{\varkappa_1 t}.$$

Further, Proposition 12 enables us to find two nonnegative constants N_2, $\varkappa_2$ so that for every $t \in (0, \infty)$ and $0 \leqq s \leqq s_0$

$$(15) \qquad |m(t, s)| \leqq N_2 e^{\varkappa_2 t},$$

$$(16) \qquad |m_t(t, s)| \leqq N_2 e^{\varkappa_2 t},$$

$$(17) \qquad |m_{tt}(t, s)| \leqq N_2 e^{\varkappa_2 t},$$

$$(18) \qquad |m_{ttt}(t, s)| \leqq N_2 e^{\varkappa_2 t},$$

$$(19) \qquad |m_{tttt}(t, s)| \leqq N_2 e^{\varkappa_2 t},$$

(20) $$|m_{ttt}(t, s) - 1| \leqq N_2 t e^{\varkappa_2 t}.$$

Consequently, we see from (9)–(20) that there exist two nonnegative constants $N, \varkappa$ for which the estimates (1)–(6) hold.

Theorem 6. *If the operator A is unbounded and if $a > 0$, $\alpha \geqq 0$ and $\alpha^2 > 4a$, then there exist two nonnegative constants $N, \varkappa$ so that for every $x \in \mathsf{D}(A^{1/2})$ there exists a Duhamel solution u of the Timoshenko type equation for which*

(1) $$u'''(0_+) = x,$$

(2) $$\|u^{(j)}(t)\| \leqq N e^{\varkappa t} \|u'''(0_+)\| \quad \textit{for every} \quad t \in (0, \infty) \quad \textit{and} \quad j \in \{0, 1, 2, 3\}.$$

Proof. Immediately from Propositions 5 and 13 by means of operational calculus as in Theorem 5.

FURTHER PROPERTIES OF TIMOSHENKO TYPE EQUATION

Lemma 18. *If p_0, p_1, p_2, q are real constants such that*

(1) $$p_0 \geqq 0,$$

(2) $$2p_0 q + p_1 \geqq 0,$$

then for every $r \geqq q$

(3) $$p_0 r^2 + p_1 r + p_2 \geqq p_0 q^2 + p_1 q + p_2.$$

Lemma 19. *Let $\vartheta \in \langle 0, \infty)$. If*

(1) $$a > 0, \quad \alpha \geqq 0, \quad \alpha^2 \geqq 4a,$$

(2) $$\alpha \vartheta^{1/2} + \beta \geqq 0,$$

(3) $$2a\vartheta^{1/2} + b \geqq 0,$$

(4) $$a\vartheta + b\vartheta^{1/2} + c \geqq 0,$$

(5) $$(\alpha^2 - 4a)\,\vartheta^{1/2} + (\alpha\beta - 2b) \geqq 0,$$

(6) $$(\alpha^2 - 4a)\,\vartheta + 2(\alpha\beta - 2b)\,\vartheta^{1/2} + s^2 - 4c \geqq 0$$

then for every $s \geqq \vartheta$

(7) $$\operatorname{Re} \lambda_{1,2,3,4}(s) = 0.$$

Proof. By Lemma 18 we obtain from (1), (3), (4)

(8) $$as + bs^{1/2} + c \geqq 0 \quad \text{for every} \quad s \geqq \vartheta$$

and (1), (5), (6) imply

$$(9) \qquad (\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + \beta^2 - 4c \geqq 0 \quad \text{for every} \quad s \geqq \vartheta\,.$$

It follows immediately from (8) and (9) that

$$(10) \qquad 0 \leqq \left[\left(\frac{\alpha s^{1/2} + \beta}{2}\right)^2 - (as + bs^{1/2} + c)\right]^{1/2} \leqq \left|\frac{\alpha s^{1/2} + \beta}{2}\right|.$$

On the other hand, (1), (2) imply that

$$(11) \qquad \alpha s^{1/2} + \beta \geqq 0 \quad \text{for every} \quad s \geqq \vartheta\,.$$

Using (10) and (11) we obtain easily

$$(12) \qquad \mu_{12}(s) \leqq 0 \quad \text{for} \quad s \geqq \vartheta\,.$$

Now (7) follows immediately from (12).

Proposition 14. *Let m be the Fourier function of the Timoshenko type equation and let $\vartheta \in \langle 0, \infty)$. If the conditions (1)–(6) of Lemma 19 hold, then for every $t \in (0, \infty)$ and $s \geqq \vartheta$*

$$(1) \qquad |m(t, s)| \leqq \frac{t^3}{6}\,,$$

$$(2) \qquad |m_t(t, s)| \leqq \frac{t^2}{2}\,,$$

$$(3) \qquad |m_{tt}(t, s)| \leqq t\,,$$

$$(4) \qquad |m_{ttt}(t, s)| \leqq (1 + |a| + |b| + |c| + |\alpha| + |\beta|)\,(1 + s^{1/4})\, t + 1\,,$$

$$(5) \qquad |m_{tttt}(t, s)| \leqq (1 + |a| + |b| + |c| + |\alpha| + |\beta|)^2\,(1 + s^{1/4})^2\, t + (1 + |a| + |b| + |c| + |\alpha| + |\beta|)\,(1 + s^{1/4})\,,$$

$$(6) \qquad |m_{ttt}(t, s) - 1| \leqq 2(1 + |a| + |b| + |c| + |\alpha| + |\beta|)\,(1 + s^{1/4})\, t\,.$$

Proof. An immediate consequence of Lemma 19 and Proposition 6.

Theorem 7. *If the operator A is unbounded and the conditions (1)–(6) of Lemma 19 are fulfilled, then for every $x \in D(A)$, there exists a Duhamel solution u of the Timoshenko type equation for which*

$$(1) \qquad u'''(0_+) = x\,,$$

(2) $$\|u^{(j)}(t)\| \leqq \frac{t^{3-j}}{(3-j)!}\,\|u'''(0_+)\|$$

for every $t \in (0, \infty)$ *and* $j \in \{0, 1, 2\}$.

Proof. An immediate consequence of Proposition 14 by means of operational calculus (cf. the proof of Theorem 5).

Lemma 20. *Let* $\vartheta \in \langle 0, \infty)$. *If*

(1) $$a > 0\,, \quad \alpha \geqq 0\,, \quad \alpha^2 > 4a\,,$$

(2) $$\alpha\vartheta^{1/2} + \beta \geqq 0\,,$$

(3) $$2a\vartheta^{1/2} + b \geqq 0\,,$$

(4) $$a\vartheta + b\vartheta^{1/2} + c > 0\,,$$

(5) $$(\alpha^2 - 4a)\,\vartheta^{1/2} + (\alpha\beta - 2b) \geqq 0\,,$$

(6) $$(\alpha^2 - 4a)\,\vartheta + 2(\alpha\beta - 2b)\,\vartheta^{1/2} + \beta^2 - 4c > 0\,,$$

then

(7) $$\operatorname{Re}\lambda_{1,2,3,4}(s) = 0 \quad \text{for} \quad s \geqq \vartheta\,,$$

(8) *there exists a constant* $L_1 > 0$ *such that*
$$|\lambda_1(s)^2 - \lambda_2(s)^2| \geqq L_1 s^{1/2} \quad \text{for every} \quad s \geqq \vartheta\,,$$

(9) *there exists a constant* $L_2 > 0$ *such that* $|\lambda_{1,2}(s)| \geqq L_2$ *for every* $s \geqq \vartheta$.

Proof. The statement (7) follows immediately from Lemma 19.

By Lemmas 11 and 7 we can write

(10) $$|\lambda_1(s)^2 - \lambda_2(s)^2| = |\mu_1(s) - \mu_2(s)| =$$
$$= |[(\alpha^2 - 4a)\,s + 2(\alpha\beta - 2b)\,s^{1/2} + (\beta^2 - 4c)]^{1/2}| \quad \text{for every} \quad s \in \langle 0, \infty)\,.$$

Using Lemma 18, we obtain from (5), (6)

(11) $$(\alpha^2 - 4a)\,s + 2(\alpha\beta - 2b)\,s^{1/2} + \beta^2 - 4c > 0 \quad \text{for every} \quad s \geqq \vartheta\,.$$

On the other hand, it follows from (1) that

(12) $$\lim_{s\to\infty} \frac{(\alpha^2 - 4a)\,s + 2(\alpha\beta - 2b)\,s^{1/2} + \beta^2 - 4c}{s} \neq 0\,.$$

Hence by (11) and (12)

(13) there exists a constant $L_1 > 0$ so that
$$[(\alpha^2 - 4a)\,s + 2(\alpha\beta - 2b)\,s^{1/2} + \beta^2 - 4c]^{1/2} \geqq L_1 s^{1/2} \quad \text{for every} \quad s \geqq \vartheta\,.$$

The statement (8) follows from (10) and (13).

By Lemma 14 we can write

$$(14) \qquad |\lambda_{1,2}(s)|^2 = |\mu_{1,2}(s)| =$$

$$= \left| -\frac{\alpha s^{1/2} + \beta}{2} \pm \left[\left(\frac{\alpha s^{1/2} + \beta}{2} \right)^2 - \left(as + bs^{1/2} + c \right) \right]^{1/2} \right| \geqq$$

$$\geqq \tfrac{1}{2}\left\{ |\alpha s^{1/2} + \beta| - \left| [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2} \right| \right\}$$

for every $s \in \langle 0, \infty)$.

By (1) and (2)

$$(15) \qquad \alpha s^{1/2} + \beta \geqq 0 \quad \text{for every} \quad s \geqq \vartheta .$$

Hence we obtain from (11), (14) and (15) that

$$(16) \qquad |\lambda_{1\,2}(s)|^2 \geqq$$

$$\geqq \frac{1}{2} \frac{(\alpha s^{1/2} + \beta)^2 - [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]}{\alpha s^{1/2} + \beta + [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2}} =$$

$$= 2 \frac{as + bs^{1/2} + c}{\alpha s^{1/2} + \beta + [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2}}$$

for every $s \geqq \vartheta$.

Using Lemma 18 we obtain from (3), (4) that

$$(17) \qquad as + bs^{1/2} + c > 0 \quad \text{for every} \quad s \geqq \vartheta .$$

Now by (11), (15) and (17)

$$(18) \qquad \frac{as + bs^{1/2} + c}{\alpha s^{1/2} + \beta + [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2}} > 0$$

for every $s \geqq \vartheta$.

On the other hand, we obtain easily from (1) that

$$(19) \qquad \lim_{s \to \infty} \frac{as + bs^{1/2} + c}{\alpha s^{1/2} + \beta + [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2}} = \infty .$$

It follows easily from (18) and (19) that there exists a constant $L_2 > 0$ so that

$$(20) \qquad 2 \frac{as + bs^{1/2} + c}{\alpha s^{1/2} + \beta + [(\alpha^2 - 4a)\, s + 2(\alpha\beta - 2b)\, s^{1/2} + (\beta^2 - 4c)]^{1/2}} > L_2^2$$

for every $s \geqq \vartheta$.

The statement (9) follows from (16) and (20).

Proposition 15. *Let m be the Fourier function of the Timoshenko type equation and $\vartheta \in (0, \infty)$. If the conditions (1)–(6) of Lemma 20 are fulfilled, then there exists a constant L such that for every $t \in (0, \infty)$ and $s \geqq \vartheta$*

$$(1) \qquad |m(t, s)| \leqq L \frac{1}{s^{1/2}},$$

$$(2) \qquad |m_t(t, s)| \leqq L \frac{1}{s^{1/2}},$$

$$(3) \qquad |m_{tt}(t, s)| \leqq L \frac{1}{s^{1/4}},$$

$$(4) \qquad |m_{ttt}(t, s)| \leqq L,$$

$$(5) \qquad |m_{tttt}(t, s)| \leqq L s^{1/4},$$

$$(6) \qquad |m_{ttt}(t, s) - 1| \leqq L t s^{1/4}.$$

Proof. By Lemmas 14 and 20 we can write for $t \in (0, \infty)$ and $s \geqq \vartheta$

$$(7) \quad m(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^{-1} \sinh(\lambda_1(s) t) - \lambda_2(s)^{-1} \sinh(\lambda_2(s) t)],$$

$$(8) \quad m_t(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\cosh(\lambda_1(s) t) - \cosh(\lambda_2(s) t)],$$

$$(9) \quad m_{tt}(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s) \sinh(\lambda_1(s) t) - \lambda_2(s) \sinh(\lambda_2(s) t)],$$

$$(10) \quad m_{ttt}(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^2 \cosh(\lambda_1(s) t) - \lambda_2(s)^2 \cosh(\lambda_2(s) t)],$$

$$(11) \quad m_{tttt}(t, s) = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^3 \sinh(\lambda_1(s) t) - \lambda_2(s)^3 \sinh(\lambda_2(s) t)],$$

$$(12) \quad m_{ttt}(t, s) - 1 = [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} [\lambda_1(s)^2 (\cosh(\lambda_1(s) t) - 1) - \lambda_2(s)^2 (\cosh(\lambda_2(s) t) - 1)] =$$

$$= [\lambda_1(s)^2 - \lambda_2(s)^2]^{-1} \left[\lambda_1(s)^3 \int_0^t \sinh(\lambda_1(s) \tau) \, d\tau - \lambda_1(s)^3 \int_0^t \sinh(\lambda_2(s) \tau) \, d\tau\right].$$

Let us denote $L_0 = 1 + |a| + |b| + |c| + |\alpha| + |\beta|$ and let us fix the constants L_1, L_2 from Lemma 20.

Using Lemmas 12 and 20 we obtain from (7)–(12) that for every $t \in (0, \infty)$ and $s \geqq \vartheta$

$$(13) \qquad |m(t, s)| \leqq \frac{2L_2}{L_1 s^{1/2}},$$

$$(14) \qquad |m_t(t, s)| \leqq \frac{2}{L_1 s^{1/2}},$$

(15) $$|m_{tt}(t, s)| \leqq \frac{2L_0(1 + s^{1/4})}{L_1 s^{1/2}} = \frac{2L_0}{L_1}\left(\frac{1}{s^{1/2}} + \frac{1}{s^{1/4}}\right),$$

(16) $$|m_{ttt}(t, s)| \leqq \frac{2L_0^2(1 + s^{1/4})^2}{L_1 s^{1/2}} = \frac{2L_0^2}{L_1}\left(1 + \frac{1}{s^{1/4}}\right)^2 \leqq$$
$$\leqq \frac{2L_0^2}{L_1}\left(1 + \frac{1}{\vartheta^{1/4}}\right)^2,$$

(17) $$|m_{tttt}(t, s)| \leqq \frac{2L_0^3(1 + s^{1/4})^3}{L_1 s^{1/2}} = \frac{2L_0^3}{L_1} s^{1/4} \frac{(1 + s^{1/4})^3}{s^{3/4}} \leqq$$
$$\leqq \frac{2L_0^3}{L_1}\left(1 + \frac{1}{\vartheta^{1/4}}\right)^3 s^{1/4},$$

(18) $$|m_{ttt}(t, s) - 1| \leqq \frac{2L_0^3(1 + s^{1/4})^3}{L_1 s^{1/2}}\, t \leqq \frac{2L_0^3}{L_1}\left(1 + \frac{1}{\vartheta^{1/4}}\right)^3 s^{1/4} t.$$

It is clear that the statement of our Proposition is an immediate consequence of (13)–(18).

Theorem 8. *If the operator A is unbounded and if there exists a positive constant ϑ such that $\langle Ax, x\rangle \geqq \vartheta\|x\|$ for every $x \in D(A)$ and the conditions (1)–(6) of Lemma 20 are fulfilled, then there exists a nonnegative constant L such that for every $x \in D(A^{1/2})$ we can find a Duhamel solution u of the Timoshenko type equation for which*

(1) $$u'''(0_+) = x,$$

(2) $$\|u^{(j)}(t)\| \leqq L \quad \textit{for every} \quad t \in (0, \infty) \quad \textit{and} \quad j \in \{0, 1, 2, 3\}.$$

Proof. Immediately from Propositions 5 and 15 by means of operational calculus (cf. the proof of Theorem 6).

EXAMPLE

Let us denote by $L_2(0, 1)$ the Hilbert space of all square integrable complex valued functions x on the interval $(0, 1)$ with the norm $\|x\| = (\int_0^1 |x(\eta)|^2 \,d\eta)^{1/2}$.

It is clear that the scalar product on $L_2(0, 1)$ is of the form $\langle x, y\rangle = \int_0^1 x(\eta) \,.\, \overline{y(\eta)} \,d\eta$.

Moreover, we shall say that a function x is feebly differentiable in $L_2(0, 1)$ if there exists a function $z \in L_2(0, 1)$ so that $x(b) - x(a) = \int_a^b z(\eta)\,d\eta$ for every $0 < a \leqq b < 1$. The function z will be denoted by x' and called the feeble derivative of x in $L_2(0, 1)$.

It is evident that every function feebly differentiable in $L_2(0, 1)$ is continuous on $(0, 1)$ and possesses limits at the points 0 and 1.

Let us now define an operator Γ in $L_2(0, 1)$ as follows: $x \in D(\Gamma)$ if and only if $x \in \in L_2(0, 1)$, x is four times feebly differentiable in $L_2(0, 1)$ and $x(0) = x(1) = x''(0) = = x''(1) = 0$; then we take $\Gamma x = x''''$.

Proposition 16. *The operator Γ is selfadjoint and strictly positive in $L_2(0, 1)$.*

Proof. Let us write for $x \in L_2(0, 1)$ and $k \in \{1, 2, \ldots\}$, $x_k = \int_0^1 x(\eta) \sin k\eta \, d\eta$. It is easy to verify that

$$x \in D(\Gamma) \quad \text{if and only if} \quad \sum_{k=1}^{\infty} k^4 |x_k|^2 < \infty \; ; \tag{1}$$

$$\Gamma \, x(\eta) = \sum_{k=1}^{\infty} k^4 x_k \sin k\eta \quad \text{for every} \quad x \in D(\Gamma) \, , \tag{2}$$

the sum being taken in the sense of $L_2(0, 1)$.

It follows immediately from (1), (2) that

$$\Gamma^{-1} \text{ exists and is bounded} \, . \tag{3}$$

Since the symmetry of Γ is almost evident, (3) implies the statement of our Proposition.

Theorem 9. *The above theory may be applied to Timoshenko type equation with $A = \Gamma$.*

Proof. An immediate consequence of Proposition 16.

Proposition 17. $x \in D(\Gamma^{1/2})$ *if and only if* $x \in L_2(0, 1)$, *x is twice feebly differentiable in $L_2(0, 1)$ and $x(0) = x(1) = 0$; then $\Gamma^{1/2}x = x''$.*

Proof. Let us define an operator Δ_0 in $L_2(0, 1)$ as follows: $x \in D(\Delta_0)$ if and only if $x \in L_2(0, 1)$, x is twice feebly differentiable in $L_2(0, 1)$ and $x(0) = x(1) = 0$; then $\Delta_0 x = x''$.

By the same method as in Proposition 16 we prove that

$$\Delta_0 \text{ is an selfadjoint, strictly positive operator in } L_2(0, 1) \, . \tag{1}$$

On the other hand, it is almost immediate that

$$\Delta^2 = \Gamma \, . \tag{2}$$

Now (1) and (2) imply the statement of our Proposition.

Theorem 10. *Let $u \in (0, \infty) \to L_2(0, 1)$. Then the function u is a solution of the Timoshenko type equation with $A = \Gamma$ if and only if*

(1) $$u \text{ is four times differentiable on } (0, \infty),$$

(2) *$u(t)$ is four times feebly differentiable in $L_2(0, 1)$ and* $u(t)(0) = u(t)(1) =$
$$= u(t)_{\xi\xi}(0) = u(t)_{\xi\xi}(1) = 0 \text{ for every } t \in (0, \infty),$$

(3) *$u''(t)$ is twice feebly differentiable in $L_2(0, 1)$ and* $u''(t)(0) = u''(t)(1) = 0$
$$\text{for every } t \in (0, \infty),$$

(4) $u''''(t)(\xi) + \alpha\, u''(t)_{\xi\xi}(\xi) + \beta\, u''(t)\, \xi + a\, u(t)_{\xi\xi\xi\xi}(\xi) + b\, u(t)_{\xi\xi}(\xi) + c\, u(t)(\xi) =$
$$= 0 \text{ for every } t \in (0, \infty) \text{ and } 0 < \xi < 1 .$$

Proof is easy by means of Proposition 17.

APPENDIX

The purpose of this appendix is to introduce and examine the concept of strict solution, strict definiteness, strict extensivity and strict exponentiality. These strengthened concepts are of principal importance in the general theory of equations of the type $u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0$ (cf. [8]). We shall show that the above theory may be developed also on the base of these strengthened concepts and that the results remain essentially unchanged.

A solution u of the Timoshenko type equation will be called strict if the function Au is continuous on $(0, \infty)$ and bounded on $(0, 1)$.

Proposition A 1. *Every strict solution of the Timoshenko type equation is also a solution.*

Lemma A 1. *For every $x \in D(A)$, $\|A^{1/2}x\| \leqq \|x\| + \|Ax\|$.*

Proposition A 2. *A function $u \in (0, \infty) \to H$ is a strict solution of the Timoshenko type equation if and only if*

(I) $$u \text{ is four times differentiable on } (0, \infty),$$

(II) $$u''(t) \in D(A^{1/2}), \quad u(t) \in D(A) \quad \text{for every} \quad t \in (0, \infty),$$

(III) *the functions $(\alpha A^{1/2} + \beta)\, u''$ and $(aA + bA^{1/2} + c)\, u$ are continuous on*
$$(0, \infty) \text{ and bounded on } (0, 1),$$

(IV) $$u''''(t) + (\alpha A^{1/2} + \beta)\, u''(t) + (aA + bA^{1/2} + c)\, u(t) = 0$$
$$\text{for every} \quad t \in (0, \infty).$$

Proof. An immediate consequence of properties of strict solutions by virtue of Lemma A 1.

Remark 5. The properties (I)–(IV) from Proposition A 2 are used as definition properties of a solution in general theory (cf. [8]).

We shall say that the Timoshenko type equation is strictly definite if every null strict solution of the Timoshenko type equation is identically zero.

Theorem A 1. *The Timoshenko type equation is always strictly definite.*

Proof. Immediate consequence of Theorem 1 and Proposition A 1.

Proposition A 3. *Let m be the Fourier function of the Timoshenko type equation and $\mathscr{E}$ the spectral resolution of the operator A. For every $x \in H$ and $K \in \langle 0, \infty)$, the function $u \in (0, \infty) \to H$ defined for $t \in (0, \infty)$ by*

$$u(t) = \int_0^K m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, x \tag{1}$$

is a Duhamel strict solution of the Timoshenko type equation such that

$$u'''(0_+) = \mathscr{E}(\langle 0, K \rangle)\, x\,. \tag{2}$$

Proof. An immediate consequence of Proposition 6 by means of operational calculus.

We shall say that the Timoshenko type equation is strictly extensive if there exists a dense subset $Z \subseteq H$ such that for every $x \in Z$ there exists a Duhamel strict solution u of the Timoshenko type equation such that $u'''(0_+) = x$.

Theorem A 2. *The Timoshenko type equation is always strictly extensive.*

Proof. Almost identical to the proof of Theorem 2 on the base of Proposition A 3.

We shall say that the Timoshenko type equation is strictly exponential if there exist two nonnegative constants M, ω such that for every strict Duhamel solution u of the Timoshenko type equation, it is for every $t \in (0, \infty)$

$$\|u(t)\| \leqq M e^{\omega t} \|u'''(0_+)\|\,, \tag{I}$$

$$\left\| \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho A\, u(v)\, \mathrm{d}v\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau \right\| \leqq M e^{\omega t} \|u'''(0_+)\|\,. \tag{II}$$

Proposition A 4. *If $a \neq 0$, then the Timoshenko type equation is strictly exponential if and only if there exist two nonnegative constants M_0, ω_0 such that for every strict Duhamel solution u of the Timoshenko type equation, it is for every $t \in (0, \infty)$*

$$\left\| \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho (\alpha A^{1/2} + \beta)\, u''(\eta)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau \right\| \leqq M_0 e^{\omega_0 t} \|u'''(0_+)\|\,, \tag{1}$$

(2) $$\left\| \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho (aA + bA^{1/2} + c)\, u(\eta)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau \right\| \leqq M_0 e^{\omega_0 t} \|u'''(0_+)\| \,.$$

Proof. The "only if" part is easy. Since by assumption on A we can write $\alpha A^{1/2} + \beta = \alpha A^{-1/2}A + \beta A^{-1}A$ and $aA + bA^{1/2} + c = aA + bA^{-1/2}A + cA^{-1}A$, the properties (I) and (II) from the above definition of strict exponentiality give immediately the properties (1) and (2).

The converse "if" part is somewhat more difficult.

The property (I) follows immediately from the properties (1) and (2) with regard to the fact that we can write for every solution u and every $t \in (0, \infty)$

$$u(t) = u'''(0_+) - \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho (\alpha A^{1/2} + \beta)\, u''(\eta)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau -$$
$$- \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho (aA + bA^{1/2} + c)\, u(\eta)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau \,.$$

It remains to prove (II). To this aim, let us denote by $\mathscr{E}$ the spectral resolution of the operator A.

First we shall find two operators P and C such that

(3) P is an orthogonal projector, $PA \subseteq AP$ and PAP is a bounded operator,

(4) C is a bounded operator.

(5) $$Ax = PAx + C(aA + bA^{1/2} + c)\, x \quad \text{for every} \quad x \in D(A)\,.$$

Indeed, since $a \neq 0$, we can find an $s_0 \in \langle 0, \infty)$ so that

(6) $$|as + bs^{1/2} + c| \geqq 1 \quad \text{for every} \quad s \geqq s_0\,.$$

Now let us write

(7) $$\chi(s) = 1 \quad \text{for} \quad 0 \leqq s < s_0\,, \qquad \chi(s) = 0 \quad \text{for} \quad s \geqq s_0\,.$$

It is easy to see from the assumption $a \neq 0$ and from (6) and (7) that

(8) the function $(1 - \chi(s))\, s(as + bs^{1/2} + c)^{-1}$ is bounded on $\langle 0, \infty)$.

Further, we can write for $s \in \langle 0, \infty)$

(9) $$s = \chi(s)\, s + (1 - \chi(s))\, s =$$
$$= \chi(s)\, s + (1 - \chi(s))\, s(as + bs + c)^{-1}\, (as + bs^{1/2} + c)\,.$$

Let us now define for $x \in H$

(10) $$Px = \int_0^\infty \chi(s)\, \mathscr{E}(\mathrm{d}s)\, x\,,$$

(11) $$Cx = \int_0^\infty (1 - \chi(s))\, s(as + bs + c)^{-1}\, \mathscr{E}(\mathrm{d}s)\, x\,.$$

The required properties (3)–(5) of the operators P and C follow easily from (6)–(11).

Now there is no difficulty in obtaining the desired property (II) from (1), (2) and from (I) already proved by means of (3)–(5).

Theorem A 3. *If the Timoshenko type equation is strictly exponential, then it is also exponential.*

Proof. An easy consequence of Propositions 9 and A 3.

Lemma A 2. *If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 > 4a$, then there exists a nonnegative constant N_0 such that*

(1) $$\frac{s}{1 + |\lambda_1(s)|^2\, |\lambda_2(s)|^2} \leqq N_0 \quad \text{for every} \quad s \in \langle 0, \infty)\,.$$

Proof. We obtain easily from the definition of $\mu_{12}(s)$ that

(2) $$\frac{|\mu_{12}(s)|}{s^{1/2}} \xrightarrow[s\to\infty]{} \left| -\frac{\alpha}{2} \pm \left[\left(\frac{\alpha}{2}\right)^2 - a\right]^{1/2} \right|.$$

By our hypothesis

(3) $$\left| -\frac{\alpha}{2} \pm \left[\left(\frac{\alpha}{2}\right)^2 - a\right]^{1/2} \right| > 0\,.$$

Hence we obtain from (2) and (3) that

(4) $$\frac{s}{1 + |\lambda_1(s)|^2\, |\lambda_2(s)|^2} \xrightarrow[s\to\infty]{}$$
$$\xrightarrow[s\to\infty]{} \frac{1}{\left| -\frac{\alpha}{2} + \left[\left(\frac{\alpha}{2}\right)^2 - a\right]^{1/2} \right| \cdot \left| -\frac{\alpha}{2} - \left[\left(\frac{\alpha}{2}\right)^2 - a\right]^{1/2} \right|} = \frac{1}{a}\,.$$

Statement (1) follows then easily from (4).

Lemma A 3. *If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$, then there exists a nonnegative constant $\varkappa$ such that for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$*

(1) $$|\cosh \lambda_{12}(s)\, t| \leqq e^{\varkappa t}\,,$$

(2) $$|\lambda_{12}(s)|^2 \left| \int_0^t \int_0^\tau \cosh \lambda_{12}(s)\, \sigma \, \mathrm{d}\sigma\, \mathrm{d}\tau \right| \leqq 2e^{\varkappa t}\,.$$

Proof. By Lemma 17 there exists a nonnegative constant $\varkappa$ such that

$$\operatorname{Re} \lambda_{12}(s) \leqq \varkappa \quad \text{for every} \quad s \in \langle 0, \infty) . \tag{3}$$

Now, (1) follows immediately from (3).
Further, we have evidently

$$\lambda_{12}(s)^2 \int_0^t \int_0^\tau \cosh \lambda_{12}(\sigma) \, d\sigma \, d\tau = \cosh \lambda_{12}(s) \, t - 1 \tag{4}$$

$$\text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty) .$$

Now (2) follows immediately from (3).

Proposition A 5. *Let m be the Fourier function of the Timoshenko type equation. If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$, then there exist two nonnegative constants M, ω such that for every $t \in (0, \infty)$ and $s \in \langle 0, \infty)$*

$$|m(t, s)| \leqq M e^{\omega t} , \tag{1}$$

$$\left| s \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s) \, d\eta \, d\varrho \, d\sigma \, d\tau \right| \leqq M e^{\omega t} . \tag{2}$$

Proof. Proposition 11 implies that there exist two nonnegative constants M_1, ω_1 so that

$$|m(t,s)| \leqq M_1 \, e^{\omega_1 t} \quad \text{for every} \quad t \in (0, \infty) \text{ and } s \in \langle 0, \infty) . \tag{3}$$

By Lemma 13, we can write the following identity which may be verified directly or by means of Laplace transformation:

$$\int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s) \, d\eta \, d\varrho \, d\sigma \, d\tau = \tag{4}$$

$$= \int_0^t \left(\int_0^{t-\tau} \int_0^\sigma \int_0^\varrho \cosh \lambda_1(s) \, \eta \, d\eta \, d\varrho \, d\sigma \right) .$$

$$\cdot \left(\int_0^\tau \int_0^\sigma \int_0^\varrho \cosh \lambda_2(s) \, \eta \, d\eta \, d\varrho \, d\sigma \right) d\tau \quad \text{for} \quad \text{every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty) .$$

Now we obtain easily from (4) by means of Lemma A 3

$$(1 + |\lambda_1(s)|^2 \, |\lambda_2(s)|^2) \left| \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s) \, d\eta \, d\varrho \, d\sigma \, d\tau \right| \leqq \tag{5}$$

$$\leqq \frac{t^7}{7!} e^{\varkappa t} + \frac{2t^3}{3!} e^{\varkappa t} = \left(\frac{t^7}{7!} + \frac{t^3}{3!} \right) e^{\varkappa t}$$

$$\text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty) .$$

Let N_0 be the constant from Lemma A 2 so that

(6) $$\frac{s}{1 + |\lambda_1(s)^2| |\lambda_2(s)|^2} \leqq N_0 \quad \text{for} \quad s \in \langle 0, \infty) .$$

It follows from (5) and (6) that

(7) $$s \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau =$$

$$= \frac{s}{1 + |\lambda_1(s)|^2 |\lambda_2(s)|^2} (1 + |\lambda_1(s)|^2 |\lambda_2(s)|^2) \,.$$

$$. \left| \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau \right| \leqq N_0 \left(\frac{t^7}{7!} + \frac{t^3}{3!} \right) e^{\varkappa t}$$

$$\text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty) .$$

It follows from (7) that then there exist two nonnegative constants M_2, ω_2 so that

(8) $$\left| s \int_0^t \int_0^\tau \int_0^\sigma \int_0^\varrho m(\eta, s)\, \mathrm{d}\eta\, \mathrm{d}\varrho\, \mathrm{d}\sigma\, \mathrm{d}\tau \right| \leqq M_2 e^{\omega_2 t}$$

$$\text{for every} \quad t \in (0, \infty) \quad \text{and} \quad s \in \langle 0, \infty) .$$

Taking $M = \max(M_1, M_2)$ and $\omega = \max(\omega_1, \omega_2)$, we obtain (1) and (2) immediately from (3) and (8).

Theorem A 4. *If $a > 0$, $\alpha \geqq 0$ and $\alpha^2 \geqq 4a$, then the Timoshenko type equation is strictly exponential.*

Proof. Let u be an arbitrary strict Duhamel solution of the Timoshenko type equation.

If m is the Fourier function of the Timoshenko type equation and $\mathscr{E}$ the spectral resolution of the operator A, then by Proposition A 1 and Proposition 9 it is

(1) $$u(t) = \lim_{k \to \infty} \int_0^k m(t, \sigma)\, \mathscr{E}(\mathrm{d}\sigma)\, (u'''(0_+)) \quad \text{for every} \quad t \in R^+ .$$

Using Proposition A 5 we obtain from (1) that there exist two nonnegative constants M, ω such that for every $t \in (0, \infty)$

(2) $$\|u(t)\| = \lim_{k \to \infty} \left\| \int_0^k m(t, \vartheta)\, \mathscr{E}(\mathrm{d}\vartheta)\, u'''(0_+) \right\| \leqq$$

$$\leqq \lim_{k \to \infty} M e^{\omega t} \|\mathscr{E}(\langle 0, k \rangle)\, u'''(0_+)\| = M e^{\omega t} \|u'''(0_+)\| ,$$