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SUBADDITIVE MEASURES AND SMALL SYSTEMS

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By a subadditive measure (see e.g. [1], [2], [3]) we mean a subadditive, monotone, non-negative real valued set-function μ defined on a ring and upper semicontinuous in $\emptyset$. It can be easily proved that μ is upper and lower semicontinuous in any set and therefore also σ-subadditive.

We shall assume that μ is a subadditive measure on a σ-ring $\mathscr{S}$. Let $\mathscr{N}_n$ be the family of all sets $E \in \mathscr{S}$ for which $\mu(E) < 2^{-n}$. Then all the properties of "small systems" (see Section 1 and also [4], [5], [6], [7], [8], [12], [14]) are satisfied. Originally, small systems were introduced for generalizations of some properties of measures, nevertheless, the results obtained can be applied also to any subadditive measure.

Section 1 contains, besides axioms and related results, a theorem on representation of small systems by subadditive measures. In Section 2 we present similar results for "subadditive integral" and "small systems" of functions. Finally, in Section 3 we produce small systems of sets from small systems of functions.

1. REPRESENTATION THEOREM

There are various systems of axioms for "small systems". The following one corresponds with our representation theorem and it was used in the paper [8].

1.1. Axioms. *Let $\mathscr{S}$ be a σ-ring of subsets of a set X. We shall assume that to any $n = 0, 1, 2, \ldots$ a system $\mathscr{N}_n \subset \mathscr{S}$ is given in such a way that the following axioms are satisfied:*

I. *$\emptyset \in \mathscr{N}_n$ for all n.*

II. *If $E_i \in \mathscr{N}_i$ $(i = n + 1, n + 2, \ldots)$ then $\bigcup_{i=n+1}^{\infty} E_i \in \mathscr{N}_n$.*

III. *If $E_i \in \mathscr{N}_0$, $E_i \supset E_{i+1}$ $(i = 1, 2, \ldots)$ and $\bigcap_{i=1}^{\infty} E_i = \emptyset$ then to any n there is m such that $E_m \in \mathscr{N}_n$.*

IV. *If* $E \subset F$, $F \in \mathcal{N}_n$, $E \in \mathcal{S}$ *then* $E \in \mathcal{N}_n$.

V. $\mathcal{N}_{n+1} \subset \mathcal{N}_n$ *for all* n.

Many results in various papers were obtained by the help of the following condition weaker than II: To any n there is a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, \ldots)$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n$. On the other hand, we shall use here a system of axioms a little stronger than the system 1.1. Of course, the systems induced by any measure or subadditive measure fulfil also the stronger axioms (with $\mathcal{N}_0 = \{E \in \mathcal{S};\ \mu(E) < \infty\}$, $\mathcal{N}_n = \{E \in \mathcal{S};\ \mu(E) < 2^{-n}\}$).

1.2. Axiom II*. *If* $E_i \in \mathcal{N}_{r_i}$ $(i = 1, \ldots, k)$ *where* $\sum_{i=1}^{k} 2^{-r_i} \leqq 2^{-n}$ *and* $E \in \mathcal{S}$, $E \subset \bigcup_{i=1}^{k} E_i$, *then* $E \in \mathcal{N}_n$.

1.3. Theorem. *The axiom* II* *implies* IV. *If* $\mathcal{N}_0 = \mathcal{S}$ *then the axioms* II*, III *and* V *imply* II. *The axioms* I–V *do not imply* II*.

Proof. Let $E \subset F$, $F \in \mathcal{N}_n$, $E \in \mathcal{S}$. Since $2^{-n} \leqq 2^{-n}$ we have $E \in \mathcal{N}_n$ according to II*, hence IV is proved.

Put $r_i = 2i$ $(i = 1, 2, \ldots)$. Let $E_i \in \mathcal{N}_{2i}$, $i \geqq n + 1$. Since

$$\bigcup_{i=n+1}^{n+k} E_i \subset \bigcup_{i=n+1}^{n+k} E_i \quad \text{and} \quad \sum_{i=n+1}^{n+k} 2^{-2i} \leqq 2^{-2n-1}$$

we have according to II*

$$\bigcup_{i=n+1}^{n+k} E_i \in \mathcal{N}_{2n+1}\,.$$

Put $F_k = \bigcup_{i=n+1}^{n+k} E_i$, $E = \bigcup_{i=n+1}^{\infty} E_i - \bigcap_{j=n+1}^{\infty} E_j$. Then $F_k \in \mathcal{N}_{2n+1}$ $(k = 1, 2, \ldots)$. On the other hand $E - F_k \searrow \emptyset$ $(k \to \infty)$. According to III there is k such that

$$E - F_k \in \mathcal{N}_{2n+2}\,.$$

Finally $\bigcap_{j=n+1}^{\infty} E_j \subset E_{n+2} \in \mathcal{N}_{2n+4} \subset \mathcal{N}_{2n+3}$, hence

$$E = \bigcap_{j=n+1}^{\infty} E_j \cup F_k \cup (E - F_k) \in \mathcal{N}_{2n}$$

and II is proved. The last assertion follows from the following example.

1.4. Example. Let $X = \langle 0, 1 \rangle$, $\mathcal{S}$ the family of all Borel subsets of $\langle 0, 1 \rangle$, μ the Lebesgue measure. Put $\mathcal{N}_n = \{E \in \mathcal{S};\ \mu(E) < 2^{-n-1}\}$, $\mathcal{N}_2 = \{E \in \mathcal{S};\ \mu(E) < 1/3\}$, $\mathcal{N}_1 = \{E \in \mathcal{S};\ \mu(E) < \frac{1}{2}\}$, $\mathcal{N}_0 = \mathcal{S}$. Then all the axioms I–V are satisfied but II* does not hold. Namely, $E_1 = \langle 0, \frac{1}{4} \rangle \in \mathcal{N}_2$, $E_2 = \langle \frac{1}{4}, \frac{1}{2} \rangle \in \mathcal{N}_2$, $E = \langle 0, \frac{1}{2} \rangle \subset E_1 \cup E_2$, $2^{-2} + 2^{-2} \leqq 2^{-1}$, but $E \notin \mathcal{N}_1$.

1.5. Definition. A non-negative function $\mu : \mathcal{S} \to R$ is said to be *equivalent to a sequence* $\{\mathcal{N}_n\}_{n=1}^{\infty}$ of subfamilies of $\mathcal{S}$ if the following two conditions are satisfied:

A. To any $\varepsilon > 0$ there is a positive integer n such that $E \in \mathcal{N}_n$ implies $\mu(E) < \varepsilon$.

B. To any positive integer n there is $\varepsilon > 0$ such that $\mu(E) < \varepsilon$ implies $E \in \mathcal{N}_n$.

1.6. Representation theorem. *Let* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *be a sequence of subfamilies of a* σ*-ring* $\mathcal{S}$ *satisfying the axioms* II*, III *and* V. *Let* $\mathcal{N}_0$ *be closed under finite unions. Then there is a subadditive measure* $\mu : \mathcal{S} \to R$ *equivalent to the sequence* $\{\mathcal{N}_n\}_{n=0}^{\infty}$.

Proof. Define first a function $\delta : \mathcal{S} \to R$ in the following way. If $E \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$ then $\delta(E) = 0$, if $E \notin \mathcal{N}_0$ then $\delta(E) = \infty$ and if $E \in \mathcal{N}_n - \mathcal{N}_{n+1}$ for some n then $\delta(E) = 2^{-n}$. Further, put for any $E \in \mathcal{S}$

$$\mu(E) = \inf \{ \sum_{i=1}^{k} \delta(E_i) \, ; \quad E_i \in S \, , \quad E \subset \bigcup_{i=1}^{k} E_i \, , \quad k \text{ positive integer} \} \, .$$

Evidently $\mu(E) \leqq \delta(E)$, hence $\mu(E) \leqq 2^{-n}$ for $E \in \mathcal{N}_n$. μ is clearly monotone, non-negative and subadditive. We have to prove that μ is upper continuous in $\emptyset$.

Let $E_n \supset E_{n+1}$, $\mu(E_n) < \infty$ $(n = 1, 2, \ldots)$, $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Since $\mu(E_1) < \infty$ there are $F_j \in \mathcal{N}_0$ such that $E_1 \subset \bigcup_{j=1}^{p} F_j$, hence $E_1 \in \mathcal{N}_0$. Therefore $E_n \in \mathcal{N}_0$ $(n = 1, 2, \ldots)$. Let $\varepsilon > 0$. Take n such that $2^{-n} < \varepsilon$. Then according to III there is such m that $E_m \in \mathcal{N}_n$. Hence for sufficiently large m

$$\mu(E_m) \leqq \delta(E_m) \leqq 2^{-n} < \varepsilon$$

and therefore

$$\lim \mu(E_m) = 0 \, .$$

Now we prove the equivalency of μ and $\{\mathcal{N}_n\}_{n=0}^{\infty}$. Let $\varepsilon > 0$. Take n such that $2^{-n} < \varepsilon$. If $E \in \mathcal{N}_n$ then $\mu(E_n) \leqq 2^{-n} < \varepsilon$. Let us point out that we have not used yet the axiom II*.

Finally, let n be a positive integer, Put $\varepsilon = 2^{-n}$. If $\mu(E) < 2^{-n}$ then there are $E_i \in \mathcal{N}_{r_i}$ $(i = 1, \ldots, k)$ such that

$$E \subset \bigcup_{i=1}^{k} E_i \, , \quad \sum_{i=1}^{k} 2^{-r_i} < 2^{-n} \, .$$

According to II* we have $E \in \mathcal{N}_n$.

2. SMALL SYSTEMS OF FUNCTIONS

Such systems (analogous to systems of small sets) were studied in [9], [10], [13] and [15]. Here we shall work with the following systems of axioms (see [9]):

2.1. Axioms. *Let $\mathcal{M}$ be the family of measurable functions (with respect to a measurable space (X, S)). Let $\{\mathcal{F}_n\}_{n=0}^{\infty}$ be a sequence of subfamilies of S satisfying the following conditions:*

i. *$0 \in \mathcal{F}_n$ for every n; $f \in \mathcal{F}_n \Leftrightarrow -f \in \mathcal{F}_n$.*

ii. *If $f_i \in \mathcal{F}_i$, $f_i \geqq 0$ $(i = n, \ldots, n + r)$, then $\sum_{i=n}^{n+r} f_i \in \mathcal{F}_{n-1}$.*

iii. *Let $f_i \in \mathcal{F}_0$, $f_i \geqq f_{i+1}$ $(i = 1, 2, \ldots)$, $\lim_{i \to \infty} f_i(x) = 0$ for every $x \in X$ (in this case we write shortly $f_i \searrow 0$). Then to any n there is m such that $f_m \in \mathcal{F}_n$.*

iv. *If $f \in \mathcal{M}$, $g \in \mathcal{F}_n$ and $|f| \leqq |g|$, then $f \in \mathcal{F}_n$.*

v. *$\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for every n.*

2.2. Example. Let $\mathcal{F}_0$ be the family of all integrable functions (with respect to a measure μ), $\mathcal{F}_n = \{f \in \mathcal{F}_0;\ \int |f| \, d\mu < 2^{-n}\}$. Evidently all asumptions i–v are satisfied.

More generally, we can construct a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ by the help of a function $J : \mathcal{F}_0 \to R$ with certain properties.

2.3. Definition. Let $\mathcal{M}$ be the family of measurable functions, $\mathcal{F}_0 \subset \mathcal{M}$. A mapping $J : \mathcal{F}_0 \to R$ is called *a subadditive integral* (see also [9]) if it has the following properties:

1. $\mathcal{F}_0$ is an additive group (with respect to the usual addition); $J(0) = 0$; $J(f + g) \leqq J(f) + J(g)$ for all non-negative f, g.

2. If $f, g \in \mathcal{F}_0$, $f \leqq g$ then $J(f) \leqq J(g)$; if $f \in \mathcal{M}$, $g \in \mathcal{F}_0$ and $|f| \leqq g$ then $f \in \mathcal{F}_0$.

3. If $f_n \searrow 0$, $f_n \in \mathcal{F}_0$ $(n = 1, 2, \ldots)$, *then* $J(f_n) \searrow 0$.

2.4. Theorem. *Let J be a subadditive integral. Put $\mathcal{F}_n = \{f \in \mathcal{F}_0;\ J(|f|) < 2^{-n}\}$. Then $\{\mathcal{F}_n\}_{n=1}^{\infty}$ fulfils the axioms i–v. Moreover, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ fulfils the following stronger conditions* ii*. *If $0 \leqq f \leqq \sum_{i=1}^{p} f_i$, $f_i \in \mathcal{F}_{r_i}$ $(i = 1, \ldots, p)$ and $\sum_{i=1}^{p} 2^{-r_i} \leqq 2^{-n}$, then $f \in \mathcal{F}_n$;* ii**. *If $f_i \in \mathcal{F}_i$, $f_i \geqq 0$ $(i = n, n + 1, \ldots)$ then $\bigcup_{i=n}^{\infty} f_i \in \mathcal{F}_{n-1}$.*

Proof. i and ii follows from 1, iii form 3, iv from 2. The property v follows immediately from the definition.

If $0 \leqq f \leqq \sum_{i=1}^{p} f_i$, $f_i \in \mathcal{F}_{r_i}$ $(i = 1, \ldots, p)$, $\sum_{i=1}^{p} 2^{-r_i} \leqq 2^{-n}$, then $J(f) \leqq \sum_{i=1}^{p} J(f_i) \leqq \sum_{i=1}^{p} 2^{-r_i} \leqq 2^{-n}$, hence $f \in \mathcal{F}_n$.

Before proving ii** we prove first that $f_n \nearrow f$ implies $J(f_n) \nearrow J(f)$. Indeed, $f_n \nearrow f$ implies $f - f_n \searrow 0$, hence $J(f - f_n) \searrow 0$. But

$$0 \leqq J(f) - J(f_n) \leqq J(f - f_n),$$

hence also $J(f_n) \nearrow J(f)$.

Finally, we prove ii**. Evidently $J(\sum_{i=n}^{n+r} |f_i|) \leqq \sum_{i=n}^{n+r} J(|f_i|) < 2^{-n+1}$. But $g_r = \sum_{i=n}^{n+r} |f_i| \nearrow \sum_{i=n}^{\infty} |f_i|$, hence $J(\sum_{i=n}^{\infty} |f_i|) = \lim_{r\to\infty} J(g_r) \leqq 2^{-n+1}$. Therefore also $\sum_{i=n}^{\infty} f_i \in \mathscr{F}_n$.

2.5. Theorem. *Let $\{\mathscr{F}_n\}_{n=0}^{\infty}$ be a sequence satisfying the axioms* ii*, iii, iv *and* v. *Then there is a subadditive integral $J: \mathscr{F}_0 \to R$ equivalent to the sequence $\{\mathscr{F}_n\}_{n=0}^{\infty}$, i.e., such that to any $\varepsilon > 0$ there exists m such that ($f \in \mathscr{F}_n \Rightarrow J(|f|) < \varepsilon$) and to any n there exists $\varepsilon > 0$ such that ($J(|f|) < \varepsilon \Rightarrow f \in \mathscr{F}_n$).*

Proof. Put $\delta(f) = 2^{-n}$ if $f \in \mathscr{F}_n - \mathscr{F}_{n-1}$ $(n = 2, 3, \ldots)$, $\delta(f) = 0$ if $f \in \bigcap_{n=1}^{\infty} \mathscr{F}_n$. Further, for $f \geqq 0$ we define

$$J(f) = \inf \{ \sum_{i=1}^{k} \delta(f_i) ; \quad f \leqq \sum_{i=1}^{k} f_i \}$$

and

$$J(f) = J(f^+) - J(f^-)$$

for any $f \in \mathscr{F}_0$. Evidently $\delta(f) \geqq J(f) \geqq 0$ for $f \geqq 0$, hence $0 \leqq J(0) \leqq \delta(0) = 0$. Also the other properties from 1 and 2 are clear for nonnegative functions. In the general case they can be obtained by the decomposition $J(f) = J(f^+) - J(f^-)$.

Let $f_n \searrow 0$, $\varepsilon > 0$. Choose n_0 such that $2^{-n_0} < \varepsilon$ and m_0 such that $f_{m_0} \in \mathscr{F}_{n_0}$. If $m > m_0$, then $0 \leqq f_m \leqq f_{m_0}$, hence $J(f_m) \leqq J(f_{m_0}) \leqq \delta(f_{m_0}) < 2^{-n_0} < \varepsilon$, therefore $\lim_{m\to\infty} J(f_m) = 0$.

Finally, we prove the equivalency of J and $\{\mathscr{F}_n\}_{n=0}^{\infty}$. Take $\varepsilon > 0$ and n such that $2^{-n+1} < \varepsilon$. Let $f \in F_n$. Then according to iv also $f^+, f^- \in \mathscr{F}_n$. Therefore

$$J(|f|) \leqq J(f^+) + J(f^-) \leqq \delta(f^+) + \delta(f^-) \leqq 2.2^{-n} < \varepsilon .$$

On the other hand, let n be a positive integer. Put $\varepsilon = 2^{-n-1}$. Let $J(|f|) < \varepsilon$. Then there are $f_i \in \mathscr{F}_{r_i}$ $(i = 1, \ldots, p)$ such that

$$J(|f|) \leqq \sum_{i=1}^{p} \delta(f_i) < \varepsilon = 2^{-n-1} .$$

Then $|f| \in \mathscr{F}_{n+1}$ according to ii*, $f^+, f^- \in \mathscr{F}_{n+1}$ according to iv and $f = f^+ - f^- \in \mathscr{F}_n$ according to ii*.

3.1. Theorem. *Let $\{\mathscr{F}_n\}_{n=0}^{\infty}$ be a sequence of systems of measurable functions satisfying conditions* i, iii, iv, v. *Then $\mathscr{N}_n = \{E;\ \chi_E \in \mathscr{F}_n\}$, $n = 0, 1, 2, \ldots$ satisfies conditions* I, III, IV, V. *If $\{\mathscr{F}_n\}_{n=0}^{\infty}$ satisfies* ii** *then $\{\mathscr{N}_n\}_{n=0}^{\infty}$ satisfies* II. *If $\{\mathscr{F}_n\}_{n=0}^{\infty}$ satisfies* ii* *then $\{\mathscr{N}_n\}_{n=0}^{\infty}$ satisfies* II*, *hence* II *as well.*

Proof. The properties I, IV and V are evident. Prove the condition III. Let $E_n \searrow \emptyset$. Then $\chi_{E_n} \searrow 0$, hence to any m there exists n such that $\chi_{E_n} \in \mathscr{F}_m$. Therefore to any m there is n such that $E_n \in \mathscr{N}_m$.

Now let ii** be satisfied. Let $E_i \in \mathscr{N}_i$ $(i = n, n+1, \ldots)$. Then $\chi_{E_i} \in \mathscr{F}_i$, hence $\sum_{i=n}^{\infty} \chi_{E_i} \in \mathscr{F}_{n-1}$. But $\chi_{\cup E_i} \leqq \sum_{i=n}^{\infty} \chi_{\cup E_i}$, hence $\chi_{\cup E_i} \in \mathscr{F}_{n-1}$ and $\bigcup_{i=n}^{\infty} E_i \in \mathscr{N}_{n-1}$.

The implication $\mathscr{F}_n$ satisfies ii* $\Rightarrow$ $\mathscr{N}_n$ satisfies II* is obvious.

3.2. Theorem. *Let $\{\mathscr{N}_n\}_{n=0}^{\infty}$ satisfy* I–V. *Then there is $\{\mathscr{F}_n\}_{n=0}^{\infty}$ such that $\mathscr{N}_n \subset \{E;\ \chi_E \in \mathscr{F}_n\}$ and $\{\mathscr{F}_n\}_{n=0}^{\infty}$ satisfies* i, ii, iv, v *and* iii *with f_1 simple (i.e. $f_1 = \sum_{i=1}^{r} c_i \chi_{E_i}$, $\bigcup_{i=1}^{r} F_i \in \mathscr{N}_0$).*

Proof. For $E \in \mathscr{S}$ put $|E| = \inf\{2^{-n};\ E \in \mathscr{N}_n\}$. Further

$$\overline{\mathscr{F}}_n = \{f;\ f = \sum_{i=1}^{k} c_i \chi_{E_i},\ E_i \in \mathscr{S},\ \sum_{i=1}^{k} |c_i|\,|E_i| \leqq 2^{-n}\}\,,$$

$$\mathscr{F}_n = \{f;\ f \text{ measurable}, \exists f_i \in \overline{\mathscr{F}}_n,\ f_i \nearrow |f|\}\,.$$

Evidently i and v holds. First we prove iv. Let f, g be simple, $g \in \overline{\mathscr{F}}_n$, $|f| \leqq |g|$. If $f = \sum c_i \chi_{E_i}$, $g = \sum d_i \chi_{E_i}$, E_i disjoint, then $|c_i| \leqq |d_i|$, hence $\sum |c_i|\,|E_i| \leqq \sum |d_i|\,|E_i| \leqq 2^{-n}$, since $g \in \overline{\mathscr{F}}_n$. It follows $f \in \overline{\mathscr{F}}_n$. Now let f, g be arbitrary, measurable, $f_i \nearrow |f|$, $g_i \nearrow |g|$, $g_i \in \overline{\mathscr{F}}_n$ $(i = 1, 2, \ldots)$. Put $h_i = \min(f_i, g_i)$. Then $|h_i| \leqq |g_i|$, hence $h_i \in \overline{\mathscr{F}}_n$. Since $h_i \nearrow |f|$ we get $f \in \mathscr{F}_n$.

Let $f_i \in \overline{\mathscr{F}}_i$ $(i = n, \ldots, n + r)$, $f_i = \sum_{j=1}^{k_i} c_i^j \chi_{E_i^j}$, $\sum_{j=1}^{k_i} |c_i^j|\,|E_i^j| \leqq 2^{-i}$. Then

$$\sum_{i=n}^{n+r} f_i = \sum_{i=n}^{n+r} \sum_{j=1}^{k_i} c_i^j \chi_{E_i^j}\,, \quad \sum_{i=n}^{n+r} \sum_{j=1}^{k_i} |c_i^j|\,|E_i^j| \leqq \sum_{i=n}^{n+r} 2^{-i} < 2^{-n+1}\,,$$

hence $\sum_{i=n}^{n+r} f_i \in \overline{\mathscr{F}}_n$.

If $f_i \in \mathscr{F}_i$ $(i = n, n+1, \ldots, n+r)$, then there are $f_i^j \in \overline{\mathscr{F}}_n$ such that $f_i^j \nearrow |f_i|$. But $\sum_{i=n}^{n+r} f_i^j \nearrow \sum_{i=n}^{n+r} |f_i|$ $(j \to \infty)$, hence $\sum_{i=n}^{n+r} |f_i| \in \mathscr{F}_{n-1}$ and also $\sum_{i=n}^{n+r} f_i \in \mathscr{F}_{n-1}$. Hence the condition ii is proved.