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ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

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ON L^p -SOLUTIONS OF THE DIFFERENTIAL EQUATION

$$y'' = q(t)y$$

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1.1. Consider a differential equation

$$(q) \quad y'' = q(t)y, \quad q \in C^0[a, b], \quad b \leq \infty, \quad q(t) < 0, \quad t \in [a, b)$$

where $C^n[a, b]$ (n being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b]$.

Let y_1 be a non-trivial solution of (q) vanishing at $t \in [a, b)$ and y_2 a non-trivial one the derivative of which vanishes at t . If $\varphi(t), \psi(t), \chi(t), \omega(t)$ is the first zero respectively of y_1, y_2', y_1', y_2 lying to the right from t , then $\varphi, \psi, \chi, \omega$ is called the basic central dispersion of the 1-st, 2-nd, 3-rd, 4-th kind, respectively (briefly, dispersion of the 1-st, 2-nd, 3-rd, 4-th kind).

Throughout the paper we shall deal with oscillatory ($t \rightarrow b_-$) differential equations (i.e., every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$).

Let δ be the dispersion of the k -th kind, $k = 1, 2, 3, 4$. Then δ has the following properties (see [4] § 13)

- (1)
- 1) $\delta \in C^3[a, b)$ if $k = 1$
 $\delta \in C^1[a, b)$ if $k = 2, 3$ or 4
 - 2) $\delta(t) > t$ on $[a, b)$
 - 3) $\delta'(t) > 0$ on $[a, b)$
 - 4) $\lim_{t \rightarrow b-} \delta(t) = b$.

Let n be a positive integer. If δ_n is the n -th iteration of the dispersion δ of the k -th kind, then δ_n has the same properties (1), see [4] § 13.

We shall need also some other properties of dispersions. Let y be a non-trivial solution of (q) and let φ_n, ψ_n be the n -th iteration of the dispersion φ, ψ of the 1-st or 2-nd kind, respectively. Then we have (see [4] § 13):

$$(2) \quad \begin{aligned} \varphi'_n(t) &= y^2(\varphi_n(t))/y^2(t) && \text{for } y(t) \neq 0 \\ &= y'^2(t)/y'^2(\varphi_n(t)) && \text{for } y(t) = 0 \\ \psi'_n(t) &= \frac{q(t)}{q(\psi_n(t))} \cdot \frac{y'^2(\psi_n(t))}{y'^2(t)} && \text{for } y'(t) \neq 0 \\ &= \frac{q(t)}{q(\psi_n(t))} \cdot \frac{y^2(t)}{y^2(\psi_n(t))} && \text{for } y'(t) = 0. \end{aligned}$$

1.2. First we summarize the results that we shall need in the sequel. See [1], [6], [2] (Theorems 5, 9, 10).

Theorem 1. Let (q), $q \in C^0[a, b]$, $q(t) < 0$, $t \in [a, b]$ be an oscillatory ($t \rightarrow b_-$) differential equation and $\varphi_n(\psi_n)$ the n -th iteration of its dispersion φ (ψ) of the first (second) kind. Let $t_0 \in [a, b]$.

a) Every solution of (q) is bounded on $[t_0, b]$ if and only if a constant N exists such that

$$\varphi'_n(x) \leq N, \quad x \in [t_0, \varphi(t_0)], \quad n = 1, 2, 3, \dots$$

b) Every solution of (q) belongs to $L^p[t_0, b]$, $p > 0$ if and only if

$$\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi'_n(t)]^{1+p/2} dt < \infty$$

holds.

c) If q is non-increasing (non-decreasing), then

$$\frac{q(t)}{q(\delta(t))} \leq \delta'(t) \leq 1 \quad (\delta'(t) \geq 1), \quad t \in [a, b]$$

holds where δ is the dispersion of the k -th kind of (q), $k = 1, 2$.

d) Let $0 > q(t) \geq \text{const} > -\infty$. If there exists a solution y of (q) tending to zero for $t \rightarrow b_-$, then every solution of (q) linearly independent of y is unbounded on $[a, b]$.

e) Let $0 > \text{const.} \geq q(t) > -\infty$. If there exists a solution y of (q) the derivative of which tends to zero $t \rightarrow b_-$, then the derivative of every solution of (q) linearly independent of y is unbounded on $[a, b]$.

f) Consider the following assertions on $[a, b]$:

A) The sequence of absolute values of local extremes of (the derivative of) an arbitrary solution of (q) is non-increasing.

B) The sequence of absolute values of local extremes of the derivative of an arbitrary solution (of an arbitrary solution) is non-decreasing.

C) $\frac{q(\psi(t))}{q(t)} \psi'(t) \geq 1$ ($\varphi(t) - t$ is non-decreasing).

D) $\varphi(t) - t$ is non-increasing $\left(\frac{q(\psi(t))}{q(t)} \psi'(t) \leq 1 \right)$.

Then $A \Leftrightarrow C \Rightarrow D \Leftrightarrow B$ holds.

2. This paragraph deals with the relation of the dispersions of the 1-st and 2-nd kind of (q) and the property of every solution (of the derivative of every solution) of (q) to belong to $L^p[a, b]$, $p > 0$. Theorem 1b) gives the necessary and sufficient condition for every solution to belong to $L^p[a, b]$, $p > 0$. The situation for the derivative of an arbitrary solution is described by the following

Theorem 2. Let (q), $q \in C^0[a, b]$, $q(t) < 0$, $t \in [a, b]$ be oscillatory on $[a, b]$ and let ψ_n be the n -th iteration of the dispersion ψ of the 2-nd kind. Then the derivative of every solution of (q) belongs to $L^p[a, b]$, $p > 0$ if and only if

$$(4) \quad \sum_{n=0}^{\infty} \int_a^{\psi(a)} |q(\psi_n(t))|^{p/2} \psi_n'(t)^{1+p/2} dt < \infty$$

holds.

Proof. Let the condition (4) be satisfied. According to (3) we have for an arbitrary solution y

$$\begin{aligned} \int_a^b |y'(t)|^p dt &= \sum_{n=0}^{\infty} \int_{\psi_n(a)}^{\psi_{n+1}(a)} |y'(t)|^p dt = \sum_{n=0}^{\infty} \int_a^{\psi(a)} |y'(\psi_n)|^p \psi_n' dt = \\ &= \sum_{n=0}^{\infty} \int_a^{\psi(a)} \psi_n'^{1+p/2} \left(|y'(t)|^2 \frac{|q(\psi_n)|}{|q(t)|} \right)^{p/2} dt \leq M \sum_{n=0}^{\infty} \int_a^{\psi(a)} |q(\psi_n(t))|^{p/2} \psi_n'^{1+p/2} dt < \infty \end{aligned}$$

where

$$M = \max_{t \in [a, \psi(a)]} \left| \frac{y'^2(t)}{q(t)} \right|^{p/2}.$$

We can see that y' belongs to $L^p[a, b]$, $p > 0$. Let y' belong to $L^p[a, b]$, $p > 0$ for an arbitrary solution y . Let y_1, y_2 be two linearly independent solutions of (q) such that $y_1' \neq 0$ on $[a, t_1]$, $y_2' \neq 0$ on $[t_1, \psi(a)]$, $t_1 = (a + \psi(a))/2$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} \int_a^{\psi(a)} |q(\psi_n)|^{p/2} \psi_n'^{1+p/2} dt &= \sum_{n=0}^{\infty} \left[\int_a^{t_1} \left| \frac{y_1'(\psi_n)}{y_1'(t)} \right|^p |q(t)|^{p/2} \psi_n' dt + \right. \\ &\left. + \int_{t_1}^{\psi(a)} \left| \frac{y_2'(\psi_n)}{y_2'(t)} \right|^p |q(t)|^{p/2} \psi_n' dt \right] \leq M_1 \cdot \left(\int_a^b |y_1'|^p dt + \int_a^b |y_2'|^p dt \right) < \infty \end{aligned}$$

where

$$M_1 = \max \left(\max_{t \in [a, t_1]} \left| \frac{q(t)}{y_1'^2(t)} \right|^{p/2}, \max_{t \in [t_1, \psi(a)]} \left| \frac{q(t)}{y_2'^2(t)} \right|^{p/2} \right)$$

and we can see that the condition (4) is satisfied.

Lemma 1. Let (q) , $q \in C^0[a, b]$, $q(t) < 0$, $t \in [a, b]$ be oscillatory on $[a, b]$ and let y be an arbitrary solution. Let $\varphi_n, \psi_n, \chi_n, \omega_n$ be the n -th iteration of the dispersion $\varphi, \psi, \chi, \omega$ of the 1-st, 2-nd, 3-rd, 4-th kind, respectively. Let $t_0, t_1 \in [a, b]$, $y(t_0) = 0$, $y'(t_1) = 0$.

a) The solution y belongs to $L^p[a, b]$, $p > 0$ if and only if

$$(5) \quad \sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) < \infty$$

holds.

b) The derivative of y belongs to $L^p[a, b]$, $p > 0$ if and only if

$$(6) \quad \sum_{n=0}^{\infty} |y'(\omega_{n+1}(t_1))|^p (\psi_{n+1}(t_1) - \psi_n(t_1)) < \infty$$

holds.

Proof. a) Let y belong to $L^p[a, b]$, $p > 0$. Then

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) = \\ & = \sum_{n=0}^{\infty} \left[\int_{\varphi_n(t_0)}^{\chi_{n+1}(t_0)} (t - \varphi_n(t_0)) \frac{|y(\chi_{n+1}(t_0))|^p}{\chi_{n+1}(t_0) - \varphi_n(t_0)} dt + \right. \\ & \left. + \int_{\chi_{n+1}(t_0)}^{\varphi_{n+1}(t_0)} (t - \varphi_{n+1}(t_0)) \frac{|y(\chi_{n+1}(t_0))|^p}{\chi_{n+1}(t_0) - \varphi_{n+1}(t_0)} dt \right] \leq \sum_{n=0}^{\infty} \int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} |y(t)|^p dt < \infty \end{aligned}$$

(because $|y|^p$ has not smaller values on the interval $[\varphi_n(t_0), \chi_{n+1}(t_0)]$ or on $[\chi_{n+1}(t_0), \varphi_{n+1}(t_0)]$ than the function the graph of which is the line segment connecting the points $(\varphi_n(t_0), |y(\varphi_n(t_0))|^p)$ and $(\chi_{n+1}(t_0), |y(\chi_{n+1}(t_0))|^p)$ or $(\chi_{n+1}(t_0), |y(\chi_{n+1}(t_0))|^p)$ and $(\varphi_{n+1}(t_0), |y(\varphi_{n+1}(t_0))|^p)$, respectively. Thus we can see that (5) is valid.

Let (5) be valid. Then

$$\begin{aligned} \int_a^b |y(t)|^p dt &= M + \sum_{n=0}^{\infty} \int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} |y(t)|^p dt \leq M + \sum_{n=0}^{\infty} \int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} |y(\chi_{n+1}(t_0))|^p dt = \\ &= M + \sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) < \infty, \quad M = \int_a^{t_0} |y(t)|^p dt \end{aligned}$$

and the theorem is proved in this case.

b) The statement for y' can be proved in the same way. We only use ψ, ω instead of φ, χ .

Theorem 3. Let $(q), q \in C^0[a, \infty), q(t) < 0, t \in [a, \infty)$ be oscillatory on $[a, \infty), q$ monotone.

- a) If there exists a solution y belonging to $L^p[a, \infty), p > 0$, then $\lim_{t \rightarrow \infty} y(t) = 0$.
- b) If every solution belongs to $L^p[a, \infty), p > 0$, then every solution converges to zero for $t \rightarrow \infty$ and $\lim_{t \rightarrow \infty} q(t) = -\infty$.
- c) If there exists a solution y such that y' belongs to $L^p[a, \infty), p > 0$, then y' converges to zero for $t \rightarrow \infty$.
- d) If the derivative of every solution belongs to $L^p[a, \infty), p > 0$, then $\lim_{t \rightarrow \infty} q(t) = 0$ and the derivative of every solution tends to zero for $t \rightarrow \infty$.

Proof. a) Let y be a non-trivial solution of (q) such that $y \in L^p[a, \infty), p > 0$. Let $t_0 \in [a, \infty), y(t_0) = 0$. According to Theorem 1c) f) the sequence of absolute values of local extremes of y is monotone. Hence $\lim_{n \rightarrow \infty} |y(\chi_n(t_0))| = M \geq 0$ where χ_n is the n -th iteration of the dispersion χ of the 3-rd kind of (q) . If $M = 0$, then $\lim_{t \rightarrow \infty} y(t) = 0$. If $M \neq 0$, then there exists a constant M_1 such that $|y(\chi_n(t_0))| \geq M_1 > 0, n = 1, 2, \dots$ and according to Lemma 1 we have

$$\sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) \geq M_1 \sum_{n=0}^{\infty} (\varphi_{n+1}(t_0) - \varphi_n(t_0)) = \infty.$$

However, this contradicts our assumption.

- c) This case can be proved in the same way as a).
- b) d) The statement follows from a) c) and Theorem 1d) e).

Remark 1. A result of Bellman [3] § 6.8 concerns problems of this paragraph.

Let $a \in C^0[t_0, \infty), b \in C^0[t_0, \infty), |b(t)| \leq \text{const.} < \infty$ for $t \in [t_0, \infty)$. Let $p > 1$ be a number and $p' = p/(p-1)$. If every solution of $y'' = a(t)y$ belongs to $L^p[t_0, \infty)$ and $L^{p'}[t_0, \infty)$, then every solution of $y'' = (a(t) + b(t))y$ has the same property.

For $p = 2$ the statement $\lim_{t \rightarrow \infty} q(t) = -\infty$ from Theorem 3b) follows from this result by indirect proof: Let $\lim_{t \rightarrow \infty} q(t) = -C > -\infty$. Put $a(t) = q(t), b(t) = -1 - q(t)$. Then every solution of $y'' = -y$ belongs to $L^2[t_0, \infty)$ but this is not true.

3. In the last paragraph we shall prove some new results concerning the existence of integral $\int_a^b y(t) dt$ where y is a non-trivial solution of (q) .

Lemma 2. Let (q) , $q \in C^0[a, b]$ be oscillatory on $[a, b]$ and let y be its solution. Let φ_n be the n -th iteration of its dispersion φ of the 1-st kind. Let $t_0 \in [a, b)$. Then

$$(7) \quad \int_{t_0}^b y(t) dt = \sum_{n=0}^{\infty} (-1)^n \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2} y(t) dt.$$

Proof. According to (2) we have

$$\begin{aligned} \int_{t_0}^b y(t) dt &= \sum_{n=0}^{\infty} \int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} y(t) dt = \sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} y(\varphi_n(t)) \varphi_n'(t) dt = \\ &= \sum_{n=0}^{\infty} (-1)^n \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}(t) y(t) dt \end{aligned}$$

and thus the statement is valid.

Theorem 4. Let (q) , $q \in C^0[a, b)$, $b < \infty$ be a differential equation, q non-increasing, $\lim_{t \rightarrow b-} q(t) = -\infty$. Let y be an arbitrary solution of (q) . Then

$$(8) \quad \left| \int_a^b y(t) dt \right| = M_y = \text{const.} < \infty$$

holds.

Proof. As $\lim_{t \rightarrow b-} q(t) = -\infty$, the equation (q) is oscillatory on $[a, b)$. Let y be a non trivial solution of (q) . Let $t_0 \in [a, b)$, $y(t_0) = 0$, $q(t) < 0$, $t \in [t_0, b)$. As $\varphi' \leq 1$ on $[t_0, b)$ (see Theorem 1c)) we have

$$\left| \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2} y(t) dt \right| \leq \left| \int_{t_0}^{\varphi(t_0)} \varphi_{n-1}'(t) y(t) dt \right|, \quad n = 2, 3, \dots$$

and according to the alternating series test the infinite series in (7) converges if and only if

$$(9) \quad \lim_{n \rightarrow \infty} \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}(t) y(t) dt = 0$$

holds. Hence $\int_a^b y(t) dt$ converges iff the condition (9) is valid.

Let $c < 0$ be a number. As $\lim_{t \rightarrow b-} q(t) = -\infty$, there exists a number t_1 , $t_1 \in [a, b)$ such that $q(t) < c$, $t \in [t_1, b)$. Then the Sturm Comparison Theorem for the equations (q) and $y'' = c \cdot y$ implies $0 < \varphi(t) - t \leq \pi/\sqrt{-c}$. Thus $\lim_{t \rightarrow b-} (\varphi(t) - t) = 0$. According to Theorem 1a) c) an arbitrary solution of (q) is bounded on $[a, b)$ and