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MEASURABILITY OF FUNCTIONS WITH VALUES IN PARTIALLY ORDERED SPACES

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INTRODUCTION

There are many ways in which the measurability of real-valued functions can be defined. Some of them can be adopted for functions with values in a partially ordered set but then cease to be equivalent. In the present paper we discuss several types of measurability of such functions, study their interrelations and also the mathematical structure of some of such classes of functions under certain additional conditions on the image space Y, e.g. if Y is a linear space or a lattice.

1. NOTATION AND NOTIONS

Throughout the paper, $\mathscr{S}$ will denote a σ-algebra of subsets of an abstract set $X \in \mathscr{S}$ and Y will be a set partially ordered by $\leqq$ which is a binary relation satisfying

(u_1) $(\forall a \in Y)\, a \leqq a$

(u_2) $(\forall a, b, c \in Y)\,((a \leqq b \wedge b \leqq c) \Rightarrow a \leqq c)$

(u_3) $(\forall a, b \in Y)\,((a \leqq b \wedge b \leqq a) \Rightarrow a = b)$.

We shall use the notation $a < b$ iff $a \leqq b$ and $a \neq b$. Given $a \in Y$, symbols $[y \leqq a]$, $[y \geqq a]$, $[y < a]$ and $[y > a]$ will denote the sets $\{y \in Y;\ y \leqq a\}$, $\{y \in Y;\ y \geqq a\}$, $\{y \in Y;\ y < a\}$ and $\{y \in Y;\ y > a\}$, respectively.

In some results, Y will be assumed to be a partially ordered linear space, that is, a real linear space with a partial ordering $\leqq$ satisfying, besides (u_1), (u_2) and (u_3), also

(v_1) $(\forall a, b, c \in Y)\,(a \leqq b \Rightarrow a + c \leqq b + c)$

(v_2) $(\forall a, b \in Y)\,(\forall t \in R)\,((t > 0 \wedge a \leqq b) \Rightarrow ta \leqq tb)$

where R denotes the field of reals.

We shall also consider the order topology $\mathscr{G}$ on Y whose subbasis consists of all sets $[y < a]$ and $[y > a]$, for $a \in Y$. The topology $\mathscr{G}$ enables us to define dense subsets of Y and to declare Y separable iff there is a countable dense subset of Y.

We say that a partially ordered set Y is conditionally σ-complete iff every countable set $Z \subset Y$ bounded from above by some $b \in Y$ (i.e. $(\forall y \in Z)\ y \leqq b$) has a supremum.

If f is a function defined on X with values in Y and M is a subset of Y, then by $f^{-1}M$ we shall denote the set $\{x \in X; f(x) \in M\}$.

Generalizing the concepts introduced in [1] and [2]*) we define the classes of upper, lower and weakly measurable functions respectively as

$$\mathscr{M}_< = \{f : X \to Y;\ (\forall a \in Y) f^{-1}[y < a] \in \mathscr{S}\}$$
$$\mathscr{M}_> = \{f : X \to Y;\ (\forall a \in Y) f^{-1}[y > a] \in \mathscr{S}\}$$
$$\mathscr{M}_w = \mathscr{M}_< \cap \mathscr{M}_> .$$

Analogously we define classes $\mathscr{M}_=$, $\mathscr{M}_\leqq$ and $\mathscr{M}_\geqq$, of which the last two in the case of a linearly ordered Y coincide with $\mathscr{M}_>$ and $\mathscr{M}_<$ respectively. We put $\mathscr{M} = \mathscr{M}_\leqq \cap \mathscr{M}_\geqq$.

Interpreting the family of Borel sets in Y as the σ-algebra $\sigma(\mathscr{G})$ generated by the order topology $\mathscr{G}$ we may introduce Borel measurable functions as those in $\mathscr{B} = \{f : X \to Y;\ (\forall G \in \mathscr{G}) f^{-1}G \in \mathscr{S}\}$.

In order to define yet some other types of measurability we need the notion of simple function. A function $f : X \to Y$ is called simple iff its range is a finite set $F \subset Y$ and for every $y \in Y$ we have $f^{-1}\{y\} \in \mathscr{S}$. The collection of all simple functions $f : X \to Y$ will be denoted by $\mathscr{J}$.

If f and f_n $(n = 1, 2, \ldots)$ are functions from X into Y, we write $f_n \nearrow f$ $(f_n \searrow f)$ iff for every $x \in X$ we have $f_n(x) \leqq f_{n+1}(x)$ $(f_n(x) \geqq f_{n+1}(x))$, $n = 1, 2, \ldots$ and $f(x) = \sup_n f_n(x)$ $(f(x) = \inf_n f_n(x))$. Now define

$$M_\nearrow = \{f : X \to Y;\ (\exists f_n \text{ simple}, n = 1, 2, \ldots) f_n \nearrow f\},$$
$$M_\searrow = \{f : X \to Y;\ (\exists f_n \text{ simple}, n = 1, 2, \ldots) f_n \searrow f\}.$$

The functions in $\mathscr{M}_s = \mathscr{M}_\nearrow \cap \mathscr{M}_\searrow$ will be called strongly measurable.

It is easy to observe that any class of Y-valued functions is partially ordered, if we put $f \leqq g$ iff $(\forall x \in X)\ f(x) \leqq g(x)$.

2. GENERAL RESULTS

Theorem 2.1. *For any partially ordered set Y we have*

$$\mathscr{M} = \mathscr{M}_w \cap \mathscr{M}_= .$$

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*) See also [3], part 1, example 2.

Proof. Let $f \in \mathcal{M} = \mathcal{M}_{\leqq} \cap \mathcal{M}_{\geqq}$. Then $f^{-1}[y < a] = f^{-1}[y \leqq a] \setminus f^{-1}[y \geqq \geqq a] \in \mathcal{S}$, hence $f \in \mathcal{M}_{<}$. Since analogously $f \in \mathcal{M}_{>}$, we infer $f \in \mathcal{M}_{w}$. To prove $f \in \mathcal{M}_{=}$ we realize that $f^{-1}\{a\} = f^{-1}[y \leqq a] \cap f^{-1}[y \geqq a]$.

For the converse inclusion, let $f \in \mathcal{M}_w \cap \mathcal{M}_{=}$. Then $f^{-1}[y \leqq a] = f^{-1}[y < a] \cup \cup f^{-1}\{a\} \in \mathcal{S}$ and analogously $f^{-1}[y \geqq a] \in \mathcal{S}$, therefore $f \in \mathcal{M}_{\leqq} \cap \mathcal{M}_{\geqq} = \mathcal{M}$.

Theorem 2.2. *For any partially ordered set Y we have*

$$\mathcal{B} \subset \mathcal{M}_w .$$

Proof. Since all the sets $[y < a]$ and $[y > a]$ are open, they are also Borel sets and hence for any Borel measurable function we have $f^{-1}[y < a] \in \mathcal{S}$ and $f^{-1}[y > > a] \in \mathcal{S}$.

As shown by Example 3.1 of [1], $\mathcal{B} = \mathcal{M}_w$ does not hold in general. Nevertheless, the following theorem states a sufficient condition for the two classes to coincide.

Theorem 2.3. *Let Y be such a partially ordered set that the order topology $\mathcal{G}$ has a countable basis. Then*

$$\mathcal{B} = \mathcal{M}_w .$$

Proof. In view of Theorem 2.2 it is sufficient to prove that for every $f \in \mathcal{M}_w$ and any $U \in \mathcal{G}$ we have $f^{-1}U \in \mathcal{S}$. Since $\mathcal{G}$ has a countable basis, the open set U can be written in the form $U = \bigcup_{n=1}^{\infty} U_n$ where U_n are finite intersections of sets of the type $[y < a]$ or $[y > a]$. Therefore $U \in \sigma(\{[y < a], [y > a];\ a \in Y\})$ and hence $f^{-1}U \in \mathcal{S}$.

The converse of the last theorem is not true as can be shown by the following example in which $\mathcal{G}$ has no countable basis and yet $\sigma(\mathcal{G}) = \sigma(\{[y < a], [y > a]; a \in Y\})$.

Example 2.1. Let Y be the space of all real-valued functions on $I = \langle 0, 1\rangle$ and put

$$f \leqq g \Leftrightarrow ((\forall x \in I)\, f(x) < g(x)) \vee ((\forall x \in I)\, f(x) = g(x)\, .$$

There is no difficulty in verifying that Y is a partially ordered real linear space. We show, nevertheless, that the order topology $\mathcal{G}$ on Y has no countable basis. Let N denote the set of all integers, N^I the collection of all integer-valued functions on I. Then for every $p \in N^I$, the set $D_p = \{f \in Y;\ p < f < p + 1\}$ is an open set in Y. The family $\{D_p;\ p \in N^I\}$ is disjoint and uncountable which proves that $\mathcal{G}$ has no countable basis.

If we denote by $\mathcal{T}$ the σ-algebra generated by the class $\{[y < a], [y > a]; a \in Y\}$, it is sufficient to prove now that every open subset of Y is in $\mathcal{T}$. Every basis element of $\mathcal{G}$ is a finite intersection of sets of the form $[f < a]$ or $[f > a]$. Owing to well known properties of real numbers every basis element C of $\mathscr{G}$ is a collection of $f \in Y$ characterized by exactly one of the following statements

(i) $(\exists a \in Y) f < a$

(ii) $(\exists a \in Y) f > a$

(iii) $(\exists a, b \in Y)\, a < f < b$.

Now every set G in $\mathscr{G}$ is a union of such sets and, since on the real line every union of intervals can be expressed as a countable union of intervals, G can be written as a countable union of some basis elements and hence $G \in \mathscr{T}$ as claimed.

Lemma 2.1. *Let $f_n \in \mathscr{M}_{\leqq}$ ($f_n \in \mathscr{M}_{\geqq}$) for $n = 1, 2, \ldots$. Let there be a function $f : X \to Y$ such that for each $x \in X$ we have $f(x) = \bigvee\{f_n(x);\ n = 1, 2, \ldots\}$ ($f(x) = \bigwedge\{f_n(x);\ n = 1, 2, \ldots\}$). Then $f \in \mathscr{M}_{\leqq}$ ($f \in \mathscr{M}_{\geqq}$).*

Proof. $f_n \in \mathscr{M}_{\leqq}$ means that $f_n^{-1}[y \leqq a] \in \mathscr{S}$ for each n and every $y \in Y$. Then $f^{-1}[y \leqq a] = \{x \in X; f(x) \leqq a\} = \{x \in X; \bigvee_n f_n(x) \leqq a\} = \bigcap_n \{x \in X; f_n(x) \leqq a\} = \bigcap_n f_n^{-1}[y \leqq a] \in \mathscr{S}$. The dual assertion is obtained analogously.

Theorem 2.4. $\mathscr{M}_{\nearrow} \subset \mathscr{M}_{\leqq}$, $\mathscr{M}_{\searrow} \subset \mathscr{M}_{\geqq}$, $\mathscr{M}_s \subset \mathscr{M}$.

Proof. For the first statement, suppose that f is a function in $\mathscr{M}_{\nearrow}$. Hence there are simple f_n, $n = 1, 2, \ldots$ with $f_n \nearrow f$. Evidently every simple function is in $\mathscr{M}_{\leqq}$ and by Lemma 2.1 it follows from $f_n \nearrow f$ that $f \in \mathscr{M}_{\leqq}$. The second statement is analogous and the last one follows from the first two.

After we have shown that every strongly measurable function is measurable, the question arises whether the converse is true. We are going to prove a sufficient condition for a measurable function to be strongly measurable.

Lemma 2.2. *Suppose Y is a lattice. Let there exist a countable set $Q \subset Y$ such that every $y \in Y$ is equal to $\bigvee\{q \in Q;\ q \leqq y\}$. Then every $f \in \mathscr{M}_{\geqq}$ bounded from below by some $b \in Q$ is in $\mathscr{M}_{\nearrow}$.*

Proof. Let $f \in \mathscr{M}_{\geqq}$, $f(x) \geqq b$ for each $x \in X$. Enumerate $Q = \{q_n;\ n = 1, 2, \ldots\}$, and define simple functions $g_n(x) = q_n$ iff $f(x) \geqq q_n$, and $g_n(x) = b$ otherwise. Put $f_n(x) = \bigvee\{g_i(x);\ i \leqq n\}$. Since Y is a lattice, the supremum defining $f_n(x)$ exists and each f_n is a simple function. Evidently $f_n \leqq f_{n+1}$ for each n and $(\forall x) f(x) = \bigvee\{q_n;\ q_n \leqq f(x)\} = \bigvee_n g_n(x) = \bigvee_n f_n(x)$.

Lemma 2.3. *Let Y be a lattice and $Q \subset Y$ countable and such that $(\forall y \in Y)\, y = \bigwedge\{q \in Q;\ q \geqq y\}$. Let $f \in \mathscr{M}_{\leqq}$ and $(\exists b \in Q)(\forall x \in X) f(x) \leqq b$. Then $f \in \mathscr{M}_{\searrow}$.*

Proof. Dual to that of Lemma 2.2.

Definition 2.1. We shall say that a partially ordered set Y is *quasi-separable* iff there is a countable set $Q \subset Y$ such that for every $y \in Y$ we have $y = \bigvee\{q \in Q; q \leqq y\} = \bigwedge\{q \in Q; q \geqq y\}$. Such a set Q will be called quasi-dense in Y.

Lemma 2.4. *Let Y be a conditionally σ-complete partially ordered set and let $\mathscr{G}$ denote the order topology. Then every set Q dense in the topological space $(Y, \mathscr{G})$ is quasi-dense in Y. Hence every separable conditionally σ-complete partially ordered set is quasi-separable.*

Proof. Take $y \in Y$. Since $A = \{q \in Q; q \leqq y\}$ is bounded from above by y and countable, there exists $a = \bigvee A$. If it were $a < y$, there would exist $q \in Q$ with $a < q < y$. But then $q \in A$, hence $q \leqq a$, a contradiction which proves that $y = \bigvee\{q \in Q; q \leqq y\}$. The proof of $y = \bigwedge\{q \in Q; q \geqq y\}$ is analogous and the second part of the theorem follows immediately.

Remark. In case Y is a partially ordered linear space, any one of the conditions $(\exists Q$ countable$)$ $(\forall y \in Y)$ $y = \bigvee\{q \in Q; q \leqq y\}$ and $(\exists Q$ countable$)$ $(\forall y \in Y)$ $y = \bigwedge\{q \in Q; q \geqq y\}$ implies the other one and hence the quasi-separability of Y.

Theorem 2.5. *Let Y be a quasi-separable (or separable and conditionally σ-complete) lattice. Then every bounded function in $\mathscr{M}$ is in $\mathscr{M}_s$.*

Proof. A direct application of the last three Lemmas.

Theorem 2.6. *Suppose that Y is a lattice. Let $f_n \in \mathscr{M}_{\nearrow}$, $n = 1, 2, \ldots$ and $f_n \nearrow f : X \to Y$. Then $f \in \mathscr{M}_{\nearrow}$. Dually, $\mathscr{M}_{\searrow} \ni f_n \searrow f$ implies $f \in \mathscr{M}_{\searrow}$.*

Proof. There are simple $g_n^m \nearrow f_n$ $(m \to \infty)$. Put $g_n = \bigvee\{g_i^n; i \leqq n\}$. Then g_n are simple and $g_n \nearrow f$.

Theorem 2.7. *Under the hypotheses of Lemma 2.2 (2.3) every function in $\mathscr{M}_{\searrow}(\mathscr{M}_{\nearrow})$ bounded from below (from above) is in $\mathscr{M}_{\nearrow}(\mathscr{M}_{\searrow})$.*

Proof. To prove that part of the theorem which is not in brackets, let $f \in \mathscr{M}_{\searrow}$. By Theorem 2.4 we have $f \in \mathscr{M}_{\geqq}$ and under the hypotheses of Lemma 2.2 it follows from f being bounded that $f \in \mathscr{M}_{\nearrow}$ as claimed. The dual part of the theorem is obtained by a dual proof.

Corollary. *If $\mathscr{A}$ denotes the family of bounded functions from X into Y, where Y is a partially ordered space which is quasi-separable, or separable and conditionally σ-complete, then $\mathscr{A} \cap \mathscr{M}_{\nearrow} \cap \mathscr{M}_{\searrow} = \mathscr{A} \cap \mathscr{M}_{\nearrow} = \mathscr{A} \cap \mathscr{M}_{\searrow}$.*

3. WEAKLY MEASURABLE FUNCTIONS INTO A LINEAR SPACE

Throughout this section Y will be a partially ordered real linear space.

Lemma 3.1. *Let $\mathscr{L}$ denote any of the classes $\mathscr{M}_{\leqq}$, $\mathscr{M}_{\geqq}$, $\mathscr{M}$, $\mathscr{M}_{<}$, $\mathscr{M}_{>}$, $\mathscr{M}_{w}$, $\mathscr{M}_{=}$, $\mathscr{M}_{\nearrow}$, $\mathscr{M}_{\searrow}$, $\mathscr{M}_{s}$ and $\mathscr{B}$. For $a \in Y$ and $f : X \to Y$ let $f + a$ be the function attaining the value $f(x) + a$ in each $x \in X$. Then for every $a \in Y$ and every $f \in \mathscr{L}$ we have $f + a \in \mathscr{L}$.*

Proof. For $\mathscr{L} = \mathscr{M}_{\leqq}, \mathscr{M}_{\geqq}, \mathscr{M}_{<}, \mathscr{M}_{>}, \mathscr{M}_{=}$ the assertion is trivial, hence also for $\mathscr{L} = \mathscr{M}_w$ and $\mathscr{M}$. In the case $\mathscr{L} = \mathscr{M}_{\nearrow}$ it is sufficient to observe that whenever f_n is a simple function then so is $f_n + a$ and that $\bigvee_n (f_n(x) + a) = (\bigvee_n f_n(x)) + a$. The proof for $\mathscr{L} = \mathscr{M}_{\searrow}$ is analogous and the last two cases yield the assertion for $\mathscr{L} = \mathscr{M}_s$.

In the case $\mathscr{L} = \mathscr{B}$, since every partially ordered linear space with the order topology is a topological group, we have

$$\{x \in X;\ f(x) + a \in G\} = \{x \in X;\ f(x) \in G - a\} \in \mathscr{S}$$

for any open set G due to $G - a$ being also an open set.

Lemma 3.2. *Let $\mathscr{L}$ denote any of the classes $\mathscr{M}_w$, $\mathscr{M}$ and $\mathscr{M}_s$. If $f \in \mathscr{L}$ and $t \in R$, then $tf \in \mathscr{L}$. (R is the field of reals.)*

Proof. Let $f \in \mathscr{M}_w$, that is, $\{x; f(x) < a\} \in \mathscr{S} \ni \{x; f(x) > a\}$ for each $a \in Y$. If $t > 0$ we have $\{x;\ tf(x) < a\} = \{x; f(x) < a/t\} \in \mathscr{S}$ and $\{x;\ tf(x) > a\} = \{x; f(x) > a/t\} \in \mathscr{S}$. If $t < 0$ we get $\{x;\ tf(x) < a\} = \{x; f(x) > a/t\} \in \mathscr{S}$ and similarly $\{x;\ tf(x) > a\} \in \mathscr{S}$. Finally, if $t = 0$, then tf is the constant function equal to the zero element O of Y and hence $tf \in \mathscr{M}_w$.

The proof for $\mathscr{L} = \mathscr{M}$ is obtained by rewriting the last one, replacing $<$ and $>$ by $\geqq$ and $\geqq$ respectively.

In the case $\mathscr{L} = \mathscr{M}_s$ it is sufficient to realize that if f_n is simple, then so is tf_n and, for $t > 0$, $f_n \nearrow f$ implies $tf_n \nearrow tf$, $f_n \searrow f$ implies $tf_n \searrow tf$, whereas for $t < 0$ we have $tf_n \searrow tf$ or $tf_n \nearrow tf$ whenever $f_n \nearrow f$ or $f_n \searrow f$, respectively. For $t = 0$ and any $f \in \mathscr{M}_s$ we have $tf = O \in \mathscr{M}_s$.

Theorem 3.1. *If Y is separable, then $\mathscr{M}_w$ is a real linear space. The partial ordering $\leqq$ of $\mathscr{M}_w$ satisfies (v_1) and (v_2).*

Proof. By Lemma 3.2, if $f \in \mathscr{M}_w$ and $t \in R$, then $tf \in \mathscr{M}_w$. To prove that for $f, g \in \mathscr{M}_w$ we have $f + g \in \mathscr{M}_w$ we first show that for any $f, g \in \mathscr{M}_w$ the set $\{x \in X; f(x) < g(x)\}$ is in $\mathscr{S}$. In fact, if Q is a countable dense subset of Y, then $\{x; f(x) < g(x)\} = \bigcup\{\{x; f(x) < q < g(x)\};\ q \in Q\}$ is a countable union of sets in $\mathscr{S}$, and hence is itself in $\mathscr{S}$. Now if f and g are in $\mathscr{M}_w$, then so is $-g + a$ for any $a \in Y$

and hence $\{x; f(x) + g(x) < a\} = \{x; f(x) < -g(x) + a\} \in \mathscr{S}$ and $\{x; f(x) + g(x) > a\} = \{x; f(x) > -g(x) + a\} \in \mathscr{S}$. We have thus proved that $\mathscr{M}_w$ is a real linear space. The properties (v_1) and (v_2) are immediate consequences of the definition of $\leqq$ in $\mathscr{M}_w$ (see Section 1).

4. STRONGLY MEASURABLE FUNCTIONS INTO A LATTICE

Theorem 4.1. *Let Y be a quasi-separable and conditionally σ-complete lattice. Then the family of all bounded functions in $\mathscr{M}_s$ is a lattice.*

Proof. Let $f, g \in \mathscr{M}_s$. Then there are simple $f_n \nearrow f$, $F_n \searrow f$, $g_n \nearrow g$, $G_n \searrow g$. To prove $f \vee g \in \mathscr{M}_s$ we show that $f_n \vee g_n \nearrow f \vee g$ and $F_n \wedge G_n \searrow f \wedge g$. Evidently $f \geqq f_n$ and $g \geqq g_n$ imply $f \vee g \geqq f_n \vee g_n$ for each n. Thus, $f \vee g$ is an upper bound for the non-decreasing sequence of $f_n \vee g_n$. To prove that it is the supremum, suppose $h \geqq f_n \vee g_n$ for every n. Then $(\forall n)\ h \geqq f_n$ and hence $h \geqq f$. Similarly, $h \geqq g_n$ for each n and hence $h \geqq g$. Since $h \geqq f$ and $h \geqq g$ we infer $h \geqq f \vee g$ and so $f \vee g$ is the supremum of $\{f_n \vee g_n;\ n = 1, 2, \ldots\}$. Now since we have proved $f \vee g \in \mathscr{M}_{\nearrow}$, owing to the boundedness of both f and g and quasi-separability of Y we infer by Theorem 2.7 that $f \vee g \in \mathscr{M}_{\searrow}$. Thus $f \vee g \in \mathscr{M}_s$ and is bounded. The proof that $f \wedge g$ belongs to $\mathscr{M}_s$ and is bounded is analogous.

Theorem 4.2. *Let Y be a quasi-separable real linear lattice. Then the family $\mathscr{M}_s \cap \mathscr{A}$ of all bounded and strongly measurable functions into Y is a partially ordered linear space ($f \leqq g$ means $f(x) \leqq g(x)$ for each x).*

Proof. Take $f, g \in \mathscr{M}_s \cap \mathscr{A}$. There are simple $f_n \nearrow f$, $g_m \nearrow g$, $F_n \searrow f$, $G_m \searrow g$. For any m we have $f_n + g_m \nearrow f + g_m$, $F_n + G_m \searrow f + G_m$ with $n \to \infty$. Thus $f + g_m \in \mathscr{M}_{\nearrow} \cap \mathscr{A}$, $f + G_m \in \mathscr{M}_{\searrow} \cap \mathscr{A}$ and hence by Corollary to Theorem 2.7 functions $f + g_m$ and $f + G_m$ are in $\mathscr{M}_s \cap \mathscr{A}$ for each $m = 1, 2, \ldots$ Since evidently $f + g_m \nearrow f + g$ and $f + G_m \searrow f + g$ with increasing m, we deduce from Theorem 2.6 that $f + g \in \mathscr{M}_{\nearrow} \cap \mathscr{M}_{\searrow}$. Due to f and g being bounded we obtain $f + g \in \mathscr{M}_s \cap \mathscr{A}$. Applying now Lemma 3.2 with $\mathscr{L} = \mathscr{M}_s$ to a function $f \in \mathscr{M}_s$ and $t \in R$ we have $tf \in \mathscr{M}_s$, and realizing that tf is also bounded we complete the proof that $\mathscr{M}_s \cap \mathscr{A}$ is a linear space. Conditions (v_1) and (v_2) are verified without difficulty.

Example 4.1. As an example of a quasi-separable linear lattice we may take any finitely dimensional real linear space Y with the ordering defined by $a \leqq b$ iff for all coordinates $a^n \leqq b^n$. Then Y is evidently a linear lattice. It is conditionally σ-complete since for any countable set $\{a_m;\ m = 1, 2, \ldots\}$ of vectors which is bounded from above by b we have in each coordinate $a_m^n \leqq b^n$ and it follows from well-known properties of real numbers that there is $a^n = \sup_m a_m^n$. Then a defined by coordinates a^n is the supremum of $\{a_m;\ m = 1, 2, \ldots\}$. Indeed, we have

$$(\forall n)\,(\forall m)\ a_m^n \leqq a^n$$

and if for some c with $c \geqq a_m$ $(m = 1, 2, \ldots)$ the inequality $c \geqq a$ were not true, then there would exist n such that $c^n < a^n$ and then, since $a^n = \sup_m a^n_m$, we would have $c^n < a^n_m$ for some m, which contradicts $(\forall m)\, c \geqq a_m$. In a similar way we may prove that the countable set Q of all vectors with rational coordinates only is quasi-dense in Y, actually, $(\forall y \in Y)\, y = \bigvee\{q \in Q;\ q \leqq y\} = \bigwedge\{q \in Q;\ q \geqq y\}$.

It is worth noticing that if the dimension of Y is greater than one, than Y is not separable in the topological sense, since the order topology in Y is discrete. Indeed, for every $y \in Y$ we have $\{y\} = Y_1 \cap Y_2$ where

$$Y_i = \{z \in Y;\ y^i - 1 < z^i < y^i + 1,\ z^j = y^j \text{ for } j \neq i\} \quad (i = 1, 2)$$

are open sets.

5. AN INTEGRATION THEORY FOR BOUNDED STRONGLY MEASURABLE FUNCTIONS

As an application of general results we construct in outline an integration theory of Daniell type. In this section Y denotes a conditionally σ-complete quasi-separable linear lattice.

Denote by $\mathscr{K}$ the family of all bounded strongly measurable functions, i.e., $\mathscr{K} = \mathscr{A} \cap \mathscr{M}_s = \mathscr{A} \cap \mathscr{M}_{\nearrow} = \mathscr{A} \cap \mathscr{M}_{\searrow}$ (see Corollary of Theorem 2.7). Let $\mathscr{K}_0$ be the family of all simple functions $f : X \to Y$ (see Section 1). We start with a function $\mathscr{I}_0 : \mathscr{K}_0 \to Y$ satisfying the following conditions:

(1) $\mathscr{I}_0(uf + vg) = u\,\mathscr{I}_0(f) + v\,\mathscr{I}_0(g)$ for $u, v \in R$, $f, g \in \mathscr{K}_0$.

(2) If $f \leqq g$, $f, g \in \mathscr{K}_0$ then $\mathscr{I}_0(f) \leqq \mathscr{I}_0(g)$.

(3) If $f_n \in \mathscr{K}_0$ $(n = 1, 2, \ldots)$ and $f_n \searrow 0$, then $\mathscr{I}_0(f_n) \searrow \mathscr{I}_0(0) = 0$.

Lemma 5.1. *If $f_n, g_n, F_n, G_n \in \mathscr{K}_0$ $(n = 1, 2, \ldots)$, $f \in \mathscr{K}$ and $f_n \nearrow f$, $g_n \nearrow f$, $F_n \searrow f$, $G_n \searrow f$, then*

$$\bigvee \mathscr{I}_0(f_n) = \bigvee \mathscr{I}_0(g_n) = \bigwedge \mathscr{I}_0(F_n) = \bigwedge \mathscr{I}_0(G_n)\,.$$

Proof. For fixed m we have $f_n \wedge g_m \nearrow f \wedge g_m = g_m$ $(n \to \infty)$, hence $g_m - f_n \wedge g_m \searrow 0$ and therefore

$$0 = \bigwedge_n \mathscr{I}_0(g_m - f_n \wedge g_m) = \bigwedge_n (\mathscr{I}_0(g_m) - \mathscr{I}_0(f_n \wedge g_m)) =$$
$$= \mathscr{I}_0(g_m) - \bigvee_n (\mathscr{I}_0(f_n \wedge g_m)) \geqq \mathscr{I}_0(g_m) - \bigvee_m \mathscr{I}_0(f_n)\,.$$

Now we have $\mathscr{I}_0(g_m) \leqq \bigvee_n \mathscr{I}_0(f_n)$ $(m = 1, 2, \ldots)$ and hence also $\bigvee_n \mathscr{I}_0(g_m) \leqq \bigvee_n \mathscr{I}_0(f_n)$. The converse inequality and the equality

$$\bigwedge \mathscr{I}_0(G_n) = \bigwedge \mathscr{I}_0(F_n)$$