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ON ALGEBRAIC PROPERTIES OF DISPERSIONS OF THE 3RD AND 4TH KIND OF THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$

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Academician O. BORŮVKA introduced in [1] the definitions and established properties of general dispersions, giving a characterization of dispersions of the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} kind as well as of central dispersions. Further he studied the sets of dispersions of the 1^{st} and 2^{nd} kind.

The subject of the present paper was suggested by Professor M. LAITOCH who directed my attention to the possibility of a parallel study of the 3^{rd} and 4^{th} kind dispersion sets.

The opening part establishes a representation of the 3^{rd} kind dispersions by means of unimodular matrices.

In the second part we define equivalence relations $\sim$ and $\approx$ in the 3^{rd} kind dispersion set $\boldsymbol{D}_3$:

$X_3 \sim Y_3$ if and only if there exists $\varphi_\nu \in \boldsymbol{C}_1$ such that $X_3\varphi_\nu = Y_3$, where $\boldsymbol{C}_1$ is the group of central dispersions of the 1^{st} kind, $X_3, Y_3 \in \boldsymbol{D}_3$;

$X_3 \approx Y_3$ if and only if there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and at the same time $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$.

The relations turn out to be the same and hence the decompositions $\boldsymbol{D}_3/\sim$ and $\boldsymbol{D}_3/\approx$ coincide. Hence, for any coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$ we can uniquely determine a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ by $\mathscr{X}_3 = \mathscr{X}_3\boldsymbol{C}_1 = \boldsymbol{C}_3\mathscr{X}_1$. Moreover, any dispersions $X_1 \in \mathscr{X}_1$ and $X_3 \in \mathscr{X}_3$ satisfy $\mathscr{X}_3 = X_3\boldsymbol{C}_1 = \boldsymbol{C}_3X_1$.

In the next part we show the existence of a $1-1$ mapping of the set $\boldsymbol{D}_3/\sim$ onto the factor group $\boldsymbol{L}/\{\mathbf{E}, -\mathbf{E}\}$. (Any coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$ is associated with a couple of unimodular matrices $\{\mathbf{C}, -\mathbf{C}\}$). Further, $\mathscr{X}_3\mathfrak{B}_1 = \mathfrak{B}_3\mathscr{X}_1 = \mathfrak{B}_3$, where $\mathfrak{B}_3(\mathfrak{B}_1)$ is the set (the group) of the 3^{rd} kind (the 1^{st} kind) direct dispersions and $\mathscr{X}_3$ is an arbitrary element in $\mathfrak{B}_3$, $\mathscr{X}_1 \in \mathfrak{B}_1$.

The concluding part of the paper is devoted to transfering the results proved for the dispersions of the 3^{rd} kind to the case of the dispersions of the 4^{th} kind.

Basic concepts and relations. (q) will always denote an ordinary linear differential equation of the 2^{nd} order in the real domain $y'' = q(t)\,y$, where $q(t) \in C_j^2$ ($j = (a, b)$ is an open definition interval) and $q(t) < 0$ for every $t \in j$; the differential equation (q) will be always assumed oscillatory in (a, b), that is, the integrals of this equation vanish infinitely many times in both directions towards the endpoints a, b of the interval (a, b). (q_1) will always denote the associated equation of (q). (See [1].) The integral space (i.e., the space of all solutions) of the differential equation (q), (q_1) will be denoted by R, R_1, respectively. The concepts not defined in this paper were adopted from [1].

1. DISPERSIONS OF THE 3^{RD} KIND

Representation by means of unimodular matrices. Let $X_3 \in \boldsymbol{D}_3$ be an arbitrary dispersion of the 3^{rd} kind, $\boldsymbol{D}_3$ the set of all dispersions of the 3^{rd} kind. Choose a basis (U_1, V_1) of the integral space R_1 and denote its Wronskian by W_1; let $u(t)$, $v(t)$ be the functions

$$u(t) = \frac{U_1[X_3(t)]}{\sqrt{|X_3'(t)|}}, \quad v(t) = \frac{V_1[X_3(t)]}{\sqrt{|X_3'(t)|}}. \tag{1}$$

By [1, § 20, 6.3], the functions $u(t)$, $v(t)$ are linearly independent integrals of (q) and thus they form a basis of the integral space R. Their Wronskian w satisfies

$$w = W_1 \,.\, \operatorname{sgn} X_3'\,. \tag{2}$$

By [1, § 1, 9] there exists exactly one integral y of (q) for each integral y_1 of differential equation (q_1) such that

$$y_1(t) = \frac{y'(t)}{\sqrt{|q(t)|}}. \tag{3}$$

Consequently, it is possible to determine exactly one basis (U, V) of R for the basis (U_1, V_1) of R_1 such that the corresponding functions U, U_1 and V, V_1 satisfy (3). The bases (u, v), (U, V) of the same space R are connected in the following way

$$u(t) = c_{11}\,U(t) + c_{12}\,V(t)\,, \quad v(t) = c_{21}\,U(t) + c_{22}\,V(t) \tag{4}$$

and hence

$$w = W \,.\, \det \mathbf{C}\,, \tag{5}$$

where w and W are the Wronskians of the bases (u, v) and (U, V), respectively. Further,

$$W_1 = \begin{vmatrix} U_1 & V_1 \\ U_1' & V_1' \end{vmatrix} = (U'V - UV') \,.\, \operatorname{sgn} q = W\,.$$

Now by (5),

(6) $$w = W_1 \,.\, \det \mathbf{C}$$

and (2) and (6) imply

(7) $$\det \mathbf{C} = \operatorname{sgn} X_3' \,.$$

Therefore the matrix $\mathbf{C}$ is unimodular.

Theorem 1.1. *For any dispersion $X_3 \in D_3$, the unimodular matrix $\mathbf{C}$ is uniquely determined by* (4).

The theorem results from the above consideration.

Theorem 1.2. *For any unimodular matrix, there exists at least one dispersion of the 3rd kind associated with it through the relations* (4) *and* (1).

Proof. Let $\mathbf{C} = \|c_{ik}\|$ be an arbitrary unimodular matrix. Let us consider the integral $c_{21} U + c_{22} V$ and let t_0 be its arbitrary zero point. Let T_0 be a zero point of the integral V_1, such that

(8) $$\operatorname{sgn} U_1(T_0) = \operatorname{sgn} (c_{11} U(t_0) + c_{12} V(t_0))\,,$$

where $U(t)$, $V(t)$ is a basis of R, $U_1(t)$, $V_1(t)$ is the basis of R_1 such that

$$U_1(t) = \frac{U'(t)}{\sqrt{|q|}}\,, \quad V_1(t) = \frac{V'(t)}{\sqrt{|q|}}\,.$$

Let us consider the linear mapping p

$$p = [u(t) \to U_1(t),\ v(t) \to V_1(t)]\,,$$

where $u(t) = c_{11} U(t) + c_{12} V(t)$, $v(t) = c_{21} U(t) + c_{22} V(t)$. This mapping is normalized with respect to t_0, T_0 and thus uniquely determines the dispersion $X_3 \in D_3$. (See [1, § 20,2].) Further,

$$\chi_p = \frac{\det \mathbf{C} \,.\, W}{W_1} = \frac{\det \mathbf{C} \,.\, W}{W} = \det \mathbf{C} = \pm 1\,,$$

where χ_p is the characteristic of the linear mapping p. Hence by [1, § 20,6 (17)] we have for any $Y_1 \in R_1$,

$$\frac{Y_1[X_3(t)]}{\sqrt{|X_3'(t)|}} = \pm y$$

where $y \in R$ and $Y_1 = py$. The sign + or − does not depend on the choice of the integral Y_1. Therefore by (7)

$$\frac{U_1[X_3(t)]}{\sqrt{|X_3'(t)|}} = +u(t) = c_{11}\,U(t) + c_{12}\,V(t)\,,$$

$$\frac{V_1[X_3(t)]}{\sqrt{|X_3'(t)|}} = +v(t) = c_{21}\,U(t) + c_{22}\,V(t)\,.$$

The dispersion $X_3(t)$ is associated with the matrix $\mathbf{C}$ in the required manner.

The decomposition of the set D_3 determined by the equivalence relation $\sim$ or $\approx$. Now we shall introduce the relation $\sim$ in the dispersion set D_3 as follows:

Let C_1 be the group of central dispersions of the first kind, X_3, Y_3 arbitrary dispersions of the 3rd kind of D_3.

(9) $$X_3 \sim Y_3 \text{ iff there exists } \varphi_\nu \in C_1 \text{ such that } X_3\varphi_\nu = Y_3\,.$$

Theorem 1.3. *The relation* (9) *is an equivalence relation on the set* D_3.

Proof. Let X_3, Y_3, Z_3 be arbitrary dispersions in D_3. Since there exists a dispersion $\varphi_0(t) = t \in C_1$ such that $X_3\varphi_0 = X_3$, it holds $X_3 \sim X_3$ for each $X_3 \in D_3$. Let $X_3 \sim Y_3$. Then $X_3\varphi_\nu = Y_3$, $X_3\varphi_\nu\varphi_{-\nu} = Y_3\varphi_{-\nu}$ and hence $Y_3\varphi_{-\nu} = X_3$. Thus $Y_3 \sim X_3$. Let $X_3 \sim Y_3$ and $Y_3 \sim Z_3$, $X_3\varphi_\nu = Y_3$ and $Y_3\varphi_\mu = Z_3$. Therefore $X_3\varphi_\nu\varphi_\mu = Y_3\varphi_\mu = Z_3$, $X_3\varphi_\sigma = Z_3$ and thus $X_3 \sim Z_3$.

Theorem 1.4. *The relation* (9) *forms a decomposition* $D_3/\sim$. *The set* C_3 *of all central dispersions of the* 3rd *kind forms exactly one coset of* $D_3/\sim$.

Proof. a) Any central dispersion $\chi_\nu \in C_3$ ($\nu = \pm 1, \pm 2, \ldots$) can be expressed (see [1, § 12,4 (7)]) in the following manner:

$$\chi_n = \chi_1\varphi_{n-1}\,, \quad \chi_{-n} = \chi_1\varphi_{-n}\,, \quad n = 1, 2, \ldots\,.$$

This implies that any central dispersion χ_ν ($\nu = \pm 1, \pm 2, \ldots$) and the dispersion χ_1 are equivalent (in equivalence $\sim$) and thus all dispersions χ_ν belong to the same coset of $D_3/\sim$.

b) If a dispersion X_3 and an arbitrary central dispersion χ_ϱ are equivalent, then X_3 is also a central dispersion. Indeed, in this case $X_3\varphi_\nu = \chi_\varrho$, $X_3\varphi_\nu\varphi_{-\nu} = \chi_\varrho\varphi_{-\nu}$ and therefore $X_3 = \chi_\sigma \in C_3$.

Corollary 1.1. *Let* $\mathscr{X}_3$ *be an arbitrary equivalence coset of* $D_3/\sim$. *Then* $\mathscr{X}_3 = X_3C_1$, *where* X_3 *is a dispersion in the coset* $\mathscr{X}_3$. *Also* $X_3C_1 = \mathscr{X}_3C_1$.

Let us now consider the group $\boldsymbol{D}_1$ of the first kind dispersions and its cyclic subgroup $\boldsymbol{S}_1$ of central dispersions with an even index. The latter one is a normal subgroup of $\boldsymbol{D}_1$. The factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ and the group $\boldsymbol{L}$ of all unimodular matrices of the 2nd order are isomorphic. (See [1, § 21,6].) Let

$$\varphi : \boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{L}$$

be the isomorphism considered in [1]. In this isomorphism the group $\boldsymbol{S}_1$ (the set $\bar{\boldsymbol{S}}_1$ of the first kind central dispersions with an odd index) and the unity matrix $\mathbf{E}$ (the matrix $-\mathbf{E}$) correspond to each other. $\boldsymbol{S}_1 \cup \bar{\boldsymbol{S}}_1 = \boldsymbol{C}_1$ is the group of the first kind central dispersions and it is also a cyclic subgroup of the group $\boldsymbol{D}_1$. (See [1, § 21,6].)

Let us now consider the induced isomorphism

$$\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} \to \{\mathbf{E}, -\mathbf{E}\}$$

between two-element subgroups $\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$ and $\{\mathbf{E}, -\mathbf{E}\}$ of the groups $\boldsymbol{D}_1/\boldsymbol{S}_1$ and $\boldsymbol{L}$, respectively. Since $\{\mathbf{E}, -\mathbf{E}\}$ is a normal subgroup of $\boldsymbol{L}$, $\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$ is a normal subgroup of $\boldsymbol{D}_1/\boldsymbol{S}_1$ and the relative factor groups are isomorphic:

$$\varphi' : (\boldsymbol{D}_1/\boldsymbol{S}_1)/\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} \to \boldsymbol{L}/\{\mathbf{E}, -\mathbf{E}\}\,.$$

Furthermore, for arbitrary $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$, $\mathscr{X}_1 . \{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} = \{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} . \mathscr{X}_1$ and hence $\mathscr{X}_1\bar{\boldsymbol{S}}_1 = \boldsymbol{S}_i\mathscr{X}_1$, where $\boldsymbol{S}_i \in \{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$. Thus, for arbitrary $X_1 \in \mathscr{X}_1$ and $\varphi_\nu \in \bar{\boldsymbol{S}}_1$, there exists $\varphi_\mu \in \boldsymbol{S}_i$ and $\bar{X}_1 \in \mathscr{X}_1$ such that $X_1\varphi_\nu = \varphi_\mu\bar{X}_1$. Since $X_1, \bar{X}_1 \in \mathscr{X}_1$, there exists $\varphi_\varrho \in \boldsymbol{S}_1$ such that $\bar{X}_1 = \varphi_\varrho X_1$. Hence $X_1\varphi_\nu = \varphi_\mu\varphi_\varrho X_1$ and $X_1\boldsymbol{C}_1 \subseteq \boldsymbol{C}_1X_1$. The converse relation $X_1\boldsymbol{C}_1 \supseteq \boldsymbol{C}_1X_1$ can be proved by analogy. Consequently, it holds $X_1\boldsymbol{C}_1 = \boldsymbol{C}_1X_1$. Since $\boldsymbol{C}_1$ is a normal subgroup of $\boldsymbol{D}_1$, we can form the factor group $\boldsymbol{D}_1/\boldsymbol{C}_1$. Any two elements $X_1, Y_1 \in \boldsymbol{D}_1$ belong to the same coset of $\boldsymbol{D}_1/\boldsymbol{C}_1$ if and only if $X_1 \sim Y_1$, i.e., if there exists $\varphi_\nu \in \boldsymbol{C}_1$ such that $X_1 = Y_1\varphi_\nu$. Let

$$\alpha : \boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{D}_1/\boldsymbol{C}_1$$

be a mapping of the factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ onto the factor group $\boldsymbol{D}_1/\boldsymbol{C}_1$ such that each coset $X_1\boldsymbol{C}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ is mapped onto the coset $X_1\boldsymbol{S}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$. Thus $\alpha(X_1\boldsymbol{S}_1) = X_1\boldsymbol{C}_1$.

Lemma 1.1. *The mapping $\alpha : \boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{D}_1/\boldsymbol{C}_1$ is a homomorphism. The kernel of this homomorphism is the two-element subgroup $\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$ to which the element $\boldsymbol{C}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ corresponds.*

Proof. Let $X_1\boldsymbol{S}_1, Y_1\boldsymbol{S}_1$ be arbitrary elements in the group $\boldsymbol{D}_1/\boldsymbol{S}_1$. Then $\alpha(X_1\boldsymbol{S}_1 . Y_1\boldsymbol{S}_1) = \alpha(X_1Y_1\boldsymbol{S}_1\boldsymbol{S}_1) = \alpha(X_1Y_1\boldsymbol{S}_1) = X_1Y_1\boldsymbol{C}_1 = X_1Y_1\boldsymbol{C}_1\boldsymbol{C}_1 = X_1\boldsymbol{C}_1 . Y_1\boldsymbol{C}_1 = \alpha(X_1\boldsymbol{S}_1) . \alpha(Y_1\boldsymbol{S}_1)$. Thus α is a homomorphism. Further $\alpha(\boldsymbol{S}_1) = \boldsymbol{C}_1$, $\alpha(\bar{\boldsymbol{S}}_1) = \alpha(\varphi_\sigma\boldsymbol{S}_1) = \varphi_\sigma . \boldsymbol{C}_1 = \boldsymbol{C}_1$ where σ is an odd integer. If $\alpha(\mathscr{X}_1) = \boldsymbol{C}_1$ for a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$, then $\mathscr{X}_1 = X_1\boldsymbol{S}_1$ where $X_1 \in \mathscr{X}_1$ implies $\boldsymbol{C}_1 = \alpha(\mathscr{X}_1) = \alpha(X_1\boldsymbol{S}_1) = X_1\boldsymbol{C}_1$. Thus $X_1 \in \boldsymbol{C}_1$ and hence either $X_1\boldsymbol{S}_1 = \boldsymbol{S}_1$ or $X_1\boldsymbol{S}_1 = \bar{\boldsymbol{S}}_1$.

Remark 1.1. According to Lemma 1.1, there exists an isomorphism

$$\tau : D_1/C_1 \to (D_1/S_1)/\{S_1, \bar{S}_1\} .$$

If we now compose the isomorphisms τ and φ' we obtain an isomorphism

$$\varphi'\tau : D_1/C_1 \to L/\{\mathbf{E}, -\mathbf{E}\} .$$

Clearly $\varphi'\tau(C_1) = \{\mathbf{E}, -\mathbf{E}\}$. Also for any element $X_1C_1 \in D_1/C_1$, $\varphi'\tau(X_1C_1) = \{\mathbf{C}, -\mathbf{C}\}$ where $\mathbf{C}$ is a matrix in L such that $\varphi(X_1S_1) = \mathbf{C}$. ($\varphi$ is the above described isomorphism $D_1/S_1 \to L$).

Now, let us return to the set D_3 and the equivalence relation (9).

Lemma 1.2. *For each dispersion of the 3rd kind $X_3 \in D_3$ and for each central dispersion $\chi_\varrho \in C_3$ there exists a dispersion of the first kind $X_1 \in D_1$ such that*

(10) $$X_3 = \chi_\varrho X_1 .$$

Proof. For each central dispersion of the 3rd kind $\chi_\varrho(\varrho = \pm 1, \pm 2, \ldots)$ there exists a central dispersion of the 4th kind $\omega_{-\varrho} \in C_4$ such that $\omega_{-\varrho}\chi_\varrho = \chi_\varrho\omega_{-\varrho} = \varphi_0(t) = t$. (See [1, § 12,4 (6)].) Consider now the function $\omega_{-\varrho}X_3$, where $\omega_{-\varrho} \in C_4$ and $X_3 \in D_3$. Then by [1, § 21,8] $\omega_{-\varrho}X_3 \in D_1$ and there exists $X_1 \in D_1$ such that $X_1 = \omega_{-\varrho}X_3$. Also $\chi_\varrho X_1 = \chi_\varrho\omega_{-\varrho}X_3 = \varphi_0 X_3 = X_3$ and hence $X_1 \in D_1$ satisfies the equality (10).

Lemma 1.3. *If two different central dispersions of the 3rd kind $\chi_\varrho, \chi_\sigma \in C_3$ fulfil $X_3 = \chi_\varrho X_1$, $X_3 = x_\sigma Y_1$, then $X_1 = Y_1\varphi_\nu$ and thus X_1, Y_1 lie in the same coset of the factor group D_1/C_1.*

Proof. It holds $\chi_\varrho \sim \chi_\sigma$, $\chi_\varrho = \chi_\sigma\varphi_\mu$, where $\varphi_\mu \in C_1$. From $\chi_\sigma Y_1 = \chi_\varrho \, . \, X_1$ it follows that $\chi_\sigma Y_1 = \chi_\sigma\varphi_\mu X_1$ and hence $\omega_{-\sigma}x_\sigma Y_1 = \omega_{-\sigma}\chi_\sigma\varphi_\mu X_1$. Therefore $Y_1 = \varphi_\mu X_1$ and also $X_1 = Y_1\varphi_\nu$.

Corollary 1.2. *For each dispersion of the 3rd kind $X_3 \in D_3$ there exists a dispersion of the 1st kind $X_1 \in D_1$ such that $X_3 \in C_3X_1$. Thus, for each dispersion X_3 there exists exactly one coset $\mathscr{X}_1 = X_1C_1$ of the factor group D_1/C_1 such that $X_3 \in C_3\mathscr{X}_1$. Consequently, $C_3X_1 = C_3\mathscr{X}_1$ holds.*

We shall now introduce a binary relation $\approx$ in the set D_3 as follows:

(11) $X_3 \approx Y_3$ iff there exists $\mathscr{X}_1 \in D_1/C_1$ such that $X_3 \in C_3\mathscr{X}_1$ and at the same time $Y_3 \in C_3\mathscr{X}_1$.

Theorem 1.5. *The relation* (11) *is an equivalence relation in the set D_3.*

Proof. By Corollary 1.2, $X_3 \approx X_3$ holds for each $X_3 \in D_3$. Now let $X_3 \approx Y_3$. Then there exists a coset $\mathscr{X}_1 \in D_1/C_1$ such that $X_3 \in C_3\mathscr{X}_1$ and $Y_3 \in C_3\mathscr{X}_1$. Therefore

$Y_3 \approx X_3$. Let $X_3 \approx Y_3$. Then there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$. Let also $Y_3 \approx Z_3$. Then there exists $\mathscr{Y}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $Y_3 \in \boldsymbol{C}_3\mathscr{Y}_1$ and $Z_3 \in \boldsymbol{C}_3\mathscr{Y}_1$. From $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and $Y_3 \in \boldsymbol{C}_3\mathscr{Y}_1$ it follows (by Lemma 1.3) that there exists $\varphi_\nu \in \boldsymbol{C}_1$ such that $X_1 = Y_1\varphi_\nu$ and hence $\mathscr{X}_1 = \mathscr{Y}_1$. Herefrom $X_3 \approx Z_3$.

Theorem 1.6. *Two arbitrary dispersions of the* 3rd *kind* $X_3, Y_3 \in \boldsymbol{D}_3$ *fulfil* $X_3 \sim Y_3$ *if and only if* $X_3 \approx Y_3$.

Proof. a) Let $X_3 \approx Y_3$. Then there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$. Thus $X_3 = \chi_\varrho X_1$, $Y_3 = \chi_\sigma Y_1$, where $\chi_\varrho, \chi_\sigma \in \boldsymbol{C}_3$, $X_1, Y_1 \in \mathscr{X}_1$. Since it holds $X_1 = Y_1\varphi_\nu$ and $\chi_\varrho = \chi_\sigma\varphi_\mu$ where φ_ν, φ_μ are proper dispersions of $\boldsymbol{C}_1$ we have $X_3 = \chi_\sigma\varphi_\mu Y_1\varphi_\nu = \chi_\sigma Y_1 \varphi_{\mu_1}\varphi_\nu = Y_3\varphi_{\nu_1}$, thus $X_3 \sim Y_3$.

b) Let $X_3 \sim Y_3$. Then there exists φ_ν such that $X_3\varphi_\nu = Y_3$. By Corollary 1.2 there exists a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$. So $X_3 = \chi_\varrho X_1$ where $\chi_\varrho \in \boldsymbol{C}_3$ and $X_1 \in \mathscr{X}_1$. Then $Y_3 = X_3\varphi_\nu = \chi_\varrho \,.\, X_1\varphi_\nu$. Thus $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and therefore $X_3 \approx Y_3$.

Hence the decompositions $\boldsymbol{D}_3/\sim$ and $\boldsymbol{D}_3/\approx$ coincide.

Let us recall that if we consider an arbitrary dispersion $X_3 \in \boldsymbol{D}_3$ and compose it with all central dispersions of the 1st kind (i.e., with dispersions from $\boldsymbol{C}_1$) we obtain exactly one coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$. Thus $\mathscr{X}_3 = X_3\boldsymbol{C}_1$, where $X_3 \in \mathscr{X}_3$. (See Corollary 1.1.)

Theorem 1.7. *Let* X_3 *be an arbitrary dispersion in* $\boldsymbol{D}_3$ *and let* $\mathscr{X}_3 = X_3\boldsymbol{C}_1$ *be a coset of* $\boldsymbol{D}_3/\sim$. *If we compose a dispersion* $X_1 \in \boldsymbol{D}_1$ *associated with* X_3 *by* (10) *with all central dispersions of the* 3rd *kind (i.e., with dispersions in* $\boldsymbol{C}_3$) *we obtain exactly one coset* $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$. *Thus* $\mathscr{X}_3 = \boldsymbol{C}_3X_1$ *and*

$$\boldsymbol{C}_3X_1 = X_3\boldsymbol{C}_1 = \mathscr{X}_3 \,. \tag{12}$$

Proof. This theorem is an immediate consequence of those above.

Theorem 1.8. *For each coset* $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$ *there exists exactly one coset* $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ *such that*

$$\mathscr{X}_3 = \mathscr{X}_3\boldsymbol{C}_1 = \boldsymbol{C}_3\mathscr{X}_1 \,. \tag{13}$$

Proof. By Corollaries 1.1 and 1.2 it holds $X_3\boldsymbol{C}_1 = \mathscr{X}_3\boldsymbol{C}_1$ and $\boldsymbol{C}_3X_1 = \boldsymbol{C}_3\mathscr{X}_1$. Herefrom and by (12) we have (13). Let us now consider a coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$. Let $\mathscr{X}_3 = \boldsymbol{C}_3Y_1$, where $Y_1 \in \boldsymbol{D}_1$. Then $Y_1 \sim X_1$ and therefore $Y_1 \in \mathscr{X}_1$ and for a coset $\mathscr{X}_3$, the coset $\mathscr{X}_1$ is uniquely determined. The converse is evident.

The properties of the factor set $\mathfrak{D}_3$ and the factor groups $\mathfrak{L}$ and $\mathfrak{D}_1$. Let us denote the factor set $\boldsymbol{D}_3/\sim$ by $\mathfrak{D}_3$, the group $\boldsymbol{D}_1/\boldsymbol{C}_1$ by $\mathfrak{D}_1$ and the group $\boldsymbol{L}/\{\mathbf{E}, -\mathbf{E}\}$ by $\mathfrak{L}$. On the basis of the results contained in the preceding part we can express the following

Theorem 1.9. *There exists a* $1-1$ *mapping*

$$\beta : \mathfrak{D}_3 \to \mathfrak{L}$$

given in the following way: For each $\mathscr{X}_3 \in \mathfrak{D}_3$, $\beta(\mathscr{X}_3) = \{\mathbf{C}, -\mathbf{C}\}$ *where* $\{\mathbf{C}, -\mathbf{C}\} = \varphi'\tau(\mathscr{X}_1)$ *for* $C_3\mathscr{X}_1 = \mathscr{X}_3$.

Lemma 1.4. *If we compose an arbitrary dispersion* $X_3 \in \mathscr{X}_3$ *and another one* $X_1 \in \mathscr{X}_1$ *we always obtain a dispersion from the same coset* $\mathscr{Y}_3 \in \mathfrak{D}_3$.

Proof. Let X_3 and X_1 be arbitrary dispersions in $\mathscr{X}_3$ and $\mathscr{X}_1$, respectively. Then $X_3X_1 \in \mathscr{Y}_3$. Now let $X_3 \sim \bar{X}_3$, $X_1 \sim \bar{X}_1$, that is $X_3 = \varphi_\nu \bar{X}_3$, $X_1 = \varphi_\mu \bar{X}_1$. Then $\bar{X}_3\bar{X}_1 = X_3\varphi_\nu X_1\varphi_\mu = X_3X_1\varphi_{\nu_1}\varphi_\mu = X_3X_1\varphi_\sigma$ and therefore $\bar{X}_3\bar{X}_1 \sim X_3X_1$. Consequently $\bar{X}_3\bar{X}_1 \in \mathscr{Y}_3$.

Now we can introduce a multiplication of cosets from $\mathfrak{D}_3$ and $\mathfrak{D}_1$ by means of Lemma 1.4 as follows:

$$\mathscr{X}_3\mathscr{X}_1 = \mathscr{Y}_3\,,$$

where $\mathscr{Y}_3$ is the coset from $\mathfrak{D}_3$ containing the product X_3X_1, where X_3, X_1 are arbitrary elements of $\mathscr{X}_3$ and $\mathscr{X}_1$, respectively.

Lemma 1.5. *Let* β *be the mapping from Theorem* 1.9 *and* $\varphi'\tau$ *the isomorphism from Remark* 1.1. *If* $\beta(\mathscr{X}_3) = \{\mathbf{C}, -\mathbf{C}\}$ *and* $\varphi'\tau(\mathscr{Y}_1) = \{\mathbf{G}, -\mathbf{G}\}$, *where* $\mathscr{X}_3 \in \mathfrak{D}_3$, $\mathscr{Y}_1 \in \mathfrak{D}_1$ *and* $\{\mathbf{C}, -\mathbf{C}\}, \{\mathbf{G}, -\mathbf{G}\} \in \mathfrak{L}$, *then* $\beta(\mathscr{X}_3\mathscr{Y}_1) = \{\mathbf{CG}, -\mathbf{CG}\}, \{\mathbf{CG}, -\mathbf{CG}\} \in \mathfrak{L}$.

The proof is evident.

$\mathfrak{L}$ is decomposed into two equivalent subsets:

the subset of unimodular matrix cosets whose determinant is equal to $+1$ and that one whose matrices have determinant equal to -1. A consequence of this is that $\mathfrak{D}_3$ (and also $\mathfrak{D}_1$, see [1, § 21]) decomposes into equivalent subsets as well:

the set $\mathfrak{B}_3(\mathfrak{B}_1)$ of direct, i.e., increasing dispersion cosets the corresponding matrices of which have determinant equal to $+1$ (compare with (7) in the first part of this paper);

the set of indirect (decreasing) dispersion cosets the corresponding matrices of which have determinant -1.

Theorem 1.10. *Choosing an arbitrary coset* $\mathscr{X}_3 \in \mathfrak{B}_3$ *and composing it with all* $\mathscr{X}_1 \in \mathfrak{B}_1$, *we obtain again the whole set* $\mathfrak{B}_3$. *That is,* $\mathscr{X}_3\mathfrak{B}_1 = \mathfrak{B}_3$ *for any* $\mathscr{X}_3 \in \mathfrak{B}_3$.

Proof. Let $\beta(\mathscr{X}_3) = \{\mathbf{C}, -\mathbf{C}\}$. Clearly $\det \mathbf{C} = \det(-\mathbf{C}) = +1$. Let $\varphi'\tau(\mathscr{X}_1) = \{\mathbf{G}, -\mathbf{G}\}$. Clearly $\det \mathbf{G} = \det(-\mathbf{G}) = +1$. By Lemma 1.5 $\beta(\mathscr{X}_3\mathscr{X}_1) =$

$= \{\mathbf{CG}, -\mathbf{CG}\}$ and since det $\mathbf{CG} = +1$, $\mathscr{X}_3\mathscr{X}_1 \in \mathfrak{B}_3$ for each $\mathscr{X}_1 \in \mathfrak{B}_1$. Let now $\overline{\mathscr{X}}_3$ be an arbitrary element of $\mathfrak{B}_3$. Then for $\mathscr{X}_3$ there always exists $\mathscr{Y}_1 \in \mathfrak{D}_1$ such that $\overline{\mathscr{X}}_3 = \mathscr{X}_3\mathscr{Y}_1$. We now prove the relation $\mathscr{Y}_1 \in \mathfrak{B}_1$ by means of the matrix representation:

Let $\beta(\overline{\mathscr{X}}_3) = \{\mathbf{A}, -\mathbf{A}\}$ and $\varphi'\tau(\mathscr{Y}_1) = \{\mathbf{B}, -\mathbf{B}\}$. From $\overline{\mathscr{X}}_3 = \mathscr{X}_3\mathscr{Y}_1$ we obtain by Lemma 1.5 for the corresponding cosets of matrices $\{\mathbf{A}, -\mathbf{A}\} = \{\mathbf{CB}, -\mathbf{CB}\}$.

Suppose first that $\mathbf{A} = \mathbf{CB}$. Hence the elements of the matrices satisfy

$$\begin{aligned} c_{11}b_{11} + c_{12}b_{21} &= a_{11}\,, \\ c_{11}b_{12} + c_{12}b_{22} &= a_{12}\,, \\ c_{21}b_{11} + c_{22}b_{21} &= a_{21}\,, \\ c_{21}b_{12} + c_{22}b_{22} &= a_{22} \end{aligned}$$

and hence det $\mathbf{B}$ = det $\mathbf{A}$. det $\mathbf{C} = +1$.

Similarly, if $-\mathbf{A} = \mathbf{CB}$ then det $\mathbf{B}$ = det $\mathbf{A}$. det $\mathbf{C} = +1$. Evidently also det $(-\mathbf{B}) = +1$. Matrices corresponding to the coset $\mathscr{Y}_1$ have the determinant equal to $+1$, that is, $\mathscr{Y}_1 \in \mathfrak{B}_1$.

Completely analogously we could prove the following

Theorem 1.11. *Choosing an arbitrary coset $\mathscr{X}_1 \in \mathfrak{B}_1$ and composing it with all $\mathscr{X}_3 \in \mathfrak{B}_3$ we obtain again the whole set $\mathfrak{B}_3$. Further it holds*

$$\mathscr{X}_3\mathfrak{B}_1 = \mathfrak{B}_3\mathscr{X}_1 = \mathfrak{B}_3\,,$$

where $\mathscr{X}_3$ is an arbitrary element of $\mathfrak{B}_3$ and $\mathscr{X}_1$ is an arbitrary element of $\mathfrak{B}_1$.

2. DISPERSIONS OF THE 4$^{\text{TH}}$ KIND

Representation by means of unimodular matrices. The representation will be realized analogously to that of the dispersions of the 3$^{\text{rd}}$ kind. Let $X_4 \in \boldsymbol{D}_4$ be an arbitrary dispersion of the 4$^{\text{th}}$ kind and $\boldsymbol{D}_4$ the set of all dispersions of the 4$^{\text{th}}$ kind. Now choose a basis (U, V) of the integral space R and denote its Wronskian by W; let $u_1(t)$, $v_1(t)$ be the functions

$$u_1(t) = \frac{U[X_4(t)]}{\sqrt{|X_4'(t)|}}\,, \quad v_1(t) = \frac{V[X_4(t)]}{\sqrt{|X_4'(t)|}}\,. \tag{14}$$

The functions $u_1(t)$, $v_1(t)$ form a basis of the integral space R_1. Their Wronskian w_1 fulfils

$$w_1 = W\,.\,\operatorname{sgn} X_4'\,. \tag{15}$$

Following (3) we can uniquely determine a basis (u, v) of R for the basis (u_1, v_1) of R_1. Two bases (u, v) and (U, V) are connected by (4). Thus the Wronskians satisfy (5). Since $w_1 = w$ holds, we get

$$w_1 = \det \mathbf{C} \,.\, W \tag{16}$$

and therefore $\det \mathbf{C} = \operatorname{sgn} X'_4$.

We now present a number of theorems concerning the properties of the 4[th] kind dispersions without giving their proofs since they are analogous to those of the theorems for the 3[rd] kind dispersions.

Theorem 2.1. *For any dispersion $X_4 \in \boldsymbol{D}_4$, the unimodular matrix $\mathbf{C}$ is uniquely determined by* (4).

Theorem 2.2. *For any unimodular matrix there exists at least one* 4[th] *kind dispersion associated with it through the relations* (4) *and* (14).

The decomposition of the set D_4 determined by the equivalence relation $\sim$ or $\approx$. We now introduce a relation $\sim$ in the dispersion set $\boldsymbol{D}_4$ as follows:

Let C_1 be the group of central dispersions of the 1[st] kind and let X_4, Y_4 be arbitrary dispersions from $\boldsymbol{D}_4$.

$$X_4 \sim Y_4 \text{ iff there exists } \varphi_\nu \in C_1 \text{ such that } \varphi_\nu X_4 = Y_4 \,. \tag{17}$$

Theorem 2.3. *The relation* (17) *is an equivalence relation on the set $\boldsymbol{D}_4$.*

Theorem 2.5. *The relation* (17) *forms a decomposition $\boldsymbol{D}_4/\sim$. The set C_4 of all central dispersions of the* 4[th] *kind forms exactly one coset of $\boldsymbol{D}_4/\sim$.*

Corollary 2.1. *Let $\mathcal{X}_4$ be an arbitrary coset of $\boldsymbol{D}_4/\sim$. Then $\mathcal{X}_4 = C_1X_4$ where X_4 is an arbitrary dispersion in the coset $\mathcal{X}_4$. Consequently $C_1X_4 = C_1\mathcal{X}_4$.*

Lemma 2.1. *For each dispersion of the* 4[th] *kind $X_4 \in \boldsymbol{D}_4$ and for each central dispersion of the* 4[th] *kind $\omega_\varrho \in C_4$ there exists a dispersion of the* 1[st] *kind $X_1 \in \boldsymbol{D}_1$ such that*

$$X_4 = X_1\omega_\varrho \,. \tag{18}$$

Lemma 2.2. *If two different central dispersions of the* 4[th] *kind $\omega_\varrho, \omega_\sigma \in C_4$ satisfy $X_4 = X_1\omega_\varrho$, $X_4 = Y_1\omega_\sigma$, then $X_1 = Y_1\varphi_\nu$ and thus X_1, Y_1 belong to the same coset of the factor group $\boldsymbol{D}_1/C_1$.*

Corollary 2.2. *For each dispersion of the* 4[th] *kind $X_4 \in \boldsymbol{D}_4$ there exists a dispersion $X_1 \in \boldsymbol{D}_1$ such that $X_4 \in X_1C_4$. Thus, for each dispersion X_4 there exists exactly* *one coset* $\mathscr{X}_1 = X_1 C_1$ *of the factor group* D_1/C_1 *such that* $X_4 \in \mathscr{X}_1 C_4$. *Hence* $X_1 C_4 \doteq \mathscr{X}_1 C_4$ *holds.*

We now introduce a relation $\approx$ in the set D_4 as follows:

(19) $X_4 \approx Y_4$ iff there exists $\mathscr{X}_1 \in D_1/C_1$ such that $X_4 \in \mathscr{X}_1 C_4$ and at the same time $Y_4 \in \mathscr{X}_1 C_4$.

Theorem 2.5. *The relation* (19) *is an equivalence relation in the set* D_4.

Theorem 2.6. *Two arbitrary dispersions of the* 4th *kind fulfil* $X_4 \sim Y_4$ *if and only if* $X_4 \approx Y_4$.

Theorem 2.7. *Let* X_4 *be an arbitrary dispersion from* D_4 *and let* $\mathscr{X}_4 = C_1 X_4$ *be a coset of* $D_4/\sim$. *Composing the dispersion* $X_1 \in D_1$ *associated with the dispersion* X_4 *through* (18) *with all central dispersions of the* 4th *kind we obtain exactly one coset* $\mathscr{X}_4 \in D_4/\sim$. *Thus* $\mathscr{X}_4 = X_1 C_4$ *and*

$$X_1 C_4 = C_1 X_4 = \mathscr{X}_4 \,. \tag{20}$$

Theorem 2.8. *For each coset* $\mathscr{X}_4 \in D_4/\sim$ *there exists exactly one coset* $\mathscr{X}_1 \in D_1/C_1$ *such that*

$$\mathscr{X}_4 = C_1 \mathscr{X}_4 = \mathscr{X}_1 C_4 \,. \tag{21}$$

The properties of the factor set $\mathfrak{D}_4$. Let us denote the factor set $D_4/\sim$ by $\mathfrak{D}_4$.

Theorem 2.9. *Between the elements of the set* $\mathfrak{D}_4$ *and those of the group* $\mathfrak{L}$ *there exists a* $1-1$ *correspondence*

$$\gamma : \mathfrak{D}_4 \to \mathfrak{L}$$

determined as follows: For each $\mathscr{X}_4 \in \mathfrak{D}_4$, $\gamma(\mathscr{X}_4) = \{\mathbf{C}, -\mathbf{C}\}$ *where* $\{\mathbf{C}, -\mathbf{C}\} = \varphi'\tau(\mathscr{X}_1)$ *for* $\mathscr{X}_1 C_4 = \mathscr{X}_4$.

Lemma 2.3. *If we compose an arbitrary dispersion* $X_4 \in \mathscr{X}_4$ *with an arbitrary dispersion* $X_1 \in \mathscr{X}_1$ *we always obtain a dispersion from the same coset* $\mathscr{Y}_4 \in \mathfrak{D}_4$.

By means of Lemma 2.3 we can now introduce a multiplication of cosets from $\mathfrak{D}_1$ and $\mathfrak{D}_4$ as follows:

$$\mathscr{X}_1 \mathscr{X}_4 = \mathscr{Y}_4 \,,$$

where $\mathscr{Y}_4$ is the coset from $\mathfrak{D}_4$ containing the product $X_1 X_4$, where X_1 is an element of $\mathscr{X}_1$ and X_4 is an element of $\mathscr{X}_4$.

Lemma 2.4. *If* $\gamma(\mathscr{X}_4) = \{\mathbf{C}, -\mathbf{C}\}$ *and* $\varphi'\tau(\mathscr{Y}_1) = \{\mathbf{G}, -\mathbf{G}\}$, *where* $\mathscr{X}_4 \in \mathfrak{D}_4$,