Časopis pro pěstování matematiky, roč. 99 (1974), Praha

A GENERALIZATION OF A THEOREM OF TURÁN FOR VALUATED GRAPHS*)

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(Received July 26, 1973)

I. INTRODUCTION

Let n and k be two given positive integers such that $1 \leqq k \leqq n$, and let $\mathbf{K}_n$ denote a complete undirected graph with n vertices denoted by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. The set of all vertices of $\mathbf{K}_n$ will be denoted by U_n, hence $U_n = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. For an arbitrary partial graph (see [1], p. [7]) $\mathbf{G}$ of $\mathbf{K}_n$ let $\alpha(\mathbf{G})$ stand for its stability number (see [1], p. 260). The fact that $\mathbf{G}$ is a partial graph of $\mathbf{K}_n$ will be expressed by $\mathbf{G} \subset \mathbf{K}_n$.

Let us put $q =]nk^{-1}[$[1]), $r = n - k(q - 1)$, hence $0 < r \leqq k$. Further, let $\mathscr{G}_n^{(k)}$ denote the family of all partial graphs $\mathbf{G} \subset \mathbf{K}_n$ such that $\alpha(\mathbf{G}) \leqq k$. The theorem of Turán according to the formulation given in [1], p. 269 is as follows.

Theorem 1. (Turán 1941, [2]) *If $\mathbf{G} \in \mathscr{G}_n^{(k)}$ and $\mathbf{G}$ contains the minimum number of edges then $\mathbf{G}$ has the following form: $\mathbf{G}$ consists of k complete subgraphs as connectivity-components, where some r of them contain q vertices each, and the remaining $k - r$ contain $q - 1$ vertices each.*

Remark. By using this theorem we can easily obtain the converse statement. Thus we have the following assertion: *Let $\mathbf{G} \in \mathscr{G}_n^{(k)}$. Then $\mathbf{G}$ has a minimum number of edges if and only if $\mathbf{G}$ is a graph described in the theorem* 1.

The Turán theorem gives a complete solution of the considered extremal combinatorial problem. Because of depth and beauty of the theorem many authors have tried to search its various generalizations, modifications and connections or to give alternate proofs of it (see e.g. [3–10]).

Our aim is to generalize the Turán's theorem for graphs with vertices valuated by arbitrary non-negative numbers. If we put $d_j(\mathbf{G})$ for the degree (valence) of $\mathbf{u}_j$

*) Presented at the conference on graph theory, Staré Splavy, April 1973.

[1]) $]\xi[$ denotes the minimum integer ι such that $\iota \geqq \xi$; $[\xi]$ denotes the maximum integer ι' such that $\iota' \leqq \xi$.

in $\mathbf{G}(j = 1, 2, \ldots, n)$ then the extremal problem solved by the theorem can be formulated, by using optimization theory's language, as follows: Minimize:

(1) $$\frac{1}{2}\sum_{j=1}^{n} d_j(\mathbf{G})$$

subject to

(2) $$\mathbf{G} \in \mathscr{G}_n^{(k)}.$$

Let us now assign to each vertex $\mathbf{u}_j$ of $\mathbf{K}_n$ an arbitrary nonnegative number c_j (c_j will be considered as a weight of $\mathbf{u}_j$) for $j = 1, 2, \ldots, n$. Our generalization will consist in solving the following extremal problem:

Minimize:

(3) $$\sum_{j=1}^{n} c_j d_j(\mathbf{G})$$

subject to

(2) $$\mathbf{G} \in \mathscr{G}_n^{(k)}.$$

We shall not only determine the minimum value of function (3) on the set (2), but under the additional assumption $c_j > 0$ $(j = 1, 2, \ldots, n)$ we shall describe all extremal graphs $\mathbf{G}$.

Let us put $\tau_n(k; c_1, c_2, \ldots, c_n) = \min \sum_{s=1}^{k} (\text{card}\,(V_s) - 1)\, c(V_s)$, where

1) The minimum is extended over the family of all k-partitions $\{V_1, V_2, \ldots, V_k\}$ of U_n (the term k-partition of a set will denote a disjoint decomposition of the set into k nonempty classes);

2) card (V_s) denotes the number of elements of V_s $(s = 1, 2, \ldots, k)$;

3) $c(V_s) = \sum_{u_j \in V_s} c_j \quad (s = 1, 2, \ldots, k)$.

A graph $\mathbf{G} \subset \mathbf{K}_n$ will be called a *k-clique* graph if $\mathbf{G}$ contains just k connectivity-components, each of them being a complete graph (clique). Extremal graphs corresponding to Turán's theorem are special k-clique graphs; the difference between the numbers of vertices of any two their connectivity-components is, at most, 1.

Let us denote by $^*\mathscr{G}_n^{(k)}$ the family of all k-clique graphs of $\mathscr{G}_k^{(n)}$. It follows from the definition of $\tau_n(k; c_1, c_2, \ldots, c_n)$ that

$$\tau_n(k; c_1, c_2, \ldots, c_n) = \min_{\mathbf{G} \in {}^*\mathscr{G}_n^{(k)}} \sum_{j=1}^{n} c_j d_j(\mathbf{G}).$$

Moreover, as $^*\mathscr{G}_n^{(k)} \subset \mathscr{G}_n^{(k)}$ we obtain immediately

Lemma 1. *It holds that*

$$\tau_n(k; c_1, c_2, \ldots, c_n) \geqq \min_{\mathbf{G} \in \mathscr{G}_n^{(k)}} \sum_{j=1}^{n} c_j d_j(\mathbf{G}).$$

II. RESULTS

The solution of the extremal problem (3), (2) is given by the following

Theorem 2. a) *It holds that*

$$\min_{\mathbf{G}\in\mathscr{G}_n^{(k)}} \sum_{j=1}^{n} c_j d_j(\mathbf{G}) = \tau_n(k; c_1, c_2, \ldots, c_n)$$

b) *If* $c_j > 0$ $(j = 1, 2, \ldots, n)$ *and* $\mathbf{G}_0 \in \mathscr{G}_n^{(k)}$ *and* $\sum_{j=1}^{n} c_j d_j(\mathbf{G}_0) = \tau_n(k; c_1, c_2 \ldots, c_n)$ *then* $\mathbf{G}_0$ *is a k-clique graph.*

Proof. Accordingly to lemma 1, it is sufficient to prove the following assertion: Let $\mathbf{G} \in \mathscr{G}_n^{(k)}$. Then

α) $$\sum_{j=1}^{n} c_j d_j(\mathbf{G}) \geqq \tau_n(k; c_1, c_2, \ldots, c_n)$$

β) If $\sum_{j=1}^{n} c_j d_j(\mathbf{G}) = \tau_n(k; c_1, c_2, \ldots, c_n)$ and $c_j > 0$ $(j = 1, 2, \ldots, n)$ then

$$\mathbf{G} \in {}^*\mathscr{G}_n^{(k)}.$$

We shall verify α) and β) at the same time, by using an induction with respect to k.

(i) For $k = 1$ both assertions α) and β) are obvious since $\mathscr{G}_n^{(1)} = \{\mathbf{K}_n\}$.

(ii) Let us assume that $k \geqq 2$ and that our assertion holds for $k - 1$. We shall establish its validity for k. Let us choose a vertex $\mathbf{u}_\alpha$ such that

(4) $$d_\alpha(\mathbf{G}) = \min_{j=1,2,\ldots,n} d_j(\mathbf{G}),$$

and put:

$V'' = \{\mathbf{u}_j \in U_n \mid \mathbf{u}_j$ is not adjacent to $\mathbf{u}_\alpha$ in $\mathbf{G}\}$;

$V' = U_n - V''$; $a = \text{card}\,(V')$;

$\mathbf{G}'$ will be a subgraph of $\mathbf{G}$ generated by V' ;

$\mathbf{G}''$ will be a subgraph of $\mathbf{G}$ generated by V'' .

Now, we can write

(5) $$\sum_{j=1}^{n} c_j d_j(\mathbf{G}) = \sum' + \sum'' + \sum''',$$

where

$$\sum' = \sum_{\mathbf{u}_j\in V'} c_j d_j(\mathbf{G}),$$

$$\sum'' = \sum_{\mathbf{u}_j\in V''} c_j d_j(\mathbf{G}''),$$

$$\sum''' = \sum_{\mathbf{u}_j\in V''} c_j (d_j(\mathbf{G}) - d_j(\mathbf{G}'')),$$

where $d_j(\mathbf{G}'')$ denotes the degree of $\mathbf{u}_j$ in $\mathbf{G}''$.

(Thus $d_j(\mathbf{G}) - d_j(\mathbf{G}'')$ is the number of edges connecting $\mathbf{u}_j$ with V'.)

Further, let us write $V'' = \{\mathbf{u}_{j(1)}, \mathbf{u}_{j(2)}, \ldots, \mathbf{u}_{j(n-a)}\}$. From (4), (5) and from the induction hypothesis it follows

$$\text{(6)} \qquad \begin{aligned} &\textstyle\sum' \geqq (a - 1) \sum_{\mathbf{u}_j \in V'} c_j \\ &\textstyle\sum'' \geqq \tau_{n-a}(k - 1; c_{j(1)}, c_{j(2)}, \ldots, c_{j(n-a)}) \\ &\textstyle\sum''' \geqq 0 . \end{aligned}$$

Accordingly to the definition of $\tau_{n-a}(k - 1; c_{j(1)}, \ldots)$ there is a $(k - 1)$-partition $\{V_2, V_3, \ldots, V_k\}$ of V'' such that

$$\text{(7)} \qquad \tau_{n-a}(k; c_{j(1)}, \ldots, c_{j(n-a)}) = \sum_{s=2}^{k} (\operatorname{card}(V_s) - 1)\, c(V_s) .$$

Let us consider the k-partition $\{V_1^*, V_2^*, \ldots, V_k^*\}$ of U_n such that $V_1^* = V'$ and $V_s^* = V_s$ for $s = 2, 3, \ldots, k$. By combining (5), (6), (7) and the definition of $\tau_n(k; c_1, c_2, \ldots, c_n)$ we obtain

$$\text{(8)} \qquad \begin{aligned} \sum_{j=1}^{n} c_j d_j(\mathbf{G}) &= \textstyle\sum' + \sum'' + \sum''' \geqq \\ &\geqq (a - 1) \sum_{\mathbf{u}_j \in V'} c_j + \tau_{n-a}(k - 1; c_{j(1)}, \ldots, c_{j(n-a)}) = \\ &= \sum_{s=1}^{k} (\operatorname{card}(V_s^*) - 1)\, c(V_s^*) \geqq \tau_n(k; c_1, c_2, \ldots, c_n) . \end{aligned}$$

Moreover, if $\sum_{j=1}^{n} c_j d_j(\mathbf{G}) = \tau_n(k; c_1, c_2, \ldots, c_n)$ and $c_j > 0$ for $j = 1, 2, \ldots, n$ then (6) and (8) yield $\sum' = (a - 1)\, c(V')$, $\sum''' = 0$ and $\sum'' = \tau_{n-a}(k - 1; c_{j(1)}, c_{j(2)}, \ldots, c_{j(n-a)})$. From $\sum''' = 0$ it follows that there are no edges in $\mathbf{G}$ linking V' with V''. From this fact and from $\sum' = (a - 1)\, c(V')$ we obtain that $\mathbf{G}'$ is complete and, finally, we conclude from $\sum'' = \tau_{n-a}(k - 1; c_{j(1)}, c_{j(2)}, \ldots, c_{j(n-a)})$ that $\mathbf{G}''$ is a $(k - 1)$-clique graph. Hence, $\mathbf{G}$ is a k-clique graph, and this completes the induction step and the proof.

Remark. It follows from the proved theorem immediately that in the case $c_j > 0$ $(j = 1, 2, \ldots, n)$ the following procedure yields all solutions of the problem (3), (2): We consider all k-partitions $\{V_1, V_2, \ldots, V_k\}$ of U_n and check those of them for which

$$\tau_n(k; c_1, c_2, \ldots, c_n) = \sum_{s=1}^{k} (\operatorname{card}(V_s) - 1)\, c(V_s) .$$

Now, we want to show that Turán's theorem (theorem 1) follows from the theorem 2. Indeed, when putting $c_1 = c_2 = \ldots = c_n = \frac{1}{2}$ in theorem 2 then

$$\sum_{j=1}^{n} c_j d_j(\mathbf{G}) = \tfrac{1}{2} \sum_{j=1}^{n} d_j(\mathbf{G})$$

equals the number of all edges of $\mathbf{G}$. Let us assume that

$$\mathbf{G} \in \mathscr{G}_n^{(k)} \quad \text{and} \quad \sum_{j=1}^{n} d_j(\mathbf{G}) \quad \text{is minimum} .$$

Then theorem 2 yields $\mathbf{G} \in {}^*\mathscr{G}_n^{(k)}$. Let $\{V_1, V_2, \ldots, V_k\}$ be the k-partition corresponding to $\mathbf{G}$, and let us put $a_s = \text{card}(V_s)$ $(s = 1, 2, \ldots, k)$. The integer vector $(a_1, a_2, \ldots, a_k)$ must be a solution of the following extremal problem:
Minimize

$$\sum_{s=1}^{k} a_s(a_s - 1)$$

subject to

$$\sum_{s=1}^{k} a_s = n ;$$

a_s is a positive integer $(s = 1, \ldots, k)$. It is easy to see that any optimum vector $(a_1, a_2, \ldots, a_k)$ must fulfil the following conditions:

$$|a_s - a_{s'}| \leqq 1 \quad \text{if} \quad 1 \leqq s < s' \leqq k .$$

Hence, $\mathbf{G}$ has the form asserted by the theorem 1.

In the sense of the theorem 2 we have reduced the extremal problem (3), (2) to the determination and investigation of the function $\tau_n(k; c_1, c_2, \ldots, c_n)$. However, the direct method of the computation of $\tau_n(k; c_1, c_2. \ldots, c_n)$ based on its definition is rather awkward. We shall give two properties of the function $\tau_n(k; c_1, c_2, \ldots c_n)$ which considerably facilitate its computation.

Property 1. *Let* $c_1 \geqq c_2 \geqq \ldots \geqq c_n \geqq 0$[2]. *Then*

$$\begin{aligned} &\tau_n(k; c_1, c_2, \ldots, c_n) = \\ &= \min\big((a(1) - 1)(c_1 + \ldots + c_{a(1)}) + (a(2) - 1) \times \\ &\times (c_{a(1)+1} + \ldots + c_{a(1)+a(2)}) + \ldots + (a(k) - 1) \times \\ &\times (c_{a(1)+\ldots+a(k-1)+1} + \ldots + c_{a(1)+\ldots+a(k)})\big) , \end{aligned}$$

where the minimum is extended over the family of all ordered k-tuples $(a(1), a(2), \ldots$ $\ldots, a(k))$ *of positive integers such that*

$$a(1) + a(2) + \ldots + a(k) = n \quad \text{and} \quad a(1) \leqq a(2) \leqq \ldots \leqq a(k) .$$

The property 1 follows immediately from the following well-known fact (see e.g. [11], p. 261).

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[2]) These relations can be assumed without loss of generality. It is sufficient to re-number the vertices of K_n in an appropriate way.

Lemma 2. *Let $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_m$ be arbitrary real numbers satisfying inequalities*

$$p_1 \geqq p_2 \geqq \ldots \geqq p_m \quad \text{and} \quad q_1 \leqq q_2 \leqq \ldots \leqq q_m$$

and let π be an arbitrary permutation of the set $\{1, 2, \ldots, m\}$. Then

$$\sum_{j=1}^{m} p_j q_{\pi(j)} \geqq \sum_{j=1}^{m} p_j q_j .$$

The second property yielding a recursive method for computing $\tau_n(k; c_1, c_2, \ldots, c_n)$ follows immediately from the property 1.

Property 2. *Let $c_1 \geqq c_2 \geqq \ldots \geqq c_n \geqq 0$. Then*

(i) $\tau_n(1; c_1, c_2, \ldots, c_n) = (n - 1)(c_1 + c_2 + \ldots + c_n)$

(ii) For $k = 2, 3, \ldots, n : \tau_n(k; c_1, \ldots, c_2, \ldots, c_n) = \min\limits_{a=1,2\ldots,[nk^{-1}]} ((a - 1)(c_1 + \ldots \ldots + c_a) + \tau_{n-a}(k - 1; c_{a+1}, \ldots, c_n))$.

Example. $n = 7$; K_7; $k = 3$; $c_j = (8 - j)^2$ for $j = 1, 2, \ldots, 7$. $\tau_7(3; 7^2, \ldots, 1^2) = \min(\tau_6(2; 6^2, \ldots, 1^2), (7^2 + 6^2) + \tau(2; 5^2, \ldots, 1^2))$; $\tau_6(2; 6^2, \ldots, 1^2) = \min(\tau_5(1; 5^2, \ldots, 1^2); (6^2 + 5^2) + \tau_4(1; 4^2, \ldots, 1^2); 2(6^2 + 5^2 + 4^2) + \tau_3(1; 3^2, 2^2, 1^2)) = \min . (220; \overline{151}; 182) = 151$; $\tau_5(2; 5^2, \ldots, 1^2) = \min(\tau_4(1; 4^2, \ldots, 1^2); (5^2 + 4^2) + \tau_3 . (1; 3^2, 2^2, 1^2)) = \min(90; \overline{69}) = 69$; $\tau_7(3; 7^2, \ldots, 1^2) = \min(\overline{151}; 154) = 151$. Moreover, it follows from the performed computation that the extremal problem (3), (2) has in our case a unique solution: the 3-clique graph induced by the 3-partition $\{\{u_1\}, \{u_2, u_3\}, \{u_4, u_5, u_6, u_7\}\}$.

III. CONCLUDING REMARKS

1) Generalizations and modifications of the Turán theorem, discussed in the literature are usually connected with variations of the condition (2). In this paper, the set of considered graphs (2) was left without any change, but, the objective function (1) was replaced by a more general function.

2) Some extremal problems in the graph theory, resp. methods of their solution, allow essential generalizations for graphs the vertices of which are valuated by real numbers (e.g. various matching problems, shortest-path problems, network-flow problems). Our result shows that a generalization of this type can be done for the Turán theorem.

3) The identity (ii) of property 2 can be viewed as a special case of a functional equation of the dynamic programming (see e.g. [12]).