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A NOTE ON *STC*-GROUPOIDS

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Let G be a groupoid. We shall denote by L_a the left translation by $a \in G$ and by R_a the right translation, i.e., $L_a(x) = ax$ and $R_a(x) = xa$ for all $x \in G$. In his book [1] V. D. BELOUSOV introduced the class of quasigroups in which all mappings $S_{a,b} = L_b^{-1}L_a^{-1}L_{ab}$ are automorphisms. Such quasigroups were called SA-quasigroups by T. KEPKA and studied in [2]. The latter also introduced TA-quasigroups, i.e., quasigroups in which all mappings $T_{a,b} = R_a^{-1}R_b^{-1}R_{ab}$ are automorphisms. SA-quasigroups and TA-quasigroups having the property that there is an Abelian group $Q(+)$, its automorphisms f, g and $x \in Q$ such that $ab = f(a) + g(b) + x$ for all $a, b \in Q$ were described by T. Kepka and P. Němec in [3]. Here we make an attempt to generalize these ideas for groupoids. In the first part we give basic definitions and some elementary assertions, in the second part we study the basic properties of STC-groupoids. In the third section we prove, following the ideas of [2], some theorems concerning the Cartesian decomposition of STC-groupoids, and in the last section we apply our results to some classes of groupoids.

1. INTRODUCTION

Let G be a groupoid. We shall say that G is

- an LC-groupoid (LD-groupoid) if for all $a \in G$ the mapping L_a is one-to-one (onto),
- an RC-groupoid (RD-groupoid) if for all $a \in G$ the mapping R_a is one-to-one (onto),
- a C-groupoid if it is simultaneously an LC- and RC-groupoid,
- a D-groupoid if it is simultaneously an LD- and RD-groupoid,
- an S-groupoid if for all $a, b \in G$ there is an endomorphism $S_{a,b}$ such that $L_{ab} = L_aL_bS_{a,b}$,
- a T-groupoid if for all $a, b \in G$ there is an endomorphism $T_{a,b}$ such that $R_{ab} = R_bR_aT_{a,b}$,

– an SF-groupoid (SH-groupoid) if it is an S-groupoid and endomorphisms $S_{x,y}$ can be chosen so that $S_{a,b} = S_{a,c}$ ($S_{b,a} = S_{c,a}$) for all $a, b, c \in G$,
– a TF-groupoid (TH-groupoid) if it is a T-groupoid and endomorphisms $T_{x,y}$ can be chosen so that $T_{b,a} = T_{c,a}$ ($T_{a,b} = T_{a,c}$) for all $a, b, c \in G$,
– a B_1-groupoid if $a \,.\, bc = b \,.\, ac$ for all $a, b, c \in G$,
– a B_2-groupoid if $ab \,.\, c = ac \,.\, b$ for all $a, b, c \in G$,
– Abelian if $ab \,.\, cd = ac \,.\, bd$ for all $a, b, c, d \in G$,
– left distributive if $a \,.\, bc = ab \,.\, ac$ for all $a, b, c \in G$,
– right distributive if $ab \,.\, c = ac \,.\, bc$ for all $a, b, c \in G$,
– distributive if it is both left and right distributive.

An element $e \in G$ is idempotent if $ee = e$. The set of idempotent elements will be denoted by Id G, and we define further

$$E(G) = \{a \in G \mid \text{there is } b \in G \text{ such that } ba = b\}\,,$$

$$F(G) = \{a \in G \mid \text{there is } b \in G \text{ such that } ab = b\}\,,$$

$$G_a = \{b \in G \mid ba = b\}\,, \quad {}_aG = \{b \in G \mid ab = b\}\,.$$

An equivalence η on a groupoid G is called a congruence (normal congruence) if for all $a, b, c \in G$, $a \,\eta\, b$ implies $ac \,\eta\, bc$ and $ca \,\eta\, cb$ (moreover, $ca \,\eta\, cb$ implies $a \,\eta\, b$ and $ac \,\eta\, bc$ implies $a \,\eta\, b$). If G is a C-groupoid and f a homomorphism of G into a groupoid H then $f(G)$ is a C-groupoid iff the relation η defined by $a \,\eta\, b \Leftrightarrow f(a) = f(b)$ is a normal congruence on G.

Obviously, all semigroups and all distributive quasigroups are ST-groupoids (in general, if G is simultaneously an X-groupoid and a Y-groupoid then we shall say that G is an XY-groupoid). If Q is a left distributive quasigroup which is not right distributive then Q is an S-groupoid which is not a T-groupoid. An example of such quasigroup can be found in [1] or [4].

1.1. Proposition. *The Cartesian product of any system of S-groupoids is an S-groupoid.*

Proof. Obvious.

1.2. Proposition. *Let G be an S-groupoid, H an LC-groupoid and f a homomorphism of G into H. Then $f(G)$ is an SLC-groupoid.*

Proof. Let $x, y, z \in f(G)$ be arbitrary. We have $x = f(a)$, $y = f(b)$, $z = f(c)$ for properly chosen $a, b, c \in G$, so that $xy \,.\, z = x \,.\, (y \,.\, f(S_{a,b}(c)))$. As $f(G)$ is an LC-groupoid, for all $d \in G$ such that $f(d) = f(c) = z$ we get $f(S_{a,b}(c)) = f(S_{a,b}(d))$. Hence we can define $S_{x,y}(z) = f(S_{a,b}(c))$. If further $u \in f(G)$ and $e \in G$ are such that $f(e) = u$ then $S_{x,y}(zu) = f(S_{a,b}(ce)) = f(S_{a,b}(c) \,.\, S_{a,b}(e)) = f(S_{a,b}(c)) \,.\, f(S_{a,b}(e)) = S_{x,y}(z) \,.\, S_{x,y}(u)$. Thus the mapping $S_{x,y}$ is an endomorphism of the groupoid $f(G)$.

1.3. Proposition. *If G is an SLC-groupoid then all mappings $S_{a,b}$ are uniquely determined and one-to-one.*

Proof. If $ab \,.\, c = a \,.\, bd = a \,.\, be$ then $d = e$, all mappings L_x being one-to-one. Further, if $S_{a,b}(c) = S_{a,b}(d)$ then $L_{ab}(c) = a \,.\, (b \,.\, S_{a,b}(c)) = a \,.\, (b \,.\, S_{a,b}(d)) = = ab \,.\, d = L_{ab}(d)$ and hence $c = d$.

1.4. Proposition. *Let G be an SLC-groupoid and H its subgroupoid. Then H is an SLC-groupoid iff $S_{a,b}(c) \in H$ for all $a, b, c \in H$.*

Proof. The "if" part is obviously true whenever G is an S-groupoid. The "only if" part follows easily from the fact that all mappings L_a are one-to-one.

1.5. Proposition. *Let G be an SLC-groupoid and H its subgroupoid having the following property:*

If $a, b \in H$ and $x \in G$ such that $ax = b$ then $x \in H$.

Then H is an SLC-groupoid.

Proof. Let $a, b, c \in H$ be arbitrary. Then $ab \,.\, c = a \,.\, (b \,.\, S_{a,b}(c)) \in H$, hence $S_{a,b}(c) \in H$ and we can use Proposition 1.4.

1.6. Proposition. *Let G be an SLC-groupoid and $a, b, c \in G$. Then $S_{a,b}(c) \in \mathrm{Id}\, G$ iff $c \in \mathrm{Id}\, G$.*

Proof. If $S_{a,b}(c) \in \mathrm{Id}\, G$ then $S_{a,b}(cc) = S_{a,b}(c)$, so that, by Proposition 1.3, $cc = c$. The converse being obvious, the proof is completed.

1.7. Proposition. *Let G be an SLC-groupoid, $\mathrm{Id}\, G = \{r\}$ and let R_r be onto. Then $ar = a$ for all $a \in G$.*

Proof. For every $a \in G$ there is $b \in G$ such that $a = br$. By Proposition 1.6, $ar = br \,.\, r = b \,.\, (r \,.\, S_{b,r}(r)) = b \,.\, rr = br = a$.

1.8. Proposition. *Let G be an SRD-groupoid, $e \in G$ such that $ea = a$ for all $a \in G$ and let all mappings $S_{a,b}$ be onto. Then G is an LC-groupoid.*

Proof. Let $a \in G$ be arbitrary and $b \in G$ such that $ba = e$. Then for every $c \in G$, $c = ba \,.\, c = b \,.\, (a \,.\, S_{b,a}(c))$, and hence $L_b L_a S_{b,a} = 1_G$, where 1_G is the identical mapping of G onto G. As $S_{b,a}$ is onto, L_a is one-to-one.

The dual assertions for T-groupoids can be proved analogously.

1.9. Proposition. *Let G be an STD-groupoid with unit. If for all $a, b \in G$ the mappings $S_{a,b}$, $T_{a,b}$ are onto then G is a quasigroup.*

Proof. This is an immediate consequence of Proposition 1.8 and its dual.

2.1. Lemma. *Let G be an SLC-groupoid. Then*

(i) *L_e is an automorphism of G for all $e \in E(G)$.*
(ii) *If* Id $G \neq \emptyset$ *then* Id G *is a left distributive LDLC-groupoid.*
(iii) *If $a \in G$ and $r \in$* Id G *then $S_{r,a} = L_r^{-1}$.*
(iv) *If, moreover, G is an RC-groupoid then*

$$\text{Id } G = E(G) \subseteq F(G).$$

Proof. (i) If $e \in E(G)$ then there is $a \in G$ such that $ae = a$. Then $L_a = L_{ae} = L_aL_eS_{a,e}$, hence $L_eS_{a,e} = 1_G$, L_a being one-to-one, and so L_e is a one-to-one mapping of G onto G. Thus $S_{a,e} = L_e^{-1}$, and consequently L_e is an automorphism of G.

(ii) Obviously Id $G \subseteq E(G) \cap F(G)$. Let $r, s \in$ Id G be arbitrary. Since $rs = L_r(s)$, we have $rs \in$ Id G by (i). Further, there is $t \in G$ such that $L_r(t) = s$. But $r \,.\, tt = rt \,.\, rt = ss = s = rt$, and therefore $t \in$ Id G.

(iii) For all $c \in G$, $ra \,.\, c = r \,.\, (a \,.\, L_r^{-1}(c))$ by (i).

(iv) If G is a C-groupoid and $e \in E(G)$ then L_e is an automorphism of G, $e \,.\, ee = ee \,.\, ee$, and hence $e = ee$.

2.2. Lemma. *Let G be a TRC-groupoid. Then*

(i) *R_f is an automorphism of G for all $f \in F(G)$.*
(ii) *If* Id $G \neq \emptyset$ *then* Id G *is a right distributive RDRC-groupoid.*
(iii) *If $a \in G$ and $r \in$* Id G *then $T_{a,r} = R_r^{-1}$.*
(iv) *If, moreover, G is an LC-groupoid then*

$$\text{Id } G = F(G) \subseteq E(G).$$

Proof. Dual to that of Lemma 2.1.

2.3. Theorem. *Let G be an STC-groupoid. Then*

(i) Id $G = E(G) = F(G)$.
(ii) *If* Id $G \neq \emptyset$ *then* Id G *is a distributive quasigroup.*
(iii) *For all $r \in$* Id G, *L_r and R_r are automorphisms.*
(iv) *If $r, s \in$* Id G *and $G_r \cap {}_sG \neq \emptyset$ then $r = s$.*

Proof. With respect to Lemma 2.1 and Lemma 2.2, it remains only to prove the assertion (iv). We have $xr = sx = x$ for some $x \in G$. Hence $xr = s \,.\, xr = sx \,.\, sr = x \,.\, sr$, so that $sr = r$. Thus $r = s$.

2.4. Corollary. *Let G be a groupoid. Then G is an idempotent STC-groupoid iff it is a distributive quasigroup.*

Now we are in position to show that, in general, STC-groupoids are not closed under subgroupoids. Indeed, let Q be a distributive quasigroup and G its subgroupoid which is not a quasigroup. Then G is not an STC-groupoid.

2.5. Proposition. *Let G be an STC-groupoid such that* $\mathrm{Id}\ G = \emptyset$. *Then there exists a groupoid H with the following properties:*

(i) *H is an STC-groupoid with unit.*
(ii) *G is a subgroupoid in H.*
(iii) $\mathrm{card}\,(H \setminus G) = 1$.

Proof. Let $e \notin G$ be arbitrary and define a binary operation $+$ on $H = G \cup \{e\}$ by $a + b = ab$ for $a, b \in G$ and $c + e = e + c = c$ for $c \in H$. It is an easy exercise to show that $H(+)$ has the desired properties.

2.6. Definition. Let G be an SLC-groupoid. We shall say that G satisfies the condition (P_S) if $\mathrm{Id}\ G \neq \emptyset$ and for all $a, b \in G$ the mapping $S_{a,b} \mid \mathrm{Id}\ G$ is a permutation of $\mathrm{Id}\ G$. Let G be a TRC-groupoid. We shall say that G satisfies the condition (P_T) if $\mathrm{Id}\ G \neq \emptyset$ and for all $a, b \in G$ the mapping $T_{a,b} \mid \mathrm{Id}\ G$ is a permutation of the set $\mathrm{Id}\ G$. Let G be an STC-groupoid. We shall say that G satisfies the condition (P) if it satisfies (P_S) and (P_T).

2.7. Proposition. *Let G be an SLC-groupoid such that* $\mathrm{Id}\ G \neq \emptyset$ *and at least one of the following two conditions holds:*

(i) *For all $a, b \in G$, $S_{a,b}$ is an automorphism of G.*
(ii) $\mathrm{Id}\ G$ *is finite.*

Then G satisfies (P_S).

The assertion for (P_T) and (P) are analogous.

2.8. Theorem. *Let G be an STC-groupoid. If G satisfies (P_S) then there is a uniquely determined mapping e_G of G into* $\mathrm{Id}\ G$ *such that $a \,.\, e_G(a) = a$ for all $a \in G$. If G satisfies (P_T) then there is a uniquely determined mapping f_G of G into* $\mathrm{Id}\ G$ *such that $f_G(a) \,.\, a = a$ for all $a \in G$. Moreover, if G satisfies (P) then $e_G = f_G$.*

Proof. Let $a \in G$ and $r \in \mathrm{Id}\ G$ be arbitrary. As in view of Theorem 2.3 the mappings R_r, L_r are onto, there are $b, c \in G$ such that $br = a = rc$. If G satisfies (P_S) then $a = b \,.\, rr = br \,.\, S_{b,r}^{-1}(r) = a \,.\, S_{b,r}^{-1}(r)$, where $S_{b,r}^{-1}$ is the inverse mapping to $S_{b,r} \mid \mathrm{Id}\ G$. Similarly, if G satisfies (P_T) then $T_{r,c}^{-1}(r) \,.\, a = a$. Finally, if G satisfies (P) then $T_{r,c}^{-1}(r) = S_{b,r}^{-1}(r)$ by Theorem 2.3 (iv).

2.9. Proposition. *Let G be an STC-groupoid. Then G is a groupoid with unit iff* $\mathrm{card}\,(\mathrm{Id}\ G) = 1$.

Proof. It follows immediately from Theorem 2.3 and Proposition 1.7 (and its dual).

2.10. Proposition. *Let G be an STC-groupoid such that at least one of the following two conditions holds:*

(i) *There is $e \in G$ such that $ae = a$ for all $a \in G$.*
(ii) *There is $f \in G$ such that $fa = a$ for all $a \in G$.*

Then G is a groupoid with unit.

Proof. Let $r \in \mathrm{Id}\, G$ be arbitrary. Then $rr = r = re$, so that $r = e$, and hence $\mathrm{card}\,(\mathrm{Id}\, G) = 1$. Application of Proposition 2.9 completes the proof (which is analogous if the condition (ii) is assumed).

2.11. Proposition. *Let G be an STC-groupoid and $r \in \mathrm{Id}\, G$. Then*

(i) *G_r is an STC-groupoid with unit,*
(ii) *$G_r = {}_rG$,*
(iii) *For every $s \in \mathrm{Id}\, G$, $G_r \cong G_s$.*

Proof. Let $a, b \in G_r$ be arbitrary. Then $ab \,.\, r = ar \,.\, br = ab$, so that $ab \in G_r$. Let further $x, y \in G$ be such that $ax = b = ya$. Then $ax = b = br = ax \,.\, r = ar \,.\, xr = a \,.\, xr$, $ya = b = br = ya \,.\, r = yr \,.\, ar = yr \,.\, a$, and therefore $x, y \in G_r$. By Proposition 1.5, its dual and Proposition 2.10, G_r is an *STC*-groupoid and r is its unit. Hence $G_r \subseteq {}_rG$ and similarly we can prove ${}_rG \subseteq G_r$. Further, let $s \in \mathrm{Id}\, G$ be arbitrary. There is $t \in \mathrm{Id}\, G$ with $rt = s$. For all $c \in G_r$ we have $ct \,.\, s = ct \,.\, rt = cr \,.\, t = ct$, and hence $ct \in G_s$. On the contrary, if $d \in G_s$ then $R_t^{-1}(d) \,.\, t = d = ds = (R_t^{-1}(d) \,.\, t) \,.\, rt = (R_t^{-1}(d) \,.\, r) \,.\, t$. Thus R_t is the isomorphism which we have sought.

3. CARTESIAN DECOMPOSITION OF *STC*-GROUPOIDS

3.1. Theorem. *Let G be an STC-groupoid satisfying the condition (P). Then there exists a normal congruence μ on G such that $\mathrm{Id}\, G$ is one of its classes. Moreover, $G/\mu \cong G_r$ for all $r \in \mathrm{Id}\, G$.*

Proof. Let $a \in G$ and $r \in \mathrm{Id}\, G$ be arbitrary. By Theorem 2.8 and Theorem 2.3, there are $e(a), s, u \in \mathrm{Id}\, G$ with $a \,.\, e(a) = e(a) \,.\, a = a$, $e(a) \,.\, s = r = u \,.\, e(a)$. Hence $as = (e(a) \,.\, a) \,.\, s = (e(a) \,.\, s) \,.\, (a \,.\, s) = r \,.\, as = ra \,.\, rs$, $ua = u \,.\, (a \,.\, e(a)) = (u \,.\, a) \,.\, (u \,.\, e(a)) = ua \,.\, r = ur \,.\, ar$. Further, there are $t, v \in \mathrm{Id}\, G$ such that $rs \,.\, t = r \,.\, e(a)$, $v \,.\, ur = e(a) \,.\, r$. Then $ra = r \,.\, (a \,.\, e(a)) = ra \,.\, (r \,.\, e(a)) = ra \,.\, (rs \,.\, t) = (ra \,.\, rs) \,.\, S_{ra,rs}^{-1}(t)$, $ar = (e(a) \,.\, a) \,.\, r = (e(a) \,.\, r) \,.\, ar = (v \,.\, ur) \,.\, ar = T_{ur,ar}^{-1}(v) \,.\, (ur \,.\, ar)$. Therefore $ra = as \,.\, S_{ra,rs}^{-1}(t) = ax$, $ar = T_{ur,ar}^{-1}(v) \,.\, ua = ya$, where $x, y \in \mathrm{Id}\, G$ by Proposition 1.3. Thus Id $G \,.\, a = a \,.\,$ Id G. Now we shall construct a homomorphism f of G onto G_r. If $a \in G$ then there is (uniquely determined) $g(a) \in$ Id G such that $e(a) \,.\, g(a) = r$. Put $f(a) = a \,.\, g(a)$. By essentially the same argument as in [2], Theorem 3, we can show (using the fact that Id $G \,.\, a = a \,.\,$ Id G for all $a \in G$) that f is a homomorphism of G onto G_r and Id G is one of the classes of the corresponding normal congruence μ.

If G is an *STC*-groupoid satisfying the condition (P) then, by Theorem 2.8, for every $a \in G$ there is (uniquely determined) $e_G(a) \in$ Id G with $e_G(a) \,.\, a = a \,.\, e_G(a) = a$.

3.2. Theorem. *Let G be an STC-groupoid. Then $G \cong D \times E$, D being a distributive quasigroup and E an STC-groupoid with unit, iff G satisfies the condition (P) and the mapping e_G is an endomorphism of G. In this case, $G \cong$ Id $G \times G_r$ for all $r \in$ Id G.*

Proof. Let E be an *STC*-groupoid with unit e, D a distributive quasigroup and $h : G \to D \times E$ an isomorphism. Then, obviously, G satisfies the condition (P). Let $(a, b) \in D \times E$. Then $e_{D \times E}(a, b) = (a, e)$, and hence $e_{D \times E}$ is an endomorphism of $D \times E$. As $e_G = h^{-1} e_{D \times E} h$, e_G is an endomorphism of G. On the other hand, let G satisfy the condition (P) and let the mapping e_G be an endomorphism of G. Let further $r \in$ Id G be arbitrary. We shall define $h : G \to$ Id $G \times G_r$ by $h(a) = (e_G(a), f(a))$, where f is the homomorphism of G onto G_r defined in the proof of Theorem 3.1. If $h(a) = h(b)$, then $e_G(a) = e_G(b)$ and $a \,.\, g(a) = b \,.\, g(b)$. Since $e_G(a) \,.\, g(a) = r = e_G(b) \,.\, g(b)$, we have $g(a) = g(b)$ and $a = b$. Further, let $s \in$ Id G and $a \in G_r$ be arbitrary. There are $t \in$ Id G, $b \in G$ with $st = r$ and $bt = a$. Hence $bt = a = ar = bt \,.\, st = bs \,.\, t$. Thus $e_G(b) = s$, $g(b) = t$, $f(b) = a$, and therefore $h(b) = (r, a)$. Since h is obviously a homomorphism, the proof is complete.

3.3. Lemma. *Let G be an STC-groupoid satisfying the condition (P). Define a relation η on G by $a \,\eta\, b \Leftrightarrow e_G(a) = e_G(b)$. The relation η is a congruence on G iff e_G is an endomorphism of G. In this case, η is a normal congruence.*

Proof. Let e_G be an endomorphism and $a \,\eta\, b$, $c \,\eta\, d$. Then $e_G(ac) = e_G(a) \,.\, e_G(c) = e_G(b) \,.\, e_G(d) = e_G(bd)$ so that $ac \,\eta\, bd$. If $ac \,\eta\, bc$ then we have $e_G(a) \,.\, e_G(c) = e_G(b) \,.\, e_G(c)$, and hence $a \,\eta\, b$. Conversely, let η be a congruence on G and let $a, b \in G$ be arbitrary. Obviously $ab \,\eta\, e_G(ab)$, $a \,\eta\, e_G(a)$, $b \,\eta\, e_G(b)$ so that $e_G(ab) \,\eta\, e_G(a) \,.\, e_G(b)$, and therefore $e_G(e_G(ab)) = e_G(e_G(a) \,.\, e_G(b))$. But $e_G(ab)$, $e_G(a)$, $e_G(b) \in$ Id G, hence $e_G(ab) = e_G(a) \,.\, e_G(b)$.

3.4. Theorem. *Let G be an STC-groupoid. Then $G \cong D \times E$, E being an STC-groupoid with unit and D a distributive quasigroup, iff G satisfies the condition (P) and there are a congruence ν on G and $r \in$ Id G such that G_r is one of the classes of ν. In this case, $G \cong$ Id $G \times G_r$.*

Proof. Let $G \cong D \times E$. Then the statement follows immediately from Theorem 3.2 and Lemma 3.3. On the contrary, let G satisfy the condition (P), let v be a congruence on G and let $r \in \mathrm{Id}\, G$ be such that G_r is one of its classes. Let $a \in G$ be arbitrary, $s \in \mathrm{Id}\, G$ such that $rs = e_G(a)$ and $c \in G$ with $cs = a$. Then $cs = a = a \,.\, e_G(a) = cs \,.\, rs = cr \,.\, s$ so that $c\; v\; r$. Since v is a congruence, we have $a\; v\; e_G(a)$. Thus if a, b are arbitrary elements of G such that $a\; v\; b$ then $e_G(a)\; v\; e_G(b)$. There is $t \in \mathrm{Id}\, G$ with $e_G(a) \,.\, t = r$. Hence we get $e_G(b) \,.\, t \in G_r$ so that $e_G(b) \,.\, t = r = e_G(a) \,.\, t$, since $e_G(b)$, t are idempotent, and therefore $a\; \eta\; b$, where η is the relation defined in Lemma 3.3. Further, if $a\; \eta\; b$ then $e_G(a) = e_G(b)$ and hence $a\; v\; e_G(a)\; v\; e_G(b)\; v\; b$. Thus $\eta = v$. Application of Lemma 3.3 and Theorem 3.2 completes the proof.

4. *STC*-GROUPOIDS OF SOME CLASSES

4.1. Theorem. *The following two conditions for a groupoid G are equivalent:*

(i) *G is an Abelian STC-groupoid such that for every $a \in G$ there are $e(a), f(a) \in G$ with $a \,.\, e(a) = f(a) \,.\, a = a$.*

(ii) *$G \cong D \times S$, where S is a commutative C-semigroup with unit and D is an idempotent Abelian quasigroup.*

Proof. Let G satisfy (i) and let $a, b \in G$ and $r \in \mathrm{Id}\, G$ be arbitrary. Then $a \,.\, br = (a \,.\, e(a)) \,.\, br = ab \,.\, (e(a) \,.\, r)$, $ra \,.\, b = ra \,.\, (f(b) \,.\, b) = (r \,.\, f(b)) \,.\, ab$, and hence $r = S_{a,b}(e(a) \,.\, r) = T_{a,b}(r \,.\, f(b))$. Thus G satisfies the condition (P). Further, $ab = (a \,.\, e(a)) \,.\, (b \,.\, e(b)) = ab \,.\, (e(a) \,.\, e(b))$ so that $e(ab) = e(a) \,.\, e(b)$. Now, application of Theorem 3.2 (and the simple facts that an Abelian groupoid with unit is a commutative semigroup and an Abelian quasigroup is distributive iff it is idempotent) completes the proof, since (i) follows from (ii) trivially.

4.2. Proposition. *The following conditions for a groupoid G are equivalent:*

(i) *G is an SCB_1-groupoid and there is $r \in \mathrm{Id}\, G$ with R_r onto.*

(ii) *G is a TCB_2-groupoid and there is $r \in \mathrm{Id}\, G$ with L_r onto.*

(iii) *G is a commutative C-semigroup with unit.*

Proof. (i) $\Leftrightarrow$ (iii). Let G be an SCB_1-groupoid and let $s \in \mathrm{Id}\, G$ be arbitrary. Then $r \,.\, sr = s \,.\, rr = sr$ and hence $r = s$. By Proposition 1.7, for all $a, b, c \in G$, $ab = a \,.\, br = b \,.\, ar = ba$ and consequently, $a \,.\, bc = b \,.\, ac = b \,.\, ca = c \,.\, ba = ab \,.\, c$. The converse is obvious.

(ii) $\Leftrightarrow$ (iii) can be proved similarly.

4.3. Lemma. *Let G be an SFLC-groupoid with unit e. Then G is a semigroup.*

Proof. For all $a, b \in G$, $S_{a,b} = S_{a,e} = 1_G$.

4.4. Lemma. *Every idempotent SLC-groupoid is an SF-groupoid.*

Proof. Let $a, b, c \in G$ be arbitrary. Then, by Lemma 2.1, $S_{a,b} = L_a^{-1} = S_{a,c}$.

4.5. Lemma. *Let G be an SFLC-groupoid such that for every $s \in G$ there is $e(a) \in G$ with $a \,.\, e(a) = a$. Then G satisfies the condition (P_S) and $S_{a,b} = L_{e(a)}^{-1}$ for all $a, b \in G$.*

Proof. Let $a, b \in G$ be arbitrary. Then $L_a = L_{a.e(a)} = L_a L_{e(a)} S_{a,e(a)}$, so that $L_{e(a)} S_{a,e(a)} = 1_G$ and $S_{a,b} = S_{a,e(a)} = L_{e(a)}^{-1}$. Application of Lemma 2.1 completes the proof.

Similarly we can prove the dual results for *TF*-groupoids.

4.6. Theorem. *The following two conditions for a groupoid G are equivalent:*

(i) *G is an SFTFC-groupoid such that for every $a \in G$ there are $e(a), f(a) \in G$ with $a \,.\, e(a) = f(a) \,.\, a = a$.*

(ii) *$G \cong D \times S$, where D is a distributive quasigroup and S is a C-semigroup with unit.*

Proof. (i) $\Rightarrow$ (ii). Lemma 4.5 and its dual guarantee that G satisfies the condition (P) and $ab \,.\, (e(a) \,.\, e(b)) = a \,.\, (b \,.\, S_{a,b}(e(a) \,.\, e(b))) = a \,.\, (b \,.\, L_{e(a)}^{-1} L_{e(a)}(e(b))) = ab$. Now we can use Theorem 3.2 and Lemma 4.3.

(ii) $\Rightarrow$ (i). This is obvious with respect to Lemma 4.4 and its dual.

4.7. Lemma. *Let G be an SHLC-groupoid. If there is $r \in \operatorname{Id} G$ with R_r being one-to-one then $\operatorname{Id} G = \{r\}$.*

Proof. For every $s \in \operatorname{Id} G$, $L_s^{-1} = S_{s,r} = S_{r,r} = L_r^{-1}$ by Lemma 2.1 (iii). Thus $sr = rr$ and $s = r$.

4.8. Proposition. *Let G be an SHLC-groupoid. If there is $r \in \operatorname{Id} G$ such that R_r is a permutation of G then $ar = a$ for all $a \in G$.*

Proof. By Lemma 4.7 and Proposition 1.7.

4.9. Theorem. *The following three conditions for a groupoid G are equivalent:*

(i) *G is an SHTC-groupoid and $\operatorname{Id} G \neq \emptyset$.*

(ii) *G is an STHC-groupoid and $\operatorname{Id} G \neq \emptyset$.*

(iii) *G is a C-semigroup with unit.*

Proof. (i) $\Leftrightarrow$ (iii). Let G be an *SHTC*-groupoid and $e \in \operatorname{Id} G$. By Lemma 4.7,