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ON VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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In this note the Volterra-Stieltjes integral equation in the space of functions with bounded variation is dealth with. The integrals used here are Perron-Stieltjes integrals. The basic definitions and notations are the same as in [4].

1. INTRODUCTION AND AUXILIARY STATEMENTS

The Volterra-Stieltjes integral equation under consideration has the form

$$x(s) = \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) + \mathbf{y}(s)\,, \quad 0 \leqq s \leqq 1\,. \tag{1,1}$$

By $V_n(0, 1) = V_n$ we denote the space of all n-vector functions of bounded variation on $[0, 1]$, V_n equipped with the norm $\|\mathbf{x}\|_{V_n} = \|\mathbf{x}(0)\| + \operatorname{var}_0^1 \mathbf{x}$ forms a complete normed linear (Banach) space. Similarly, for an interval $[a, b]$ we define the space $V_n(a, b)$ of n-vector functions defined on $[a, b]$ with bounded variation on $[a, b]$; the corresponding norm is $\|\mathbf{x}\|_{V_n(a,b)} = \|\mathbf{x}(a)\| + \operatorname{var}_a^b \mathbf{x}$ for $\mathbf{x} \in V_n(a, b)$.

For a $k \times l$-matrix $\mathbf{A} = (a_{ij})$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ we are setting $\|\mathbf{A}\| = \max_{i=1,\ldots,k} \sum_{j=1}^{l} |a_{ij}|$.

We suppose in the following that $\mathbf{y} \in V_n$. As for the kernel $\mathbf{K}(s, t)$, we suppose that it is an $n \times n$-matrix for all $(s, t) \in [0, 1] \times [0, 1]$ $(\mathbf{K}(s, t) : [0, 1] \times [0, 1] \to L(R^n \to R^n))$ such that

$$v(\mathbf{K}) < +\infty \tag{1,2}$$

where $v(\mathbf{K})$ denotes the twodimensional variation in the sense of Vitali for the matrix $\mathbf{K}$ in the interval $I = [0, 1] \times [0, 1]$[1]).

[1]) The number $v(\mathbf{K})$ is defined by the relation $v(\mathbf{K}) = \sup \sum_i \|m_{\mathbf{K}}(J_i)\|$ where the supremum is taken over all finite systems of nonoverlapping intervals $J_i \subset I$, $J_i = [a_i, b_i] \times [c_i, d_i]$ and where $m_{\mathbf{K}}(J_i) = \mathbf{K}(b_i, d_i) - \mathbf{K}(b_i, c_i) - \mathbf{K}(a_i, d_i) + \mathbf{K}(a_i, c_i)$ (for this notion cf. [4] or [1]).

Further we assume that

$$(1,3) \qquad \operatorname{var}_0^1 \mathbf{K}(1, .) < +\infty$$

where $\operatorname{var}_0^1 \mathbf{K}(1, .)$ denotes the variation of the matrix $\mathbf{K}(s, t)$ for $s = 1$ in the interval $[0, 1]$. The variation of a matrix-valued function is defined in the usual way using the norm of a matrix.

If (1,2) and (1,3) is satisfied then by Prop. 2,3 in [4] the integral $\int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)$ exists for any $s \in [0, 1]$ and $\mathbf{x} \in V_n$.

To the kernel $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ we define a new kernel $\mathbf{K}^\Delta(s, t) : I \to \to L(R^n \to R^n)$ in the following way:

$$(1,4) \qquad \begin{aligned} \mathbf{K}^\Delta(s, t) &= \mathbf{K}(s, t) - \mathbf{K}(s, 0) \quad \text{if} \quad 0 \leqq t \leqq s \leqq 1\,, \\ \mathbf{K}^\Delta(s, t) &= \mathbf{K}(s, s) - \mathbf{K}(s, 0) = \mathbf{K}^\Delta(s, s) \quad \text{if} \quad 0 \leqq s < t \leqq 1\,. \end{aligned}$$

Evidently, $\mathbf{K}^\Delta(s, 0) = 0$ for any $s \in [0, 1]$ and $\mathbf{K}^\Delta(0, t) = 0$ for any $t \in [0, 1]$.

The kernel $\mathbf{K}^\Delta$ is the triangular kernel corresponding to the kernel $\mathbf{K}$. By Lemma 1,3,1 in [2] and by the definition of $\mathbf{K}^\Delta$ we have

$$\int_s^1 \mathrm{d}_t[\mathbf{K}^\Delta(s, t)]\, \mathbf{x}(t) = 0$$

for any $s \in [0, 1]$ and $\mathbf{x} \in V_n$. Further we have

$$\int_0^s \mathrm{d}_t[\mathbf{K}^\Delta(s, t)]\, \mathbf{x}(t) = \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)$$

for any $s \in [0, 1]$, $\mathbf{x} \in V_n$ since Theorem 1,3,6 in [2] implies

$$\int_0^s \mathrm{d}_t[\mathbf{K}^\Delta(s, t) - \mathbf{K}(s, t)]\, \mathbf{x}(t) =$$
$$= \lim_{\tau \to s-} [\mathbf{K}^\Delta(s, \tau) - \mathbf{K}(s, \tau) - \mathbf{K}^\Delta(s, s) + \mathbf{K}(s, s)]\, \mathbf{x}(s) = 0\,.$$

Hence we have

$$(1,5) \qquad \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) = \int_0^1 \mathrm{d}_t[\mathbf{K}^\Delta(s, t)]\, \mathbf{x}(t)$$

for any $s \in [0, 1]$ and $\mathbf{x} \in V_n$. By means of (1,5) the Volterra-Stieltjes integral equation (1,1) can be rewritten in the form

$$(1,6) \qquad \mathbf{x}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}^\Delta(s, t)]\, \mathbf{x}(t) + \mathbf{y}(s), \quad s \in [0, 1]\,.$$

Throughout the paper the equation (1,1) will be studied in this new form (1,6).

By the relations (1,4) a new matrix $\mathbf{K}^{\Delta} : I \to L(R^n \to R^n)$ is defined in terms of the matrix $\mathbf{K} : I \to L(R^n \to R^n)$. For $\mathbf{K}^{\Delta}$ we have

$$v(\mathbf{K}^{\Delta}) \leqq 2v(\mathbf{K}) + \operatorname{var}_0^1 \mathbf{K}(1, .) . \tag{1,7}$$

This inequality can be established as follows: Let be given a net type subdivision of the interval I, i.e., let be given a finite sequence $0 = \tau_0 < \tau_1 < \ldots < \tau_m = 1$ and let us define intervals $J_{ij} = [\tau_{i-1}, \tau_i] \times [\tau_{j-1}, \tau_j]$, $i, j = 1, 2, \ldots, m$; the system J_{ij} forms a net type subdivision of I.

Let us define

$$m_{\mathbf{K}^{\Delta}}(J_{ij}) = \mathbf{K}^{\Delta}(\tau_i, \tau_j) - \mathbf{K}^{\Delta}(\tau_i, \tau_{j-1}) - \mathbf{K}^{\Delta}(\tau_{i-1}, \tau_j) + \mathbf{K}^{\Delta}(\tau_{i-1}, \tau_{j-1}) .$$

From (1,4) we have $m_{\mathbf{K}^{\Delta}}(J_{ij}) = m_{\mathbf{K}}(J_{ij})$ for $0 \leqq j < i \leqq m$, $m_{\mathbf{K}^{\Delta}}(J_{ij}) = 0$ for $0 \leqq i < j \leqq m$ and $m_{\mathbf{K}^{\Delta}}(J_{jj}) = \mathbf{K}(\tau_j, \tau_j) - \mathbf{K}(\tau_j, \tau_{j-1})$ for $j = 1, 2, \ldots, m$. Hence

$$\begin{aligned}
\sum_{i,j=1}^{m} \|m_{\mathbf{K}^{\Delta}}(J_{ij})\| &= \sum_{i=1}^{m} \sum_{j=1}^{i} \|m_{\mathbf{K}^{\Delta}}(J_{ij})\| = \\
&= \sum_{i=1}^{m} \sum_{j=1}^{i-1} \|m_{\mathbf{K}}(J_{ij})\| + \sum_{i=1}^{m} \|m_{\mathbf{K}}(J_{ii})\| = \sum_{i=1}^{m} \sum_{j=1}^{i-1} \|m_{\mathbf{K}}(J_{ij})\| + \\
&\quad + \sum_{i=1}^{m} \|\mathbf{K}(\tau_i, \tau_i) - \mathbf{K}(\tau_i, \tau_{i-1}) - \mathbf{K}(0, \tau_i) + \mathbf{K}(0, \tau_{i-1})\| + \\
&\quad + \sum_{i=1}^{m} \|\mathbf{K}(0, \tau_i) - K(0, \tau_{i-1})\| \leqq v(\mathbf{K}) + \operatorname{var}_0^1 \mathbf{K}(0, .) .
\end{aligned}$$

The inequality (1,7) follows now from the definition of $v(\mathbf{K}^{\Delta})$ if we use the inequality $\operatorname{var}_0^1 \mathbf{K}(0, .) \leqq v(\mathbf{K}) + \operatorname{var}_0^1 \mathbf{K}(1, .)$ (see (2,14a) in [4]),

Let us mention that the assumption that the subdivision is of the net type does not cause any loss of generality in the proof since any finite subdivision can be completed to a finite net type subdivision.

Remark 1.1. From the inequality (1,7) it is easy to see that if (1,2) and (1,3) are satisfied, then $v(\mathbf{K}^{\Delta}) < \infty$. The definition (1,4) of $\mathbf{K}^{\Delta}$ ensures that $\operatorname{var}_0^1 \mathbf{K}^{\Delta}(0, .) < \infty$ and $\operatorname{var}_0^1 \mathbf{K}^{\Delta}(., 0) < \infty$. Hence the kernel $\mathbf{K}^{\Delta}(s, t)$ satisfies all the assumptions of Theorem 5,2 from [4]. Using this theorem we can conclude that for the equation (1,6) the Fredholm alternative is valid, i.e., either the equation (1,6) admits a unique solution for any $\mathbf{y} \in V_n$ or the corresponding homogeneous equation

$$\mathbf{x}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}^{\Delta}(s, t)]\, \mathbf{x}(t) , \quad s \in [0, 1]$$

admits a finite number r of linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_r \in V_n$. This alternative theorem can be formulated also in terms of the equation (1,1) and of the corresponding homogeneous equation. Our aim is to prove that in the case under consideration the first possibility of the Fredholm alternative takes place, i.e., that the equation (1,1) is really a Volterra-type equation or, in other words, that the operator $\mathbf{x} \in V_n \to \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) \in V_n$ has only one eigenvalue which is equal to zero.

Proposition 1.1. *Let $\mathbf{L}, \mathbf{M} : I \to L(R^n \to R^n)$ be such matrix valued functions defined on the interval $I = [0, 1] \times [0, 1]$ that $v(\mathbf{L}) < \infty$, $v(\mathbf{M}) < \infty$, $\mathbf{L}(s, 0) = \mathbf{M}(s, 0) = \mathbf{0}$ for any $s \in [0, 1]$, $\mathbf{L}(0, t) = \mathbf{M}(0, t) = \mathbf{0}$ for any $t \in [0, 1]$ and $\mathbf{L}(s, t) = \mathbf{L}(s, s)$, $\mathbf{M}(s, t) = \mathbf{M}(s, s)$ for all $0 \leqq s < t \leqq 1$,*

Let us define

$$(1,8) \qquad \mathbf{Q}(s, t) = \int_0^1 \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t) : I \to L(R^n \to R^n)\,.$$

The matrix $\mathbf{Q}(s, t)$ is evidently defined for any $(s, t) \in I$.

Then

$$(1,9) \qquad \mathbf{Q}(s, 0) = \int_0^1 \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, 0) = \mathbf{0} \quad \text{for any} \quad s \in [0, 1]$$

$$\mathbf{Q}(0, t) = \int_0^1 \mathrm{d}_r[\mathbf{L}(0, r)]\, \mathbf{M}(r, t) = \mathbf{0} \quad \text{for any} \quad t \in [0, 1]$$

For $0 \leqq t \leqq s \leqq 1$ we have

$$(1,10) \qquad \mathbf{Q}(s, t) = \int_0^t \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, r) + \int_t^s \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t)$$

and for $0 \leqq s < t \leqq 1$ we have

$$(1,11) \qquad \mathbf{Q}(s, t) = \int_0^s \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, r) = \mathbf{Q}(s, s)\,.$$

Further,

$$(1,12) \qquad v(\mathbf{Q}) \leqq v(\mathbf{L})\, v(\mathbf{M}) < \infty$$

and

$$(1,13) \qquad \|\mathbf{Q}(s, t)\| \leqq \int_0^s \psi_{\mathbf{M}}(r)\, \mathrm{d}\psi_{\mathbf{L}}(r) \quad \text{for any} \quad (s, t) \in I$$

where

$$(1,14) \qquad \psi_{\mathbf{L}}(\tau) = v_{[0,1]\times[0,\tau]}(\mathbf{L}),\ \psi_{\mathbf{M}}(\tau) = v_{[0,1]\times[0,\tau]}(\mathbf{M}),\ \tau \in [0, 1]$$

is the twodimensional variation (in the above mentioned sense) in the interval $[0, 1] \times [0, \tau]$ of $\mathbf{L}$, $\mathbf{M}$ respectively[2]*).*

[2]) The real function $\psi_{\mathbf{L}}(\tau)$ is introduced in [4] (see (2,15b) in [4]). We have $\psi_{\mathbf{L}}(0) = 0$, $\psi_{\mathbf{L}}(1) = v(\mathbf{L})$ and $\psi_{\mathbf{L}}$ is a nondecreasing function in [0, 1]; similarly for $\psi_{\mathbf{M}}$.

Proof. The relations (1,9) are evident. For $0 \leqq t \leqq s \leqq 1$ we can write

$$\mathbf{Q}(s, t) = \int_0^t \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t) + \int_t^s \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t) + \int_s^1 \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t)\,.$$

If $0 \leqq r \leqq t$ then by the assumptions we have $\mathbf{M}(r, t) = \mathbf{M}(r, r)$ and therefore

$$\int_0^t \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t) = \int_0^t \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, r)\,.$$

For $0 \leqq s \leqq r \leqq 1$ we obtain also by the assumptions $\mathbf{L}(s, r) = \mathbf{L}(s, s)$ and Theorem 1, 3, 7 in [2] implies $\int_s^1 \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t) = 0$. This yields the equality (1,10). The equality (1,11) can be obtained in the same way.

To prove (1,12) let be given a finite sequence $0 = \tau_0 < \tau_1 < \ldots < \tau_m = 1$. Let us construct the net type subdivision of I corresponding to this sequence, i.e., the finite system of intervals $J_{ij} = [\tau_{i-1}, \tau_i] \times [\tau_{j-1}, \tau_j]$, $i, j = 1, 2, \ldots, m$. We consider the sum $\sum_{i,j=1}^m \|m_{\mathbf{Q}}(J_{ij})\|$ where (by definition)

$$m_{\mathbf{Q}}(J_{ij}) = \mathbf{Q}(\tau_i, \tau_j) - \mathbf{Q}(\tau_i, \tau_{j-1}) - \mathbf{Q}(\tau_{i-1}, \tau_j) + \mathbf{Q}(\tau_{i-1}, \tau_{j-1}) =$$
$$= \int_0^1 \mathrm{d}_r[\mathbf{L}(\tau_i, r) - \mathbf{L}(\tau_{i-1}, r)]\, (\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1}))\,.$$

Using this expression for $m_{\mathbf{Q}}(J_{ij})$ we can write

$$\|m_{\mathbf{Q}}(J_{ij})\| \leqq \int_0^1 \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| \,\mathrm{d}\,\mathrm{var}_0^r(\mathbf{L}(\tau_i, .) - \mathbf{L}(\tau_{i-1}, .))^{3)}$$

By (1,11) we have $\mathbf{Q}(s, t) = \mathbf{Q}(s, s)$ for $0 \leqq s < t \leqq 1$ and hence

$$\sum_{i,j=1}^m \|m_{\mathbf{Q}}(J_{ij})\| = \sum_{i=1}^m \sum_{j=1}^i \|m_{\mathbf{Q}}(J_{jk})\| \leqq \tag{1,15}$$
$$\leqq \sum_{i=1}^m \int_0^1 \sum_{j=1}^i \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| \,\mathrm{d}\,\mathrm{var}_0^r(\mathbf{L}(\tau_i, .) - \mathbf{L}(\tau_{i-1}, .))\,.$$

From the assumptions on $\mathbf{M}(s, t)$ we obtain

α) $\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1}) = 0$ for $r \leqq \tau_{j-1}$

β) $\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1}) = \mathbf{M}(r, r) - \mathbf{M}(r, \tau_{j-1})$ and

$$\|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| = \|\mathbf{M}(r, r) - \mathbf{M}(r, \tau_{j-1}) - \mathbf{M}(0, r) + \mathbf{M}(0, \tau_{j-1})\| \leqq$$
$$\leqq v_{[0,r]\times[\tau_{j-1},r]}(\mathbf{M}) \quad \text{for} \quad \tau_{j-1} < r < \tau_j\,,$$

[3]) This follows from the inequality $\|\int_a^b \mathbf{A}(r)\, \mathrm{d}\mathbf{B}(r)\| \leqq \int_a^b \|\mathbf{A}(r)\|\, \mathrm{d}[\mathrm{var}_a^r\, \mathbf{B}]$ where $\mathbf{A}, \mathbf{B}$ are $n \times n$-matrices of finite variation on $[a, b]$. It is easy to prove this inequality for example using the sum definition of the Perron-Stieltjes integral. Cf. [2].

γ) $\|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| = \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1}) - \mathbf{M}(0, \tau_j) - \mathbf{M}(0, \tau_{j-1})\|$ $\leqq v_{[0,r] \times [\tau_{j-1}, \tau_j]}(\mathbf{M})$ for $\tau_j \leqq r$.

Let now some $r \in [0, 1]$ and integer i, $1 \leqq i \leqq m$ be fixed. Then either a) $r \geqq \tau_i$ or b) $r < \tau_i$. In the case a) it follows by γ) and (2,12) from [4] that

$$\sum_{j=1}^{i} \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| \leqq \sum_{j=1}^{i} v_{[0,r] \times [\tau_{j-1}, \tau_j]}(\mathbf{M}) \leqq$$

$$\leqq v_{[0,r] \times [0,\tau_i]}(\mathbf{M}) \leqq v_{[0,1] \times [0,r]}(\mathbf{M}) = \psi_{\mathbf{M}}(r) .$$

In the case b) there exists an index $j_r \leqq i$ so that $\tau_{j_r - 1} \leqq r \leqq \tau_{j_r}$. Hence from α) and β) we obtain

$$\sum_{j=1}^{i} \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| = \sum_{j_r=1}^{j_r-1} \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| +$$

$$+ \|\mathbf{M}(r, r) - \mathbf{M}(r, \tau_{j_r - 1})\| = \sum_{j=1}^{j_r-1} v_{[0,r] \times [\tau_{j-1}, \tau_j]}(\mathbf{M}) +$$

$$+ v_{[0,r] \times [\tau_{j_r-1}, r]}(\mathbf{M}) \leqq v_{[0,r] \times [0,r]}(\mathbf{M}) \leqq \psi_{\mathbf{M}}(r) .$$

Consequently, for any $r \in [0, 1]$ and $i = 1, \ldots, m$ we have

$$\sum_{j=1}^{i} \|\mathbf{M}(r, \tau_j) - \mathbf{M}(r, \tau_{j-1})\| \leqq \psi_{\mathbf{M}}(r) .$$

This inequality together with (1,15) gives

$$\sum_{i,j=1}^{m} \|m_{\mathbf{Q}}(J_{ij})\| \leqq \sum_{i=1}^{m} \int_0^1 \psi_{\mathbf{M}}(r) \, \mathrm{d}\operatorname{var}_0^r(\mathbf{L}(\tau_i, .) - \mathbf{L}(\tau_{i-1}), .)) =$$

$$= \int_0^1 \psi_{\mathbf{M}}(r) \, \mathrm{d}\big(\sum_{i=1}^{m} \operatorname{var}_0^r[\mathbf{L}(\tau_i, .) - \mathbf{L}(\tau_{i-1}, .)]\big) \leqq \int_0^1 \psi_{\mathbf{M}}(r) \, \mathrm{d}\psi_{\mathbf{L}}(r) \leqq$$

$$\leqq \psi_{\mathbf{M}}(1) \int_0^1 \mathrm{d} \, \psi_{\mathbf{L}}(r) = \psi_{\mathbf{M}}(1) \, \psi_{\mathbf{L}}(1) = v(\mathbf{L}) \, v(\mathbf{M}) .$$

The second inequality in this relation is obtained from Lemma 3,1 in [3] and from (2,16a) in [4]. The inequality (1,12) follows now immediately.

If $0 \leqq r < t \leqq 1$ then

$$\|\mathbf{M}(r, t)\| = \|\mathbf{M}(r, r)\| = \|\mathbf{M}(r, r) - \mathbf{M}(r, 0) - \mathbf{M}(0, r) + \mathbf{M}(0, 0)\| \leqq$$

$$\leqq v_{[0,r] \times [0,r]}(\mathbf{M}) \leqq \psi_{\mathbf{M}}(r) .$$

Similarly, for $0 \leqq t \leqq r \leqq 1$ we have

$$\|\mathbf{M}(r, t)\| \leqq \|\mathbf{M}(r, t) - \mathbf{M}(r, 0) - \mathbf{M}(0, t) + \mathbf{M}(0, 0)\| \leqq$$
$$\leqq v_{[0,r]\times[0,t]}(\mathbf{M}) \leqq v_{[0,1]\times[0,r]}(\mathbf{M}) = \psi_{\mathbf{M}}(r).$$

This implies

$$\|\mathbf{Q}(s, t)\| = \left\|\int_0^s \mathrm{d}_r[\mathbf{L}(s, r)]\, \mathbf{M}(r, t)\right\| \leqq \int_0^s \|\mathbf{M}(r, t)\|\, \mathrm{d}(\mathrm{var}_0^r \mathbf{L}(s, .)) \leqq$$
$$\leqq \int_0^s \psi_{\mathbf{M}}(r)\, \mathrm{d}(\mathrm{var}_0^r\, \mathbf{L}(s, .)) \leqq \int_0^s \psi_{\mathbf{M}}(r)\, \mathrm{d}(\mathrm{var}_0^r(\mathbf{L}(s, .) - \mathbf{L}(0, .))) \leqq \int_0^s \psi_{\mathbf{M}}(r)\, \mathrm{d}\psi_{\mathbf{L}}(r)$$

since for $r_1 < r_2$ it holds

$$|\mathrm{var}_0^{r_2}(\mathbf{L}(s, .) - \mathbf{L}(0, .)) - \mathrm{var}_0^{r_1}(\mathbf{L}(s, .) - \mathbf{L}(0, .))| =$$
$$= \mathrm{var}_{r_1}^{r_2}(\mathbf{L}(s, .) - \mathbf{L}(0, .)) \leqq v_{[0,s]\times[r_1,r_2]}(\mathbf{L}) \leqq$$
$$\leqq v_{[0,1]\times[r_1,r_2]}(\mathbf{L}) \leqq \psi_{\mathbf{L}}(r_2) - \psi_{\mathbf{L}}(r_1)$$

and the above inequality is a consequence of Lemma 2,1 in [3]. The proof of the proposition is complete.

Lemma 1.1. *Let $\mathbf{M} : I \to L(R^n \to R^n)$ be such a matrix valued function defined on the interval $I = [0, 1] \times [0, 1]$ that $v(\mathbf{M}) < \infty$, $\mathbf{M}(s, 0) = \mathbf{0}$ for any $s \in [0, 1]$, $\mathbf{M}(0, t) = \mathbf{0}$ for any $t \in [0, 1]$ and $\mathbf{M}(s, t) = \mathbf{M}(s, s)$ for all $0 \leqq s \leqq t \leqq 1$.*

Then

$$\mathrm{var}_0^\delta\left(\int_0^s \mathrm{d}_t[\mathbf{M}(s, t)]\, \mathbf{x}(t)\right) \leqq \int_0^\delta \|\mathbf{x}(t)\|\, \mathrm{d}\psi_{\mathbf{M}}(t) \leqq \tag{1,16}$$
$$\leqq \|\mathbf{x}(0)\|\,[\psi_{\mathbf{M}}(0+) - \psi_{\mathbf{M}}(0)] + \|\mathbf{x}\|_{V_n(0,\delta)}\,[\psi_{\mathbf{M}}(\delta) - \psi_{\mathbf{M}}(0+)]$$

for any $\delta \in (0, 1]$ and $\mathbf{x} \in V_n$ where $\psi_{\mathbf{M}}$ is defined by (1,14) and $\|\mathbf{x}\|_{V_n(0,\delta)} = \|\mathbf{x}(0)\| + \mathrm{var}_0^\delta\, \mathbf{x}$.

Proof. The kernel $\mathbf{M}(s, t)$ is a triangular one. Hence we have for any $s \in [0, \delta]$

$$\int_0^s \mathrm{d}_t[\mathbf{M}(s, t)]\, \mathbf{x}(t) = \int_0^\delta \mathrm{d}_t[\mathbf{M}(s, t)]\, \mathbf{x}(t)$$

(this can be obtained similarly as in the proof of (1,5)).

Further, (2,24) from Proposition 2,3 in [4] yields

$$\mathrm{var}_0^\delta\left(\int_0^s \mathrm{d}_t[\mathbf{M}(s, t)]\, \mathbf{x}(t)\right) = \mathrm{var}_0^\delta\left(\int_0^\delta \mathrm{d}_t[\mathbf{M}(s, t)]\, \mathbf{x}(t)\right) \leqq \int_0^\delta \|\mathbf{x}(t)\|\, \mathrm{d}\psi_{\mathbf{M}}(t). \tag{1,17}$$

Using Theorem 1,3,6 in [2] and the relations between the generalized Perron integral and the Perron-Stieltjes integral we obtain by simple computation

$$(1,18) \qquad \int_0^\delta \|\mathbf{x}(t)\| \, \mathrm{d}\psi_{\mathbf{M}}(t) = \|\mathbf{x}(0)\| \, [\psi_{\mathbf{M}}(0+) - \psi_{\mathbf{M}}(0)] + \lim_{\sigma \to 0+} \int_0^\delta \|\mathbf{x}(t)\| \, \mathrm{d}\psi_{\mathbf{M}}(t) \,.$$

Since

$$\int_0^\delta \|\mathbf{x}(t)\| \, \mathrm{d}\psi_{\mathbf{M}}(t) \leqq \sup_{t \in [\sigma, \delta]} \|\mathbf{x}(t)\| \,.\, [\psi_{\mathbf{M}}(\delta) - \psi_{\mathbf{M}}(\sigma)]$$

for any $0 < \sigma < \delta$, we can write by (1,18) and by the obvious inequlity a $\sup_{t \in [\sigma, \delta]} \|\mathbf{x}(t)\| \leqq \|\mathbf{x}(0)\| + \mathrm{var}_0^\delta \, \mathbf{x} = \|\mathbf{x}\|_{V_n(0,\delta)}$ the relation

$$\int_0^\delta \|\mathbf{x}(t)\| \, \mathrm{d}\psi_{\mathbf{M}}(t) \leqq \|\mathbf{x}(0)\| \, [\psi_{\mathbf{M}}(0+) - \psi_{\mathbf{M}}(0)] + \|\mathbf{x}\|_{V_n(0,\delta)} \lim_{\sigma \to 0+} [\psi_{\mathbf{M}}(\delta) - \psi_{\mathbf{M}}(\sigma)]$$

This inequality together with (1,17) yields (1,16).

Remark 1.2. In the proof of the next proposition the following will be essential: *If $h(t)$ is a real valued, nondecreasing, nonnegative, from the left continuous function in the interval $[a, b]$, then*

$$(1,19) \qquad \int_a^b h^k(\tau) \, \mathrm{d}h(\tau) \leqq \frac{1}{k+1} \, [h^{k+1}(b) - h^{k+1}(a)]$$

for any $k = 0, 1, \ldots$ (see Lemma 3,3 in [3]).

Proposition 1.2. *Let $\mathbf{M}(s, t) : I \to L(R^n \to R^n)$ be such a matrix valued functions defined on $I = [0, 1] \times [0, 1]$ that $v(\mathbf{M}) < \infty$, $\mathbf{M}(s, 0) = \mathbf{M}(0, t) = \mathbf{0}$ for $s \in [0, 1]$, $t \in [0, 1]$, $\mathbf{M}(s, t) = \mathbf{M}(s, s)$ for all $0 \leqq s < t \leqq 1$ and let*

$$(1,20) \qquad \lim_{t \to t_0 -} \|\mathbf{M}(s, t) - \mathbf{M}(s, t_0)\| = 0$$

for any $s \in [0,1]$ and $t_0 \in (0, 1]$ (i.e., $\mathbf{M}(s, t)$ is continuous from the left in the second variable t).

Let us define for $(s, t) \in I$

$$(1,21) \qquad \mathbf{M}^{(1)}(s, t) = \mathbf{M}(s, t) \,,$$

$$\mathbf{M}^{(q)}(s, t) = \int_0^1 \mathrm{d}_r[\mathbf{M}(s, r)] \, \mathbf{M}^{(q-1)}(r, t) = \int_0^s \mathrm{d}_r[\mathbf{M}(s, r)] \, \mathbf{M}^{(q-1)}(r, t) \,,$$

$$q = 2, 3, \ldots .$$

Then

$$(1,22)\qquad \operatorname{var}_0^1 \boldsymbol{M}^{(q)}(s,.) \leqq [\psi_{\boldsymbol{M}}(s)]^q/q! \quad \textit{for any} \quad s \in [0,1] \quad \textit{and} \quad q = 1, 2, \ldots,$$

$$(1,23)\qquad v(\boldsymbol{M}^{(q)}) \leqq v(\boldsymbol{M})^q/q! \quad \textit{for any} \quad q = 1, 2, \ldots.$$

The series $\sum_{q=1}^{\infty} \boldsymbol{M}^{(q)}(s,t)$ *converges uniformly to* $\Xi(s,t): : I \to L(R^n \to R^n)$ *i.e.*,

$$(1,24)\qquad \sum_{q=1}^{\infty} \boldsymbol{M}^{(q)}(s,t) = \Xi(s,t)$$

and $\Xi(s,0) = \Xi(0,t) = \boldsymbol{0}$ *for any* $s \in [0,1]$, $t \in [0,1]$, $\Xi(s,t) = \Xi(s,s)$ *for* $0 \leqq \leqq s < t \leqq 1$.

Further it is

$$(1,25)\qquad \lim_{l \to \infty} v\left(\sum_{q=1}^{l} \boldsymbol{M}^{(q)} - \Xi\right) = 0$$

and

$$(1,26)\qquad v(\Xi) \leqq e^{v(\boldsymbol{M})} - 1\,.$$

The matrix valued function $\Xi(s,t)$ *satisfies for any* $(s,t) \in I$ *the equation*

$$(1,27)\qquad \Xi(s,t) = \boldsymbol{M}(s,t) + \int_0^1 \mathrm{d}_r[\boldsymbol{M}(s,r)]\,\Xi(r,t) = \boldsymbol{M}(s,t) + \int_0^s \mathrm{d}_r[\boldsymbol{M}(s,r)]\,\Xi(r,t)\,.$$

Proof. Since (1,20) holds the function $\psi_{\boldsymbol{M}}(\tau) : [0,1] \to R$ is nondecreasing, continuous from the left (see Lemma 2,1 in [4]) and $\psi_{\boldsymbol{M}}(0) = 0$, $\psi_{\boldsymbol{M}}(1) = v(\boldsymbol{M})$.

We prove first the relation (1,22). Let $0 = \tau_0 < \tau_1 < \ldots < \tau_m = 1$ be a subdivision of the interval $[0,1]$ and let $s \in [0,1]$. Let $m_s \leqq m$ be such a positive integer that $\tau_{m_s-1} \leqq s < \tau_{m_s}$. Then by the assumptions about $\boldsymbol{M}(s,t)$ we have

$$\begin{aligned}
\sum_{j=1}^{m} \|\boldsymbol{M}(s,\tau_j) - \boldsymbol{M}(s,\tau_{j-1})\| &= \sum_{j=1}^{m_s-1} \|\boldsymbol{M}(s,\tau_j) - \boldsymbol{M}(s,\tau_{j-1})\| + \|\boldsymbol{M}(s,s) - (s,\tau_{m_s-1})\| = \\
&= \sum_{j=1}^{m_s-1} \|\boldsymbol{M}(s,\tau_j) - \boldsymbol{M}(s,\tau_{j-1}) - \boldsymbol{M}(0,\tau_j) + \boldsymbol{M}(0,\tau_{j-1})\| + \\
&+ \|\boldsymbol{M}(s,s) - \boldsymbol{M}(s,\tau_{m_s-1}) - \boldsymbol{M}(0,s) + \boldsymbol{M}(0,\tau_{ms-1})\| \leqq \\
&\leqq \sum_{j=1}^{m_s-1} v_{[0,s]\times[\tau_{j-1},\tau_j]}(\boldsymbol{M}) + v_{[0,s]\times[\tau_{m_s-1},s]}(\boldsymbol{M}) \leqq \\
&\leqq v_{[0,1]\times[0,s]}(\boldsymbol{M}) = \psi_{\boldsymbol{M}}(s)\,.
\end{aligned}$$

Since the subdivision $0 = \tau_0 < \ldots < \tau_m = 1$ is arbitrary we obtain by passing to the supremum over all subdivisions of [0,1] the inequality (1,22) for $q = 1$. Let now (1,22) be valid for $q-1$ and let $0=\tau_0<\tau_1<\ldots<\tau_m=1$ be an arbitrary subdivision of the interval $[0,1]$, $s\in[0,1]$. Then (see Remark 1.2)

$$\sum_{j=1}^{m}\|\boldsymbol{M}^{(q)}(s,\tau_j)-\boldsymbol{M}^{(q)}(s,\tau_{j-1})\| =$$

$$=\sum_{j=1}^{m}\left\|\int_0^s \mathrm{d}_r[\boldsymbol{M}(s,r)]\,(\boldsymbol{M}^{(q-1)}(r,\tau_j)-\boldsymbol{M}^{(q-1)}(r,\tau_{j-1}))\right\| \leqq$$

$$\leqq \int_0^s\Big(\sum_{j=1}^{m}\|\boldsymbol{M}^{(q-1)}(r,\tau_j)-\boldsymbol{M}^{(q-1)}(r,\tau_{j-1})\|\Big)\,\mathrm{d}\psi_{\boldsymbol{M}}(r) \leqq$$

$$\leqq \int_0^s \operatorname{var}_0^1 \boldsymbol{M}^{(q-1)}(r,.)\,\mathrm{d}\psi_{\boldsymbol{M}}(r) \leqq \frac{1}{(q-1)!}\int_0^s \psi_{\boldsymbol{M}}^{q-1}(r)\,\mathrm{d}\psi_{\boldsymbol{M}}(r) \leqq$$

$$\leqq \frac{1}{q!}\,[\psi_{\boldsymbol{M}}^q(s)-\psi_{\boldsymbol{M}}^q(0)] = \frac{1}{q!}\,\psi_{\boldsymbol{M}}^q(s)\,.$$

By passing to the supremum over all subdivisions of $[0,1]$ we obtain (1,22) for the value q. In this manner (1,22) is proved by induction.

To prove (1,23) let an arbitrary subdivision $0=\tau_0<\tau_1<\ldots<\tau_m=1$ be given and let $J_{ij}=[\tau_{i-1},\tau_i]\times[\tau_{j-1},\tau_j]$ $i,j=1,2,\ldots,m$ be the corresponding net type subdivision of $I=[0,1]\times[0,1]$. Let q be an arbitrary positive integer. Then we have

$$\sum_{i,j=1}^{m}\|m_{\boldsymbol{M}^{(q)}}(J_{ij})\| = \sum_{i,j=1}^{m}\|\boldsymbol{M}^{(q)}(\tau_i,\tau_j)-\boldsymbol{M}^{(q)}(\tau_i,\tau_{j-1})-\boldsymbol{M}^{(q)}(\tau_{i-1},\tau_j)+$$

$$+\boldsymbol{M}^{(q)}(\tau_{i-1},\tau_{j-i})\| =$$

$$=\sum_{i,j=1}^{m}\left\|\int_0^1 \mathrm{d}_r[\boldsymbol{M}(\tau_i,r)-\boldsymbol{M}(\tau_{i-1},r)]\,(\boldsymbol{M}^{(q-1)}(r,\tau_j)-\boldsymbol{M}^{(q-1)}(r,\tau_{j-1}))\right\| \leqq$$

$$\leqq \int_0^1\Big(\sum_{j=1}^{m}\|\boldsymbol{M}^{(q-1)}(r,\tau_j)-\boldsymbol{M}^{(q-1)}(r,\tau_{j-1})\|\Big)\,\mathrm{d}\Big(\sum_{i=1}^{m}\operatorname{var}_0^r(\boldsymbol{M}(\tau_i,.)-\boldsymbol{M}(\tau_{i-1},.))\Big) \leqq$$

$$\leqq \int_0^1 \operatorname{var}_0^1 \boldsymbol{M}^{(q-1)}(r,.)\,\mathrm{d}\psi_{\boldsymbol{M}}(r) \leqq \int_0^1 \frac{1}{(q-1)!}\,\psi_{\boldsymbol{M}}^{q-1}(r)\,\mathrm{d}\psi_{\boldsymbol{M}}(r) \leqq$$

$$\leqq \frac{1}{q!}\,\psi_{\boldsymbol{M}}^q(1) = \frac{1}{q!}\,[v(\boldsymbol{M})]^q$$

and by passing to the supremum over all decompositions of $[0,1]$ we obtain (1,23).

For any $(s,t)\in I$, $q=1,2,\ldots$ we have evidently

$$\|\boldsymbol{M}^{(q)}(s,t)\| = \|\boldsymbol{M}^{(q)}(s,t)-\boldsymbol{M}^{(q)}(s,0)-\boldsymbol{M}^{(q)}(0,t)+\boldsymbol{M}^{(q)}(0,0)\| \leqq v(\boldsymbol{M}^{(q)}) \leqq \frac{[v(\boldsymbol{M})]^q}{q!}$$

since by Proposition 1,1 $\boldsymbol{M}^{(q)}(s, 0) = \boldsymbol{M}^{(q)}(0, t) = \boldsymbol{0}$ for any $s \in [0, 1]$, $t \in [0, 1]$. This inequality yields immediately the uniform convergence of the series (1,24) to some matric valued function $\Xi(s, t) : I \to L(R^n \to R^n)$ since the majorizing series $\sum_{q=1}^{\infty} [v(\boldsymbol{M})]^q/q! = e^{v(\boldsymbol{M})} - 1$ evidently converges.

The equalities $\Xi(s, 0) = \Xi(0, t) = \boldsymbol{0}$ for $s \in [0, 1]$, $t \in [0, 1]$ are easy consequences of the relation (1,9) in Prop. 1,1. From (1,10) in the same Proposition we obtain $\Xi(s, t) = \Xi(s, s)$ for $0 \leqq s < t \leqq 1$.

Using (1,23) we obtain the inequality

$$v\Big(\sum_{q=1}^{l} \boldsymbol{M}^{(q)} - \Xi\Big) = v\Big(\sum_{q=l+1}^{\infty} \boldsymbol{M}^{(q)}\Big) \leqq \sum_{q=l+1}^{\infty} v(\boldsymbol{M}^{(q)}) \leqq \sum_{q=l+1}^{\infty} [v(\boldsymbol{M})]^q/q!$$

which implies (1,25). The inequality (1,26) can be obtained as follows:

$$v(\Xi) \leqq \sum_{q=1}^{\infty} v(\boldsymbol{M}^{(q)}) \leqq \sum_{q=1}^{\infty} [v(\boldsymbol{M})]^q/q! = e^{v(\boldsymbol{M})} - 1 .$$

The uniform convergence of the series (1,24) for any $(s, t) \in I$ implies

$$\int_0^1 \mathrm{d}_r[\boldsymbol{M}(s, r)]\, \Xi(r, t) = \int_0^1 \mathrm{d}_r[\boldsymbol{M}(s, r)] \Big(\sum_{q=1}^{\infty} \boldsymbol{M}^{(q)}(r, t)\Big) =$$

$$= \sum_{q=1}^{\infty} \int_0^1 \mathrm{d}_r[\boldsymbol{M}(s, r)]\, \boldsymbol{M}^{(q)}(r, t) = \sum_{q=1}^{\infty} \boldsymbol{M}^{(q+1)}(s, t) = \Xi(s, t) - \boldsymbol{M}(s, t) .$$

Hence $\Xi(s, t)$ satisfies (1,27).

Corollary 1.1. *Let $\boldsymbol{M}(s, t) : I \to L(R^n \to R^n)$ satisfy all assumptions from Proposition 2. Let $y \in V_n$ and let us set*

$$(\boldsymbol{T}\boldsymbol{y})(s) = \int_0^1 \mathrm{d}_t[\boldsymbol{M}(s, t)]\, \boldsymbol{y}(t) = \int_0^s \mathrm{d}_t[\boldsymbol{M}(s, t)]\, \boldsymbol{y}(t)$$

for $s \in [0, 1]$. $\boldsymbol{T} : V_n \to V_n$ is a linear operator.

If we define

$$(\boldsymbol{T}^q\boldsymbol{y})(s) = \boldsymbol{T}(\boldsymbol{T}^{q-1}\boldsymbol{y})(s)$$

for $s \in [0, 1]$ and $q = 2, 3, \ldots$ then

$$(1,28) \qquad \|(\boldsymbol{T}^q\boldsymbol{y})(s)\| \leqq \frac{\psi_{\boldsymbol{M}}^q(s)}{q!} \|\boldsymbol{y}\|_{V_n} \quad \text{for} \quad q = 1, 2, \ldots, \ s \in [0, 1] ,$$

$$(1,29) \qquad \operatorname{var}_0^1 \boldsymbol{T}^q\boldsymbol{y} \leqq \frac{1}{q!} [v(\boldsymbol{M})]^q \|\boldsymbol{y}\|_{V_n} \quad \text{for} \quad q = 1, 2, \ldots ,$$

$$(1,30) \qquad \|\boldsymbol{T}^q\boldsymbol{y}\|_{V_n} = \|\boldsymbol{T}^q\boldsymbol{y}(0)\| + \operatorname{var}_0^1 \boldsymbol{T}^q\boldsymbol{y} \leqq \frac{1}{q!} [v(\boldsymbol{M})]^q \|\boldsymbol{y}\|_{V_n} \quad \text{for} \quad q = 1, 2, \ldots .$$

Proof. For basic properties of the operator $\boldsymbol{T}$ (linearity, continuity, etc.) see Part 3 in [4]. By definition we have

$$(\boldsymbol{T}^2\boldsymbol{y})(s) = \boldsymbol{T}(\boldsymbol{T}\boldsymbol{y})(s) = \int_0^1 \mathrm{d}_r[\boldsymbol{M}(s, r)]\, \boldsymbol{T}\boldsymbol{y}(r) = \int_0^1 \mathrm{d}_r[\boldsymbol{M}(s, r)] \left(\int_0^1 \mathrm{d}_t[\boldsymbol{M}(r, t)]\, \boldsymbol{y}(t)\right).$$

The assumptions on $\boldsymbol{M}(s, t)$ and $\boldsymbol{y}$ make it possible in a similar way as in Prop. 2,4 from [4] to demonstrate the possibility of interchanging the order of integration in this expression. Hence we obtain

$$(\boldsymbol{T}^2\boldsymbol{y})(s) = \int_0^1 \mathrm{d}_t \left[\int_0^1 \mathrm{d}_r[\boldsymbol{M}(s, r)]\, \boldsymbol{M}(r, t)\right] \boldsymbol{y}(t) = \int_0^1 \mathrm{d}_t[\boldsymbol{M}^{(2)}(s, t)]\, \boldsymbol{y}(t)$$

where $M^{(2)}(s, t) : I \to L(R^n \to R^n)$ is defined in (1,21).

If we continue this procedure then for any $s \in [0, 1]$ we obtain

$$(1,31) \qquad (\boldsymbol{T}^q\boldsymbol{y})(s) = \int_0^1 \mathrm{d}_t[\boldsymbol{M}^{(q)}(s, t)]\, \boldsymbol{y}(t), \quad q = 1, 2, \ldots$$

where $\boldsymbol{M}^{(q)}(s, t)$ is given by (1,21). Hence

$$\|(\boldsymbol{T}^q\boldsymbol{y})(s)\| \leqq \int_0^1 \|\boldsymbol{y}(t)\|\, \mathrm{d}\operatorname{var}_0^t \boldsymbol{M}^{(q)}(s, .) \leqq \|\boldsymbol{y}\|_{V_n} \operatorname{var}_0^1 \boldsymbol{M}^{(q)}(s, .)$$

and the relation (1,28) follows immediately from (1,22). The inequality (1,29) is a direct consequence of (3,5) from (4) and of (1,23). Since $\psi_{\boldsymbol{M}}(0) = 0$ we obtain (1,30) from (1,28) and (1,29).

2. THE CASE OF A KERNEL WHICH IS LEFT CONTINUOUS IN THE SECOND VARIABLE

In this part we study the Volterra-Stieltjes integral equation

$$(2,1) \qquad \boldsymbol{x}(s) = \int_0^s \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}(t) + \boldsymbol{y}(s), \quad 0 \leqq s \leqq 1$$

in the space V_n with the kernel $\boldsymbol{K}(s, t) : I = [0, 1] \times [0, 1] \to L(R^n \to R^n)$ satisfying

$$(2,2) \qquad v(\boldsymbol{K}) < \infty$$

$$(2,3) \qquad \operatorname{var}_0^1 \boldsymbol{K}(l, .) < \infty$$

and

$$(2,4) \qquad \lim_{t \to t_0 -} \|\boldsymbol{K}(s, t) - \boldsymbol{K}(s, t_0)\| = 0$$

for any $s \in [0, 1]$, $t_0 \in (0, 1]$. The function $\boldsymbol{y}$ is assumed to be an element of the space V_n.

If we define for the kernel $\boldsymbol{K}$ the corresponding triangular kernel $\boldsymbol{K}^{\Delta}$ by the relations (1,4) then the kernel $\boldsymbol{K}^{\Delta}$ also satisfies the relation (2,4), i.e., we have

$$(2,5) \qquad \lim_{t \to t_0 -} \|\boldsymbol{K}^{\Delta}(s, t) - \boldsymbol{K}^{\Delta}(s, t_0)\| = 0$$

for any $s \in [0, 1]$, $t_0 \in (0, 1]$. This fact can be easily verified from the definition (1,4) of the kernel $\boldsymbol{K}^{\Delta}$. We have also (cf. (1,7))

$$(2,6) \qquad v(\boldsymbol{K}^{\Delta}) < \infty$$

and

$$(2,7) \qquad \operatorname{var}_0^1 \boldsymbol{K}^{\Delta}(0, .) = 0, \operatorname{var}_0^1 \boldsymbol{K}^{\Delta}(., 0) = 0 .$$

Hence we can conclude that the triangular kernel $\boldsymbol{K}^{\Delta}$ satisfies all assumptions of Proposition 2 and, moreover, all assumptions from Parts 3 and 4 in [4].

In view of the relation (1,5), the equation (2,1) can be considered in the form

$$(2,8) \qquad \boldsymbol{x}(s) = \int_0^1 \mathrm{d}_t[\boldsymbol{K}^{\Delta}(s, t)]\, \boldsymbol{x}(t) + \boldsymbol{y}(s), \quad 0 \leqq s \leqq 1 .$$

For this equation all the results from Parts 3 and 4 in [4] hold.

Let us now define the linear operator $\boldsymbol{T} : V_n \to V_n$ by the relation

$$(2,9) \qquad \boldsymbol{Tz}(t) = \int_0^1 \mathrm{d}_t[\boldsymbol{K}^{\Delta}(s, t)]\, \boldsymbol{z}(t) = \int_0^s \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{z}(t), \quad \boldsymbol{z} \in V_n, \quad s \in [0, 1]$$

(cf. Part 3 in [4]).

The equation (2,1) or (2,8) can be formally written in the form

$$(2,10) \qquad \boldsymbol{x} - \boldsymbol{Tx} = \boldsymbol{y}, \quad \boldsymbol{y} \in V_n .$$

We set

$$(2,11) \qquad \varphi_0(s) = \boldsymbol{y}(s), \quad \varphi_l(s) = \varphi_{l-1}(s) + \boldsymbol{T}\varphi_{l-1}(s), \quad l = 1, 2, \dots .$$

Evidently

$$(2,12) \qquad \varphi_l = \boldsymbol{y} + \sum_{q=1}^{l} \boldsymbol{T}^q \boldsymbol{y}, \quad l = 1, 2, \dots$$

where

$$(2,13) \qquad \boldsymbol{T}^q \boldsymbol{y} = \boldsymbol{T}(\boldsymbol{T}^{q-1} \boldsymbol{y}), \quad q = 2, 3, \dots .$$

Let us denote

$$(2,14) \qquad \psi(\tau) = v_{[0,1] \times [0,\tau]}(\boldsymbol{K}^{\Delta}), \quad \tau \in [0, 1]$$

(ψ is a nondecreasing, left continuous real function on $[0,1]$, $\psi(0) = 0$, $\psi(1) = v(\mathbf{K}^\Delta)$). The sequence $\{\boldsymbol{\varphi}_l\}_{l=1}^\infty \in V_n$ defined by (2,12) satisfies the relation

$$\|\boldsymbol{\varphi}_m - \boldsymbol{\varphi}_l\|_{V_n} = \|\sum_{q=l+1}^{m} \mathbf{T}^q \mathbf{y}\|_{V_n} \leqq \sum_{q=l+1}^{m} \|\mathbf{T}^q \mathbf{y}\|_{V_n} .$$

Using (1,30) from Corollary 1,1 we obtain

$$\|\boldsymbol{\varphi}_m - \boldsymbol{\varphi}_l\|_{V_n} \leqq \sum_{q=l+1}^{m} \frac{[v(\mathbf{K}^\Delta)]^q}{q!} \|\mathbf{y}\|_{V_n} .$$

This inequality implies that $\{\boldsymbol{\varphi}_l\}_{l=1}^\infty$ forms a fundamental sequence in the Banach space V_n. Hence there exists an element $\mathbf{x} \in V_n$ such that $\boldsymbol{\varphi}_l \to \mathbf{x}$ for $l \to \infty$ in V_n, i.e.,

$$\lim_{l\to\infty} \|\boldsymbol{\varphi}_l - \mathbf{x}\|_{V_n} = 0 . \tag{2,15}$$

Further, evidently

$$\boldsymbol{\varphi}_l - \mathbf{T}\boldsymbol{\varphi}_l = \mathbf{y} - \mathbf{T}^{l+1}\mathbf{y} , \quad l = 1, 2, \ldots \tag{2,16}$$

and by (1,30) also

$$\lim_{l\to\infty} \|\mathbf{T}\boldsymbol{\varphi}_l - \mathbf{T}\mathbf{x}\|_{V_n} \leqq \lim_{l\to\infty} v(\mathbf{K}^\Delta) \|\boldsymbol{\varphi}_l - \mathbf{x}\|_{V_n} = 0 .$$

By (1,30) we obtain that $\lim_{l\to\infty} \|\mathbf{T}^{l+1}\mathbf{y}\|_{V_n} = 0$. Passing to the limit $l \to \infty$ in (2,16) we therefore obtain

$$\mathbf{x} - \mathbf{T}\mathbf{x} = \lim_{l\to\infty} (\boldsymbol{\varphi}_l - \mathbf{T}\boldsymbol{\varphi}_l) = \mathbf{y} - \lim_{l\to\infty} \mathbf{T}^{l+1}\mathbf{y} = \mathbf{y} ,$$

i.e., $\mathbf{x} \in V_n$ is a solution of the equation (2,10) and hence also of the equation (2,1). In this way we have shown that the Volterra-Stieltjes integral equation (2,1) has for any $\mathbf{y} \in V_n$ a solution $\mathbf{x} \in V_n$. Since (2,15) and (2,12) hold, this solution can be written in the form

$$\mathbf{x} = \mathbf{y} + \sum_{q=1}^{\infty} \mathbf{T}^q \mathbf{y} . \tag{2.17}$$

The solution $\mathbf{x}$ is unique; in fact if $\mathbf{x}_1, \mathbf{x}_2 \in V_n$ are solutions of (2,10) then we have

$$\mathbf{x}_1 - \mathbf{x}_2 = (\mathbf{T}(\mathbf{x}_1 - \mathbf{x}_2) = \ldots = \mathbf{T}^l(\mathbf{x}_1 - \mathbf{x}_2)$$

for any $l = 1, 2, \ldots$. Hence by (1,30)

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_{V_n} \leqq \frac{1}{l!} v(\mathbf{K}^\Delta)^l \|\mathbf{x}_1 - \mathbf{x}_2\|_{V_n}$$

for any positive integer l which makes the desired unicity obvious.[4])

[4]) The unicity of the solution of the equation (2,1) is also a consequence of its existence for any right hand side $\mathbf{y} \in V_n$ and the Fredholm alternative for this equation (see Theorem 5.2 in [4]).

We return now to the definition (2,9) of the operator $\boldsymbol{T}$ and its powers (2,13). In the same manner as in the proof of Corollary 1,1 it can be shown that for $\boldsymbol{y} \in V_n$ we have

$$(2,18) \qquad \boldsymbol{T}^q \boldsymbol{y}(s) = \int_0^1 \mathrm{d}_t[\boldsymbol{K}^{\Delta}_{(q)}(s, t)]\, \boldsymbol{y}(t)\,, \quad s \in [0,]\,, \quad q = 1, 2, \ldots$$

where

$$(2,19) \qquad \boldsymbol{K}^{\Delta}_{(1)}(s, t) = \boldsymbol{K}^{\Delta}(s, t)$$

$$\boldsymbol{K}^{\Delta}_{(q)}(s, t) = \int_0^1 \mathrm{d}_r[\boldsymbol{K}^{\Delta}(s, r)]\, \boldsymbol{K}^{\Delta}_{(q-1)}(r, t)\,, \quad q = 1, 3, \ldots$$

for any $(s, t) \in I$.

In view of (2,17) and (2,18) the solution $\boldsymbol{x} \in V_n$ of the equation (2,1) can be written in the form

$$(2,20) \qquad \boldsymbol{x}(s) = \boldsymbol{y}(s) + \sum_{q=1}^{\infty} \int_0^1 \mathrm{d}_t[\boldsymbol{K}^{\Delta}_{(q)}(s, t)]\, \boldsymbol{y}(t)\,, \quad s \in [0, 1]\,.$$

Let us denote (cf. Prop. 2. (1,24)–(1,27))

$$(2,21) \qquad \boldsymbol{\Gamma}(s, t) = \sum_{q=1}^{\infty} \boldsymbol{K}^{\Delta}_{(q)}(s, t)\,, \quad (s, t) \in I\,.$$

Since (3,5) in [4] implies for any $l = 1, 2, \ldots$

$$\left\| \sum_{q=1}^{l} \int_0^1 \mathrm{d}_t[\boldsymbol{K}^{\Delta}_{(q)}(s, t)]\, \boldsymbol{y}(t) - \int_0^1 \mathrm{d}_t[\boldsymbol{\Gamma}(s, t)]\, \boldsymbol{y}(t) \right\|_{V_n} =$$

$$= \left\| \int_0^1 \mathrm{d}_t\Big[\sum_{q=1}^{l} \boldsymbol{K}^{\Delta}_{(q)}(s, t) - \boldsymbol{\Gamma}(s, t)\Big]\, \boldsymbol{y}(t) \right\|_{V_n} \leqq v(\boldsymbol{\Gamma} - \sum_{q=1}^{l} \boldsymbol{K}^{(q)}_{\Delta}) \|\boldsymbol{y}\|_{V_n}$$

and, by (1,25), $\lim_{l \to \infty} v(\boldsymbol{\Gamma} - \sum_{q=1}^{l} \boldsymbol{K}^{\Delta}_{(q)}) = 0$, we have

$$\sum_{q=1}^{\infty} \int_0^1 \mathrm{d}_t[\boldsymbol{K}^{\Delta}_{(q)}(s, t)]\, \boldsymbol{y}(t) = \int_0^1 \mathrm{d}_t[\boldsymbol{\Gamma}(s, t)]\, \boldsymbol{y}(t)\,, \quad s \in [0, 1]\,.$$

Hence (2,20) has the form (since the kernel $\boldsymbol{\Gamma}$ is by Prop. 1,2 a triangular one)

$$(2,22) \qquad \boldsymbol{x}(s) = \boldsymbol{y}(s) + \int_0^s \mathrm{d}_t[\boldsymbol{\Gamma}(s, t)]\, \boldsymbol{y}(t)\,, \quad s \in [0, 1]\,.$$

Resuming the above results together with Proposition 1,2 we obtain

Theorem 2.1. *Let the kernel* $\mathbf{K} : I \to L(R^n \to R^n)$ *satisfy* (2,2), (2,3), (2,4). *Then the Volterra-Stieltjes integral equation* (2,1) *has precisely one solution* $\mathbf{x} \in V_n$ *for any* $\mathbf{y} \in V_n$. *This solution is given by the relation* (2,22) *where the resolvent kernel* $\boldsymbol{\Gamma}(s, t) : I \to L(R^n \to R^n)$ *is given by the series* (2,21) *with* $\mathbf{K}^\Delta$ *from* (2,19). *The resolvent kernel* $\boldsymbol{\Gamma}(s, t)$ *satisfies the integral equation*

$$(2,23) \qquad \boldsymbol{\Gamma}(s, t) = \mathbf{K}^\Delta(s, t) + \int_0^1 \mathrm{d}_r[\mathbf{K}^\Delta(s, r)]\, \boldsymbol{\Gamma}(r, t)\,.$$

Remark 2.1. In the same way as above it is possible to consider the Volterra-Stieltjes integral equation of the form

$$\mathbf{x}(s) = \lambda \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) + \mathbf{y}(s)$$

where $\mathbf{y} \in V_n$, λ is a real parameter and $\mathbf{K}(s, t)$ satisfies the assumptions from Theorem 2,1. It can be shown that this equation has for any λ a unique solution, which can be expressed in the form

$$\mathbf{x}(s) = \mathbf{y}(s) + \int_0^s \mathrm{d}_t \boldsymbol{\Gamma}(s, t, \lambda)\, \mathbf{y}(t)$$

where $\boldsymbol{\Gamma}(s, t, \lambda)$ is given by the series

$$\boldsymbol{\Gamma}(s, t, \lambda) = \sum_{q=1}^{\infty} \lambda^q \mathbf{K}^\Delta_{(q)}(s, t)$$

with $\mathbf{K}^\Delta_{(q)}(s, t)$ from (2,19).

In other words, the operator $\boldsymbol{T}$ from (2,9) under the given assumptions has no eigenvalues different from zero.

Remark 2.2. Since the kernel $\mathbf{K}^\Delta(s, t)$ satisfies all assumptions of Theorem 5,2 in [4] and Theorem 2,1 does hold, the first case of the Fredholm alternative from Theorem 5,2 in [4] occurs. From the same Theorem in [4] we also obtain some conclusions about the adjoint equation in the sense of [4].

3. THE CASE OF A GENERAL KERNEL

In the previous part the assumption (2,4) was essential. In this part we drop this assumption which concerns some continuity properties of the kernel $\mathbf{K}(s, t) : [0, 1] \times \times [0, 1] \to L(R^n \to R^n)$ and replace it by a weaker one. Consequently the result is also weaker. We establish only the existence and unicity of the solution $\mathbf{x} \in V_n$ of the equation

$$(3,1) \qquad \mathbf{x}(s) = \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) + \mathbf{y}(s)\,, \quad s \in [0,1]$$

for any $\mathbf{y} \in V_n$ while no information about the analytic form of this solution is obtained.

Let us suppose (as in Part 2)

$$v(\mathbf{K}) < \infty \tag{3,2}$$

and

$$\operatorname{var}_0^1 \mathbf{K}(1, .) \leqq \infty . \tag{3,3}$$

We consider the homogeneous equation

$$\mathbf{x}(s) = \int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) , \quad s \in [0, 1] . \tag{3,4}$$

Using the limit relation for Perron-Stieltjes integrals (cf. Theorem 1,3,6 in [2]) we have

$$\int_0^s \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) = \lim_{\sigma \to s-} \int_0^\sigma \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) + [\mathbf{K}(s, s) - \mathbf{K}(s, s-)]\, \mathbf{x}(s) .$$

Hence the homogeneous equation (3,4) can be written in the form

$$\mathbf{x}(s) - \mathbf{H}(s)\, \mathbf{x}(s) \lim_{\sigma \to s-} \int_0^\sigma \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) , \quad s \in [0, 1]$$

where

$$\mathbf{H}(s) = \mathbf{K}(s, s) - \mathbf{K}(s, s-) = \mathbf{K}(s, s) - \lim_{t \to s-} \mathbf{K}(s, t) . \tag{3,5}$$

This form of (3,4) implies that for the unique determination of $\mathbf{x}(s)$ from the knowledge of $\mathbf{x}(\tau)$ for $\tau \in [0, s)$ it is necessary to assume that the inverse matrix $[\mathbf{I} - \mathbf{H}(s)]^{-1}$ exists. The above form of the equation (3,4) is not convenient for further investigation. Therefore we define

$$\mathbf{M}(s, t) = \mathbf{K}(s, t-) = \lim_{\tau \to t-} \mathbf{K}(s, \tau) . \tag{3,6}$$

We have

$$\mathbf{K}(s, t) = \mathbf{M}(s, t) + [\mathbf{K}(s, t) - \mathbf{K}(s, t-)] . \tag{3,7}$$

Since (3,2) holds we have $\operatorname{var}_0^1 \mathbf{K}(s, .) < \infty$ for any $s \in [0, 1]$ (see (2,14a) in [4]). Hence for any $s \in [0, 1]$ the difference $\mathbf{K}(s, t) - \mathbf{K}(s, t-)$ is different from zero only on an at most countable set of points $t \in [0, 1]$ and we have

$$\int_0^1 \mathrm{d}_t[\mathbf{K}(s, t) - \mathbf{K}(s, t-)]\, \mathbf{x}(t) = [\mathbf{K}(s, s) - \mathbf{K}(s, s-)]\, \mathbf{x}(s) = \mathbf{H}(s)\, \mathbf{x}(s) \tag{3,8}$$

for any $s \in [0, 1]$ (this can be easily obtained from Corollary 2,2 and Proposition 2,1 in [4]).

The homogeneous equation (3,4) assumes in view of (3,7) and (3,8) the form

$$x(s) = \int_0^s \mathrm{d}_t[M(s, t)]\, x(t) + H(s)\, x(s)$$

or

$$x(s) = [I - H(s)]^{-1} \int_0^s \mathrm{d}_t[M(s, t)]\, x(t)\,. \quad s \in [0,1] \tag{3,9}$$

where $H(s) : [0, 1] \to L(R^n \to R^n)$ is given by (3,5) and $M(s, t) : : [0, 1] \times [0, 1] \to \to L(R^n \to R^n)$ by (3,6). Let us mention that $M(s, t)$ is evidently continuous from the left in the second variable, i.e. $\lim_{\tau \to t} M(s, \tau) = M(s, t)$, $t \in (0, 1]$.

Lemma 3.1. *If (3,2) and (3,3) are satisfied then the matrix $H(s) : [0, 1] \to L(R^n \to \to R^n)$ defined by (3,5) fulfils*

$$\operatorname{var}_0^1 H < \infty \tag{3,10}$$

and there exists a sequence $\{s_i\}_{i=1}^{\infty}$, $s_i \in [0, 1]$ such that $H(s) = 0$ if $s \neq s_i$, $i = = 1, 2, \ldots$. Consequently

$$\operatorname{var}_0^1 H = 2 \sum_{i=1}^{\infty} \|H(s_i)\| < \infty \tag{3,11}$$

Proof. Let us define $\tilde{K}(s, t) = K(s, t) - K(s, 0)$. By (3,2) we have $v(\tilde{K}) < \infty$ because $v(\tilde{K}) = v(K)$ by the definition of the twodimensional variation. Further it is $\operatorname{var}_0^1 \tilde{K}(1, .) = \operatorname{var}_0^1 K(1, .)$ since $\tilde{K}(1, t_2) - \tilde{K}(1, t_1) = K(1, t_2) - K(1, t_1)$ for any $t_1, t_2 \in [0, 1]$ and we have also $\operatorname{var}_0^1 K(., 0) = 0$. Hence by Theorem 5,4 in Chapter III iu [1] the set of discontinuities of $\tilde{K}(s, t)$ lie on an at most countable set of lines parallel to the coordinate axes and therefore $\tilde{K}(s, t) - \tilde{K}(s, t-) = \tilde{K}(s, t) - \tilde{K}(s, t-)$ equals the zero matrix except a countable set of lines which are parallel to the s-axis in $[0,1] \times [0,1]$. This yields the existence of a sequence $\{s_i\}_{i=1}^{\infty}$, $s_i \in [0, 1]$ such that $H(s) = K(s, s) - K(s, s-) = 0$ for $s \neq s_i$, $i = 1, 2, \ldots$.

If $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_k = 1$ is an arbitrary finite sequence then $\|H(\sigma_i) - - H(\sigma_{i-1})\| \leq \|H(\sigma_i)\| - \|H(\sigma_{i-1})\|$ for $i = 1, 2, \ldots, k$. Further we have $\|H(s)\| = = \|K(s, s) - K(s, s-)\| \leq \|K(1, s) - K(1, s-)\| + \|K(s, s) - K(s, s-) - K(1, s) + + K(1, s-)\|$.

Hence

$$\sum_{i=1}^{k} \|H(\sigma_i) - H(\sigma_{i-1})\| \leq [\sum_{i=1}^{k} \|H(\sigma_i)\| + H(\sigma_{i-1})\|] \leq 2 \sum_{i=1}^{k} \|H(\sigma_i)\| \leq$$

$$\leq 2 \sum_{i=1}^{k} [\|K(\sigma_i, \sigma_i) - K(\sigma_i, \sigma_i-) - K(1, \sigma_i) + K(1, \sigma_i-)\| +$$

$$+ \|K(1, \sigma_i) - K(1, \sigma_i-)\|] \leq 2v(K) + 2 \operatorname{var}_0^1 K(1, .)\,.$$

Therefore by (3,2) and (3,3) we obtain (3,10) if we take the supremum over all finite decompositions of $[0, 1]$ on the left hand side of this inequality. The relation (3,11) is evident.

Lemma 3.2. *Let* (3,2) *and* (3,3) *hold and let for* $\boldsymbol{H}(s) : [0, 1] \to L(R^n \to R^n)$ *from* (3,5) *the inverse matrix* $[\boldsymbol{I} - \boldsymbol{H}(s)]^{-1}$ *exist for all* $s \in [0, 1$. *For* $\boldsymbol{z} \in V_n(0, 1)$ *we define the function* $\boldsymbol{v} : [0, 1] \to R^n$ *by the relation*

$$(3,12) \qquad \boldsymbol{v}(s) = [\boldsymbol{I} - \boldsymbol{H}(s)]^{-1}\, \boldsymbol{v}(s)\,, \quad s \in [0, 1]\,.$$

Then $\boldsymbol{v} \in V_n(0, 1)$, *i.e., the linear operator*

$$(3,13) \qquad \boldsymbol{T}\boldsymbol{z} = \boldsymbol{v}\,, \quad \boldsymbol{v} \in V_n$$

maps the Banach space V_n *into the same space* V_n. *Moreover, the operator* $\boldsymbol{T}$ *is bounded, i.e., there exists a nonnegative constant* C *such that*

$$(3,14) \qquad \|\boldsymbol{T}\boldsymbol{z}\|_{V_n} \leqq C\|\boldsymbol{z}\|_{V_n}\,.$$

Proof. Since $[\boldsymbol{I} - \boldsymbol{H}(s)]^{-1} = \boldsymbol{I} + \boldsymbol{H}(s)\,[\boldsymbol{I} - \boldsymbol{H}(s)]^{-1}$, we have $\boldsymbol{v}(s) = \boldsymbol{z}(s) + \boldsymbol{H}(s)\,[\boldsymbol{I} - \boldsymbol{H}(s)]^{-1}\,\boldsymbol{z}(s)$. We consider the second term of this equality, i.e. the function

$$\boldsymbol{u}(s) = \boldsymbol{H}(s)\,[\boldsymbol{I} - \boldsymbol{H}(s)]^{-1}\,\boldsymbol{z}(s)\,.$$

By Lemma 3,1 there is a sequence $\{s_i\}_{i=1}^{\infty}$, $s_i \in [0, 1]$ such that $\boldsymbol{H}(s) = \boldsymbol{0}$ for $s \neq s_i$ hence $\boldsymbol{u}(s) = \boldsymbol{0}$ for $s \neq s_i$. It follows

$$(3,15) \qquad \operatorname{var}_0^1 \boldsymbol{u} = 2\sum_{i=1}^{\infty} \|\boldsymbol{u}(s_i)\| = 2\sum_{i=1}^{\infty} \|\boldsymbol{H}(s_i)\,[\boldsymbol{I} - \boldsymbol{H}(s_i)]^{-1}\,\boldsymbol{z}(s_i)\| \leqq$$
$$\leqq 2\|\boldsymbol{z}\|_{V_n} \sum_{i=1}^{\infty} \|\boldsymbol{H}(s_i)\|\; \|[\boldsymbol{I} - \boldsymbol{H}(s_i)]^{-1}\|\,.$$

Since $\sum_{i=1}^{\infty} \|\boldsymbol{H}(s_i)\| < \infty$, there exists an integer $n_0 > 0$ such that $\|\boldsymbol{H}(s_i)\| < 1/2$ for all $i > n_0$. Consequently

$$\|[\boldsymbol{I} - \boldsymbol{H}(s_i)]^{-1}\| \leqq 1 + \|\boldsymbol{H}(s_i)\| + \|\boldsymbol{H}(s_i)\|^2 + \ldots = (1 - \|\boldsymbol{H}(s_i)\|)^{-1} < 2$$

for $i > n_0$. Furthermore,

$$\sum_{i=1}^{\infty} \|\boldsymbol{H}(s_i)\|\; \|[\boldsymbol{I} - \boldsymbol{H}(s_i)]^{-1}\| \leqq \sum_{i=1}^{n_0} \|\boldsymbol{H}(s_i)\| \,.\, \|[\boldsymbol{I} - \boldsymbol{H}(s_i)]^{-1}\| + 2\sum_{i=n_0+1}^{\infty} \|\boldsymbol{H}(s_i)\| \leqq$$
$$\leqq \{\sup_{i=1,\ldots,n_0} \|[\boldsymbol{I} - \boldsymbol{H}(s_i)]^{-1}\| + 2\} \sum_{i=1}^{\infty} \|H(s_i)\| = C_0 < \infty$$

Hence by (3,15)

$$\operatorname{var}_0^1 \boldsymbol{u} \leqq C_0 \|\boldsymbol{z}\|_{V_n} \, (C_0 < \infty).$$

This inequality gives

$$\operatorname{var}_0^1 \boldsymbol{v} = \operatorname{var}_0^1 \boldsymbol{z} + \operatorname{var}_0^1 \boldsymbol{u} \leqq (1 + C_0) \|\boldsymbol{z}\|_{V_n} = C_1 \|\boldsymbol{z}\|_{V_n}$$

and also $\boldsymbol{v} \in V_n$.

By the definition of the norm in V_n we have further

$$\|\boldsymbol{T}\boldsymbol{z}\|_{V_n} = \|\boldsymbol{v}(0)\| + \operatorname{var}_0^1 \boldsymbol{v} \leqq \|\boldsymbol{H}(0)[\boldsymbol{I} - \boldsymbol{H}(0)]^{-1} \boldsymbol{z}(0)\| + C_1 \|\boldsymbol{z}\|_{V_n} \leqq$$
$$\leqq (\|\boldsymbol{H}(0)[\boldsymbol{I} - \boldsymbol{H}(0)]^{-1}\| + C_1) \|\boldsymbol{z}\|_{V_n} = C \|\boldsymbol{z}\|_{V_n}$$

where obviously $C < \infty$. The proof of our Lemma is complete.

Remark 3.1. Let us mention that in Lemma 3,2 the estimate (3,14) remains valid also for a smaller interval, for example for the interval $[0, \delta]$, where $0 < \delta < 1$, i.e.

$$\|\boldsymbol{T}\boldsymbol{z}\|_{V_n[0,\delta]} \leqq C \|\boldsymbol{z}\|_{V_n[0,\delta]}.$$

Lemma 3.3. *Let* $\boldsymbol{M}(s, t) : [0, 1] \times [0, 1] \to L(R^n \to R^n)$ *satisfy* $v(\boldsymbol{M}) < \infty$, $\operatorname{var}_0^1 \boldsymbol{M}(1, .) < \infty$ *and* $\boldsymbol{M}(s, t-) = \lim_{\tau \to t-} \boldsymbol{M}(s, \tau) = \boldsymbol{M}(s, t)$ *for any* $s \in [0, 1]$ *and* $t \in (0, 1]$. *Then there exists a nondecreasing function* $\zeta : [0, 1] \to [0, +\infty)$ *which is continuous from the left in* $(0, 1]$ *such that for any* $\delta \in [0, 1]$ *and* $\boldsymbol{x} \in V_n$ *we have*

(3,16)

$$\operatorname{var}_0^\delta \left(\int_0^s \mathrm{d}_t[\boldsymbol{M}(s, t)] \, \boldsymbol{x}(t) \right) \leqq \|\boldsymbol{x}(0)\| \, (\zeta(0+) - \zeta(0)) + \|\boldsymbol{x}\|_{V_n[0,\delta]} (\zeta(\delta) - \zeta(0+)).$$

Proof. Let $0 = s_0 < s_1 < \ldots < s_l = \delta$ be an arbitrary decomposition of the interval $[0, \delta]$. If we define for $\boldsymbol{M}$ the corresponding triangular kernel $\boldsymbol{M}^\Delta$ (see (1,4)) then we have

$$\int_0^s \mathrm{d}_t[\boldsymbol{M}(s, t)] \, \boldsymbol{x}(t) = \int_0^1 \mathrm{d}_t[\boldsymbol{M}^\Delta(s, t)] \, \boldsymbol{x}(t)$$

and

$$\operatorname{var}_0^\delta \left(\int_0^s \mathrm{d}_t[\boldsymbol{M}(s, t)] \, \boldsymbol{x}(t) \right) = \operatorname{var}_0^\delta \left(\int_0^1 \mathrm{d}_t[\boldsymbol{M}^\Delta(s, t)] \, \boldsymbol{x}(t) \right).$$

Further it is

(3,17)
$$\sum_{i=1}^{l} \left\| \int_0^1 \mathrm{d}_t[\boldsymbol{M}^\Delta(s_i, t) - \boldsymbol{M}^\Delta(s_{i-1}, t)] \, \boldsymbol{x}(t) \right\| \leqq$$
$$\leqq \sum_{i=1}^{l} \int_0^1 \|\boldsymbol{x}(t)\| \, \mathrm{d} \operatorname{var}_0^t (\boldsymbol{M}^\Delta(s_i, .) - \boldsymbol{M}^\Delta(s_{i-1}, .)).$$

By the definition of $\boldsymbol{M}^{\Delta}(s, t)$ and from the inequality $s_i \leqq \delta$ for any $i = 1, \ldots, l$ it is easy to obtain that for $t > \delta$ the real function $\mathrm{var}_0^t(\boldsymbol{M}^{\Delta}(s_i, .) - \boldsymbol{M}^{\Delta}(s_{i-1}, .))$ of the variable t is constant. Hence

$$\int_0^1 \|\boldsymbol{x}(t)\| \, \mathrm{d}\, \mathrm{var}_0^t (\boldsymbol{M}^{\Delta}(s_i, .) - \boldsymbol{M}^{\Delta}(s_{i-1}, .)) = \int_0^{\delta} \|\boldsymbol{x}(t)\| \, \mathrm{d}\, \mathrm{var}_0^t(\boldsymbol{M}^{\Delta}(s_i, .) - \boldsymbol{M}^{\Delta}(s_{i-1}, .))$$

for any $i = 1, \ldots, l$.

This implies (cf. (3,17)) the inequality

$$(3,18) \qquad \sum_{i=1}^{l} \left\| \int_0^1 \mathrm{d}_t[\boldsymbol{M}^{\Delta}(s_i), t) - \boldsymbol{M}^{\Delta}(s_{i-1}, t)] \, \boldsymbol{x}(t) \right\| \leqq$$
$$\leqq \int_0^{\delta} \|\boldsymbol{x}(t)\| \, \mathrm{d} \left(\sum_{i=1}^{l} \mathrm{var}_0^t (\boldsymbol{M}^{\Delta}(s_i, .) - \boldsymbol{M}^{\Delta}(s_{i-1}, .)) \right).$$

Further, (2,16a) in [4] implies for any $t_1, t_2 \in [0, \delta]$

$$(3,19) \quad \left| \sum_{i=1}^{l} [\mathrm{var}_0^{t_2}(\boldsymbol{M}(s_i, .) - \boldsymbol{M}(s_{i-1}, .)) - \mathrm{var}_0^{t_1}(\boldsymbol{M}(s_i, .) - M(s_{i-1}, .))] \right| \leqq$$
$$\leqq |\zeta(t_2) - \zeta(t_1)|$$

where $\zeta : [0, 1] \to [0, +\infty)$ is defined by the relation

$$\zeta(\tau) = v_{[0,1] \times [0,\tau]}(\boldsymbol{M}^{\Delta})$$

for $\tau \in [0, 1]$ (see also (2,15b) in [4]). We have $\zeta(0) = 0$, $\zeta(1) = v(\boldsymbol{M}^{\Delta})$; is nondecreasing and continuous from the left in $[0, 1]$. The inequality (3,19) yields by Lemma 3,1 from [3] the inequality

$$\int_0^{\delta} \|\boldsymbol{x}(t)\| \, \mathrm{d} \, (\sum_{i=1}^{l} \mathrm{var}_0^t(\boldsymbol{M}^{\Delta}(s_i, .) - \boldsymbol{M}^{\Delta}(s_{i-1}, .)) \leqq \int_0^{\delta} \|\boldsymbol{x}(t)\| \, \mathrm{d}\zeta(t) \, .$$

Hence by (3,18) we obtain

$$(3,20) \qquad \sum_{i=1}^{l} \left\| \int_0^1 \mathrm{d}_t[\boldsymbol{M}^{\Delta}(s_i, t) - \boldsymbol{M}^{\Delta}(s_{i-1}, t)] \, \boldsymbol{x}(t) \right\| \leqq \int_0^{\delta} \|\boldsymbol{x}(t)\| \, \mathrm{d}\zeta(t)$$

and evidently also

$$(3,21) \qquad \mathrm{var}_0^{\delta} \left(\int_0^s \mathrm{d}_t[\boldsymbol{M}(s, t)] \, \boldsymbol{x}(t) \right) \leqq \int_0^{\delta} \|\boldsymbol{x}(t)\| \, \mathrm{d}\zeta(t)$$

since the estimate (3,20) is independent of the choice of the decomposition $0 = s_0 < < s_1 < \ldots < s_l = \delta$.

If we use Theorem 1,3,6 from [2] then we obtain

$$\int_0^\delta \|\boldsymbol{x}(t)\| \,\mathrm{d}\zeta(t) = \|\boldsymbol{x}(0)\| \,(\zeta(0+) - \zeta(0)) + \lim_{\sigma \to 0+} \int_\sigma^\delta \|\boldsymbol{x}(t)\| \,\mathrm{d}\zeta(t)$$

and for any $\sigma > 0$ we have evidently

$$\int_\sigma^\delta \|\boldsymbol{x}(t)\| \,\mathrm{d}\zeta(t) \leqq \sup_{t \in [0,\delta]} \|\boldsymbol{x}(t)\| \,(\zeta(\delta) - \zeta(\sigma)) \leqq \|\boldsymbol{x}\|_{V_n(0,\delta)} [\zeta(\delta) - \zeta(0+)] \,.$$

Summarizing these facts we obtain from (3,21) the inequality (3,16).

Let us now consider the homogeneous equation (3,4) in the form (3,9). Formally the equation (3,9) can be written in the form

$$\boldsymbol{x} = \boldsymbol{T}\boldsymbol{M}\boldsymbol{x} \,, \quad \boldsymbol{x} \in V_n \tag{3,22}$$

where $\boldsymbol{T} : V_n \to V_n$ is the linear operator defined by (3,13) in Lemma 3,2 and $\boldsymbol{M}$: : $V_n \to V_n$ is the linear operator defined by the relation

$$\boldsymbol{M}\boldsymbol{x} = \boldsymbol{z} \,, \quad \boldsymbol{x} \in V_n \tag{3,23}$$

where

$$\boldsymbol{z}(s) = \int_0^s \mathrm{d}_t[\boldsymbol{M}(s, t)] \,\boldsymbol{x}(t) \quad \text{for} \quad \boldsymbol{x} \in V_n \,. \tag{3,24}$$

By Lemma 3,3 evidently $\boldsymbol{z} \in V_n$ and, moreover,

$$\|\boldsymbol{M}\boldsymbol{x}\|_{V_n} = \|\boldsymbol{z}(0)\| + \mathrm{var}_0^1 \,\boldsymbol{z} = \mathrm{var}_0^1 \,\boldsymbol{z} \leqq \|\boldsymbol{x}(0)\| \,(\zeta(0+) - \zeta(0)) + \\ + \|\boldsymbol{x}\|_{V_n} (\zeta(\delta) - \zeta(0+)) \,.$$

This means that the operator $\boldsymbol{M} : V_n \to V_n$ is bounded. Since $\boldsymbol{x}(0) = \boldsymbol{0}$ for any solution $\boldsymbol{x} \in V_n$ of the homogeneous equation (3,4), we obtain for any solution of (3,4) by Lemma 3,3 the inequality

$$\|\boldsymbol{x}\|_{V_n(0,\delta)} = \|\boldsymbol{T}\boldsymbol{M}\boldsymbol{x}\|_{V_n(0,\delta)} \leqq C + \|\boldsymbol{M}\boldsymbol{x}\|_{V_n(0,\delta)} \leqq \\ \leqq C(\zeta(\delta) - \zeta(0+)) \,\|\boldsymbol{x}\|_{V_n(0,\delta)} \tag{3,25}$$

where $C \geqq 0$ is the bound of the operator $\boldsymbol{T}$ (see Lemma 3,2 and Remark 3,1), and $\zeta : [0, 1] \to [0, +\infty]$ is the function from Lemma 3,3. Therefore for sufficiently small $\delta > 0$ we have $\zeta(\delta) - \zeta(0+) < 1/(2C)$ and consequently, by (3.25)

$$\|\boldsymbol{x}\|_{V_n(0,\delta)} \leqq \tfrac{1}{2} \|\boldsymbol{x}\|_{V_n(0,\delta)} \tag{3,26}$$

holds for a sufficiently small $\delta > 0$. Hence $\|\boldsymbol{x}\|_{V_n(0,\delta)} = 0$ and this implies the existence of a $\delta > 0$ such that $\boldsymbol{x}(t) = \boldsymbol{0}$ for all $t \in [0, \delta]$.

In this way we have obtained that if $\boldsymbol{x} \in V_n$ is a solution of the homogeneous equation (3,4) then there exists a positive δ such that $\boldsymbol{x}(t)$ equals identically zero on the interval $[0, \delta]$.

If now $t^* \in [0, 1]$ is the supremum of all such positive δ that the solution $\boldsymbol{x} \in V_n$ of (3,4) equals zero on $[0, \delta]$, we have $\boldsymbol{x}(t) = \boldsymbol{0}$ for all $t \in [0, t^*)$. Hence

$$\int_0^{t^*} \mathrm{d}_t \left[\boldsymbol{M}(s, t)\right] \boldsymbol{x}(t) = \boldsymbol{0}^{5})$$

and since the matrix $\boldsymbol{I} - \boldsymbol{H}(t^*)$ is assumed to be nonsingular we obtain immediately $\boldsymbol{x}(t^*) = \boldsymbol{0}$. Now, assuming that $t^* < 1$, we can prove essentially in the same manner as above that there is a positive δ such that for the solution $\boldsymbol{x} \in V_n$ of (3,4) we have also $\boldsymbol{x}(t) = \boldsymbol{0}$ for $t \in [t^*, t^* + \delta]$ and this contradicts the assumption $t^* < 1$ and the definition of the supremum. Hence $t^* = 1$ and for any solution $\boldsymbol{x} \in V_n$ of (3,4) we have $\boldsymbol{x} = \boldsymbol{0}$.

Summarizing the above results we can formulate the following

Theorem 3.1. *Let for the kernel* $\boldsymbol{K}(s, t) : [0, 1] \times [0, 1] \to L(R^n \to R^n)$ *(3,2) and (3,3) be satisfied and let for any* $s \in [0, 1]$ *the inverse matrix* $[\boldsymbol{I} - (\boldsymbol{K}(s, s) - \boldsymbol{K}(s, s-))]^{-1}$ *exist. Then the homogeneous Volterra-Stieltjes integral equation (3,4) has only the zero solution* $\boldsymbol{x}(t) = \boldsymbol{0}$ *for any* $t \in [0, 1]$. *The corresponding nonhomogeneous equation*

$$\boldsymbol{x}(s) = \int_0^s \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}(t) + \langle(s)\,, \quad s \in [0, 1] \tag{3,27}$$

has precisely one solution $\boldsymbol{x} \in V_n$ *for any* $\boldsymbol{y} \in V_n$.

Proof. Since (3,2) and (3,3) is satisfied the operator occuring in the equation (3,4) is completely continuous (cf. Theorem 3,1 in [4]); this can be proved via the corresponding triangular kernel $\boldsymbol{K}^\Delta$ given by (1,4). Hence the Fredholm alternative for the Voltera-Stieltjes integral equation is valid (cf. Theorem 5,2 in [4]). We have proved above in this part that any solution of the homogeneous equation (3,4) is equal identically to zero and this together with the Fredholm alternative implies the existence and unicity of the solution of the nonhomogeneous equation (3,27).

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[5]) This follows from the fact that $\boldsymbol{M}(s, t)$ is continuous from the left in the variable t and therefore the integral considered does not depend on the value $\boldsymbol{x}(t^*)$.