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# MORSE-SARD THEOREM IN INFINITE DIMENSIONAL BANACH SPACES AND INVESTIGATION OF THE SET OF ALL CRITICAL LEVELS

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## INTRODUCTION

Let $f$ and $g$ be two nonlinear functionals defined on a real Banach space $X$. Consider the eigenvalue problem

$$\text{(E)} \qquad \lambda f'(u) = g'(u)\,, \quad u \in M_r(f) = \{x \in X : f(x) = r\}$$

($r > 0$ is a prescribed number, $f'$ and $g'$ denote Fréchet derivatives of $f$ and $g$, respectively). The value of the functional $g$ at the solution of (E) is called the critical level. Denote by $\Gamma$ the set of all critical levels. L. A. LJUSTERNIK and L. SCHNIRELMANN proved that the set $\Gamma$ is, under suitable assumptions, at least countable (see [1, 10, 11]). In papers [2, 3] it is proved that $\Gamma$ is a sequence of positive numbers converging to zero. While the determination of the lower bound for the number of points of the set $\Gamma$ is based on topological methods, the upper bound is found on the basis of properties of real-analytic functionals $f$ and $g$. It is our object in this paper to prove that if $f$ and $g$ are not real-analytic functionals, then the set $\Gamma$ is small, i.e., $\alpha$-Hausdorff measure of $\Gamma$ is zero, where $\alpha$ depends on differentiability of functionals $f$ and $g$. The proof is based on the Morse-Sard theorem in infinite-dimensional Banach space which was firstly for so-called "Fredholm functionals" considered by S. I. POCHOŽAJEV [13] (see Section 2). The results about the structure of the set $\Gamma$ are obtained in Section 3. Section 4 deals with the applications of previous abstract results to the boundary value problem for ordinary differential equations.

## 1. NOTATIONS AND GENERAL REMARKS

Let $X$ be a real Banach space with the norm $\|\cdot\|$, $X^*$ its dual, $\Omega$ an open set in $X$. Consider the other (real) Banach space $Y$ with the norm $\|\cdot\|_Y$ and a mapping $F$ of $\Omega$ into $Y$.

**Differentiability of mappings.** The mapping $F$ is said to have *Fréchet derivative* $\mathrm{d}F(x, \cdot)$ at the point $x \in \Omega$ if $\mathrm{d}F(x, \cdot)$ is a linear and bounded mapping of $X$ into $Y$ such that for each $h \in X$

$$F(x + h) - F(x) = \mathrm{d}F(x, h) + r(x, h),$$

where

$$\lim_{\|h\| \to 0} \frac{\|r(x, h)\|_Y}{\|h\|} = 0.$$

Further, for each $h_1, h_2 \in X$, denote

$$\mathrm{d}^2F(x, h_1, h_2) = \lim_{\xi \to 0} \frac{\mathrm{d}F(x + \xi h_2, h_1) - \mathrm{d}F(x, h_1)}{\xi}.$$

If we have defined $\mathrm{d}^{n-1}F(x, \ldots)$ as a multilinear continuous mapping of $X \times \ldots \times X$ ($(n - 1)$-times) into $Y$, then we set for each $h_1, \ldots, h_n \in X$

$$(*)\ \mathrm{d}^nF(x, h_1, \ldots, h_n) = \lim_{\xi \to 0} \frac{\mathrm{d}^{n-1}F(x + \xi h_n, h_1, \ldots, h_{n-1}) - \mathrm{d}^{n-1}F(x, h_1, \ldots, h_{n-1})}{\xi}.$$

The mapping $F$ is said to have Fréchet derivative $\mathrm{d}^nF(x, \ldots)$ of the order $n$, if $\mathrm{d}^nF(x, \ldots)$ is a multilinear continuous mapping of $X \times \ldots \times X$ ($n$-times) such that the relation $(*)$ holds uniformly for $\|h_1\| \leqq 1, \ldots, \|h_n\| \leqq 1$. We shall denote $\mathrm{d}F(x, \cdot) = F'(x)$, i.e., $\mathrm{d}F(x, h) = F'(x)(h)$ and $\mathrm{d}^nF(x, \ldots) = F^{(n)}(x)$. Let us suppose $F$ has Fréchet derivatives up to the order $n$ in $X$. If $X_1, X_2$ are subspaces of the space $X$, and $X = X_1 \oplus X_2$, $x_1 \in X_1$, $x_2 \in X_2$, then we denote for $h \in X_2$

$$F'_{x_2}(x_1, x_2)(h) = \partial_{x_2}F(x_1, x_2; h) = \lim_{\xi \to 0} \frac{F(x_1, x_2 + \xi h) - F(x_1, x_2)}{\xi}.$$

Linear mapping $\partial_{x_2}F(x_1, x_2; \cdot)$ (for $x_1, x_2$ fixed) of $X_2$ into $Y$ is said to be *partial derivative* of $F$ in $x = (x_1, x_2)$ with respect to the variable $x_2$. Analogously, we can introduce partial derivative with respect to the variable $x_1$ and the partial derivatives of the higher orders (up to the order $n$). For example, we see

$$\partial^2_{x_2,x_1}F(x_1, x_2; h_1, h_2) = \lim_{\xi \to 0} \frac{\partial_{x_2}F(x_1 + \xi h_1, x_2; h_2) - \partial_{x_2}F(x_1, x_2; h_2)}{\xi}.$$

If $f$ is a functional on $\Omega$, then

$$\mathrm{d}^2f(x, h, \cdot) = f''(x)(h, \cdot)$$

(for $x$ fixed) can be considered as a continuous linear mapping of $X$ into $X^*$. We shall denote $f''(x)(h, .) = f''(x)(h)$.

**Spaces $C^{k,\alpha}$.** Let $k$ be a positive integer, $\alpha$ a real number, $\alpha \in \langle 0, 1 \rangle$. We shall write $F \in C^{k,\alpha}(\Omega)$ if

(a) $F$ has on $\Omega$ all Fréchet derivatives up to the order $k$ and these derivatives are continuous in the variable $x$, i.e., with respect to the norm

$$\|F^{(j)}(x)\|_j = \sup_{\substack{h_i \in X, \|h_i\| = 1 \\ i = 1, \dots, j}} \|F^{(j)}(x)\,(h_1, \dots, h_j)\|_Y ;$$

(b) the derivative $F^{(k)}$ is $\alpha$-hölderian, i.e., there exists $c > 0$ such that

$$\|F^{(k)}(x) - F^{(k)}(y)\|_k \leqq c\|x - y\|^\alpha$$

for each $x, y \in \Omega$.

We shall denote $C^{k,0}(\Omega) = C^k(\Omega)$.

The mapping $F$ is said an element of the space $C^{k,\alpha}(\bar{\Omega})$ ($\bar{\Omega}$ denotes the closure of $\Omega$) if $F \in C^{k,\alpha}(\Omega)$ and the derivatives $F^{(j)}$ ($j = 0, \dots, k$) are continuously extendible on $\bar{\Omega}$.

**Proposition 1.1** (Implicit function theorem). *Let $X, Y, Z$ be real Banach spaces, $\Omega$ an open set in the space $X \times Y$, $[x_0, y_0] \in \Omega$. Consider a mapping $F \in C^{k,\alpha}(\Omega)$ of $\Omega$ into $Z$ such that there exists the mapping $[F'_y(x_0, y_0)]^{-1}$ of $Z$ onto $Y$ and $F(x_0, y_0) = 0$.*

*Then there exists a neighborhood $U(x_0)$ of the point $x_0$, and a neighborhood $U(y_0)$ of the point $y_0$ and only one mapping $\varphi$ from $U(x_0)$ into $U(y_0)$ such that*

(1.1) *$[F'_y(x, y)]^{-1}$ exists and maps $Z$ onto $Y$ for each $x \in U(x_0)$ and $y \in U(y_0)$,*

(1.2) *$F(x, \varphi(x)) = 0$ on $U(x_0)$.*

*Moreover, $\varphi \in C^{k,\alpha}(U(x_0))$.*

Proof of this assertion for $F \in C^k$ (i.e., for $\alpha = 0$) is given in the paper [6]. Let us show that it holds for $\alpha \in (0, 1\rangle$, too. Suppose that $U(x_0)$, $U(y_0)$ are neighborhoods and $\varphi$ is a mapping such that (1.1), (1.2) are fulfilled and $\varphi \in C^k(U(x_0))$. We shall prove $\varphi \in C^{k,\alpha}(U(x_0))$. It follows from (1.2)

$$\mathrm{d}F([x, \varphi(x)], h) = \partial_x F([x, \varphi(x)], h) + \partial_y F([x, \varphi(x)], \mathrm{d}\varphi(x, h))$$

for each $h \in X$. By using (1.1) we obtain

$$\varphi'(x) = -\,[F'_y(x, \varphi(x))]^{-1}\, F'_x(x, \varphi(x))\,.$$

Further, (if $k \geqq 2$),

$$\varphi''(x) = -\,[F'_y(x, \varphi(x))]^{-1}\, F''_{yx}(x, \varphi(x))\,[F'_y(x, \varphi(x))]^{-1}\,,$$

$$F'(x, \varphi(x)) - [F'_y(x, \varphi(x))]^{-1}\, F''_{yy}(x, \varphi(x))\,\varphi'(x)\,[F'_y(x, \varphi(x))]^{-1}\,,$$

$$F'_x(x, \varphi(x)) - [F'_y(x, \varphi(x))]^{-1}\, F''_{xx}(x, \varphi(x)) - [F'_y(x, \varphi(x))]^{-1}\, F''_{xy}(x, \varphi(x))\,\varphi'(x)\,.$$

It is easy to see that

$$\varphi^{(k)}(x) = \Phi_1(x) + \ldots + \Phi_p(x),$$

where $\Phi_i(x)$ (for fixed $x$) is a multilinear continuous mapping of $X \times \ldots \times X$ ($k$-times) into $Z$, which can be obtained as a suitable composition of $[F'_y(x\ \varphi(x))]^{-1}$ and of partial derivatives up to the order $k$ ($i = 1 \ldots p$). Derivatives of $F$ of the order $k$ are $\alpha$-hölderian mappings, too.

Hence, it is sufficient to show $[F'_y(x, \varphi(x))]^{-1}$ is $\alpha$-hölderian. For $x_1, x_2 \in U(x_0)$ we have ($\|\cdot\|_1$ is the norm defined in (a))

$$\begin{aligned}
&\|[F'_y(x_1, \varphi(x_1))]^{-1} - [F'_y(x_2, \varphi(x_2)]^{-1}\|_1 = \\
&\|[F'_y(x_2, \varphi(x_2))]^{-1} F'_y(x_2, \varphi(x_2)) [F'_y(x_1, \varphi(x_1))]^{-1} - \\
&- [F'_y(x_2, \varphi(x_2))]^{-1} F'_y(x_1, \varphi(x_1)) [F'_y(x_1, \varphi(x_1))]^{-1}\|_1 \leqq \\
&\leqq \|[F'_y(x_2, \varphi(x_2))]^{-1}\|_1 \cdot \|F'_y(x_2, \varphi(x_2)) - F'_y(x_1, \varphi(x_1))\|_1 \cdot \\
&\cdot \|[F'_y(x_1, \varphi(x_1))]^{-1}\|_1 \leqq c\|x_1 - x_2\|^\alpha
\end{aligned}$$

(it is easy to see that the norms $\|[F'_y(x, \varphi(x))]^{-1}\|_1$ are bounded for $x$ from a sufficiently small neighborhood $U(x_0)$ of the point $x_0$).

**Hausdorff measure.** Let $A$ be a subset of $n$-dimensional Euclidean space $E_n$ and let $s$ be a positive real number. Set for each $\varepsilon > 0$

$$\mu_{s,\varepsilon}(A) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^s,$$

the infimum being taken over all countable coverings $\{A_i\}_{i=1}^{\infty}$ of $A$ such that $\operatorname{diam} A_i < \varepsilon$. The number

$$\mu_s(A) = \lim_{\varepsilon \to 0+} \mu_{s,\varepsilon}(A)$$

is said to be *s-Hausdorff measure* of the set $A$. If $\mu_s(A) = 0$, then the set $A$ is said to be $s$-null. If $A$ is $s$-null, then $A$ is $r$-null for each $r > s$. If $s = n$, then $\mu_n(A)$ is the $n$-dimensional Lebesgue measure of the set $A$.

## 2. INFINITE-DIMENSIONAL VERSION OF THE MORSE-SARD THEOREM

The well-known theorem about real-valued functions, so called Morse-Sard theorem, says that if $\Omega$ is an open subset of Euclidean $n$-space $E_n$ and $f \in C^n(\Omega)$ is a real function, then the Lebesgue measure of the set $f(B)$ is zero, where

$$B = \{x \in \Omega : \operatorname{grad} f(x) = 0\}.$$

For further consideration, the following generalization is fundamental.

**Proposition 2.1** (see [8]). *Let $\Omega$ be an open set in $E_n$, let $f$ be a function, $f \in C^{k,\alpha}(\Omega)$ (where $k$ is positive integer, $\alpha \in \langle 0, 1 \rangle$).*

*Then the set $f(B)$ is $[n/(k + \alpha)]$-null.*

**Remark 2.1.** If $[n/(k + \alpha)] \leqq 1$, then the Lebesgue measure of the set $f(B)$ is zero. If $s < [n/(k + \alpha)]$, then we can construct a function $f \in C^{k,\alpha}(\Omega)$ such that the set $f(B)$ is not $s$-null (see [8]). If $f \in C^{\infty}(\Omega)$, then the set $f(B)$ is $s$-null for each $s > 0$, but this set need not be countable. It is proved in [14], that in the case of real-analytic function $f$ (i.e., each point $w \in \Omega$ has an open neighborhood $U$ such that the function $f$ has a power series expansion in $U$), the set $f(B)$ is countable.

In the sequel we wish to give analogous assertion as in Proposition 2.1 for functionals in infinite-dimensional Banach spaces. As the counterexample of I. Kupka (see [9]) shows, in the whole generality such assertion is not true. I. Kupka constructed the functional $f \in C^{\infty}$ on the separable Hilbert space such that the set $f(B)$ has nonzero Lebesgue measure. S. I. Pochožajev in the paper [13] introduced the notion of "*Fredholm functional*" and he proved under some assumptions that the set $f(B)$ has a zero Lebesgue measure for $f \in C^k(\Omega)$. The analog of Morse-Sard theorem for real-analytic "*Fredholm functionals*" in infinite-dimensional Banach spaces and for functionals which derivative has a finite-dimensional range is given in the paper [4]. In this Section we give the proof of Morse-Sard theorem for "*Fredholm functionals*" $f \in C^{k,\alpha}(\Omega)$, $\Omega$ is an open subset in infinite-dimensional Banach space.

We recall that the linear operator $A$ defined on the Banach space $X$ with values in Banach space $Y$ is said to be *Fredholm operator* if the following conditions are fulfilled:

(i) $R = A(X)$ is a closed subspace of $Y$,
(ii) $Y/R$ has a finite dimension,
(iii) $Z = A^{-1}(0)$ is a finite-dimensional subspace of $X$.

Note that if $A = L + M$, where $L$ is an isomorphism of $X$ onto $Y$ and $M$ is linear completely continuous mapping of $X$ into $Y$, then $A$ is Fredholm operator (theorem due to L. Schwartz – see e.g. [5, Appendix B]).

**Definition 2.1.** Let $X$, $Y$ be two Banach spaces, $\Omega \subset X$ an open subset and $x_0 \in \Omega$. The mapping $F : \Omega \to Y$ is said to be *Fredholmian* at the point $x_0$ if $F$ has Fréchet derivative $F'(x_0)$ at the point $x_0$ and $F'(x_0)$ is a Fredholm operator. Denote by $N(F, x_0)$ the dimension of the space

$$\{h \in X : F'(x_0)(h) = 0\} .$$

The functional $f : \Omega \to E_1$ is said to be *Fredholm functional* at the point $x_0 \in \Omega$ if $f$ has Fréchet derivative $f'$ on some open neighborhood $U(x_0) \subset \Omega$ of the point $x_0$ and the mapping $f' : U(x_0) \to X^*$ is a Fredholmian operator at the point $x_0$. (From the definition of Fredholm functional $f$ follows that there exists $f''(x_0)$).

If $f : \Omega \to E_1$ is a Fredholm functional at $x_0 \in \Omega$ denote by $N(f, x_0)$ the dimension of the subspace

$$\{h \in X : f''(x_0)(h) = 0\}$$

(i.e., $N(f, x_0) = N(f', x_0)$).

The main theorem (Theorem 2.2) is not lucid at the first sight. This is the reason for the formulation of the following theorem, which is its special case. Theorem 2.2 is useful for the proof of Theorem 3.2, which is necessary for some more complicated applications (see the proof of Theorem 4.1 for $p > 2$).

If $\varphi$ is a given functional defined on $\Omega$, then we denote

$$B = \{y \in \Omega : \varphi'(y) = 0\}.$$

**Theorem 2.1.** *Let $\varphi$ be a functional defined on an open subset $\Omega$ of a Hilbert space $H$. Let $k \geqq 1$ be a positive integer, $\alpha \in \langle 0, 1 \rangle$. Suppose that $\varphi \in C^{k+1,\alpha}(\Omega)$, $y_0 \in B$ and $\varphi$ is Fredholm functional at the point $y_0$.*

*Then there exists a neighborhood $V(y_0) \subset \Omega$ of the point $y_0$ such that*

$$\varphi(B \cap V(y_0))$$

*is $[N(\varphi, y_0)/(k + \alpha)]$ – null.*

**Corollary 2.1.** *Suppose that $k \geqq 1$ is an integer, $\alpha \in \langle 0, 1 \rangle$ and $\varphi$ is a functional defined on an open subset $\Omega$ of a separable Hilbert space $H$. Let $\varphi \in C^{k+1,\alpha}(\Omega)$ and denote for positive integer $n$*

$$B_n = \{y \in B : \varphi \text{ is Fredholm functional at the point } y,\ N(\varphi, y) \leqq n\}.$$

*Then the set $\varphi(B_n)$ is $[n/(k + \alpha)]$-null.*

Proof. Assume that Theorem 2.1 is proved. For each $y_0 \in B_n$ let $V(y_0)$ be an open neighborhood from the assertion of Theorem 2.1. The system $\{V(y_0)\}_{y_0 \in B_n}$ forms an open covering of the set $B_n$. Therefore we can select a countable covering $\{V(y_i)\}_{i=1}^{\infty}$, for the space $H$ is separable. Since the sets $\varphi(B \cap V(y_i))$ $(i = 1, 2, \ldots)$ are $[n/(k + \alpha)]$-null, the assertion follows from $\varphi(B_n) \subset \bigcup_{i=1}^{\infty} \varphi(B \cap V(y_i))$.

**Corollary 2.2.** *Let the assumptions of Corollary 2.1 be fulfilled. Suppose $\varphi \in C^{\infty}(\Omega)$ and denote $B_F = \bigcup_{n=1}^{\infty} B_n$.*

*Then the set $\varphi(B_F)$ is $s$-null for each $s > 0$.*

(This follows immediately from corollary 2.1.)

We shall consider two Banach spaces $Y_1, Y_2$ satisfying the following condition (Y): there exists a bilinear form $\langle .,. \rangle$ on $Y_1 \times Y_2$ such that $\langle .,. \rangle$ is continuous on $Y_2$ for each fixed $y_1 \in Y_1$ and if $y_2 \in Y_2$, $\langle y, y_2 \rangle = 0$ for each $y \in Y_1$, then $y_2 = 0$.

For example, the spaces $Y_1 = C_0^{2,\alpha}(\langle 0, 1\rangle)$ (the space of all functions from the class $C^{2,\alpha}(\langle 0, 1\rangle)$ which values in the points 0,1 are zero) and $Y_2 = C^{0,\alpha}(\langle 0, 1\rangle)$ satisfy the condition (Y) with the bilinear form

$$\langle u, v\rangle = \int_0^1 u(t)\, v(t)\, \mathrm{d}t\,.$$

**Theorem 2.2.** *Let $Y_1$, $Y_2$ be two Banach spaces satisfying condition* (Y), *$\Omega$ an open set in $Y_1$. Let $\varphi$ be a functional on $\Omega$, $\varphi \in C^{k,\alpha}(\Omega)$. Suppose for each $y \in \Omega$ there exists $\Phi(y) \in Y_2$ (under our assumptions there exists only one) such that*

$$(\Phi) \qquad \varphi'(y)\,(h) = \langle h, \Phi(y)\rangle$$

*for each $y \in \Omega$, $h \in Y_1$.*

*Let $k$ be a positive integer, $\alpha \in \langle 0, 1\rangle$ and $y_0 \in B$. Suppose that $\Phi \in C^{k,\alpha}(\Omega)$ and $\Phi$ is Fredholmian at the point $y_0$.*

*Then there exists a neighborhood $V(y_0) \subset \Omega$ of the point $y_0$ such that the set*

$$\varphi(B \cap V(y_0))$$

*is $[N(\Phi, y_0)/(k + \alpha)]$-null.*

**Remark 2.2.** Theorem 2.2 implies Theorem 2.1 by the setting $Y_1 = Y_2 = H$, $\langle .,.\rangle$ the inner product in $H$ and $\Phi = \varphi'$.

**Proof of Theorem 2.2.** Define $F = \Phi'(y_0)$ (i.e., $F$ is a linear mapping of $Y_1$ into $Y_2$). The subspace $R = F(Y_1)$ is closed and the space $Y_2/R$ is finite-dimensional (see Definition 2.1). Hence, there exists a projection $P_R$ of $Y_2$ onto $R$, i.e., a bounded linear mapping such that $P_R^2 = P_R$. Denote

$$Z_1 = \{y \in Y_1 : F(y) = 0\}\,.$$

The space $Z_1$ is finite-dimensional, $\dim Z_1 = N(\Phi, y_0)$, for the mapping $\Phi$ is Fredholmian at the point $y_0$. Thus, there exists a closed subspace $Z_2$ of $Y_1$ such that $Z_1 \oplus Z_2 = Y_1$. For each $y \in B$ it is $\Phi(y) = 0$ and thus

$$0 = \Phi(y) - \Phi(y_0) = F(y - y_0) + r(y)\,,$$

where

$$\lim_{y \to y_0} \frac{r(y)}{\|y - y_0\|_{Y_1}} = 0\,.$$

Hence,

$$0 = F(y - y_0) + P_R\, r(y)\,.$$

For $y \in Y_1$ we shall write $y = [z_1, z_2]$, where $z_i \in Z_i$. For $y \in \Omega \subset Y_1 = Z_1 \times Z_2$ define

$$A([z_1, z_2]) = A(y) = F(y - y_0) + P_R\, r(y)\,.$$

We have $A \in C^{k,\alpha}(\Omega)$ and

$$A'_{z_2}(z_1^0, z_2^0) = F\,, \quad ([z_1^0, z_2^0] = y_0)\,.$$

The linear operator $A'_{z_2}(z_1^0, z_2^0)$ is an isomorphism of $Z_2$ onto $R$ and therefore there exists $[A'_{z_2}(z_1^0, z_2^0)]^{-1}$. Implicit function theorem (see Proposition 1.1) implies that there exists a neighborhood $U(z_1^0) \subset Z_1$ of the point $z_1^0$, a neighborhood $U(z_2^0) \subset Z_2$ of the point $z_2^0$ (such that $[U(z_1^0) \times U(z_2^0)] \subset \Omega$) and unique mapping $\omega$ from $U(z_1^0)$ into $U(z_2^0)$ such that

$$A(z_1, \omega(z_1)) = 0 \tag{2.1}$$

for each $z_1 \in U(z_1^0)$.

Moreover, $\omega \in C^{k,\alpha}(U(z_1^0))$.

Define

$$\varphi_0(z_1) = \varphi([z_1, \omega(z_1)])$$

for $z_1 \in U(z_1^0)$ and

$$D = \{z_1 \in U(z_1^0) : \varphi_0'(z_1) = 0\}\,.$$

It is easy to see $\varphi_0 \in C^{k,\alpha}(U(z_1^0))$. If $[z_1, z_2] \in B \cap [U(z_1^0) \times U(z_2^0)]$, then we obtain from (2.1) that $z_2 = \omega(z_1)$ and thus

$$\varphi_0'(z_1) = \varphi'_{z_1}([z_1, z_2]) + \varphi'_{z_2}([z_1, z_2])\, \omega'(z_1) = 0\,.$$

Hence,

$$\varphi(B \cap [U(z_1^0) \times U(z_2^0)]) \subset \varphi_0(D \cap U(z_1^0))$$

and with respect to Proposition 2.1 there exists a neighborhood $U_0(z_1^0) \subset U(z_1^0)$ such that the set $\varphi_0(D \cap U_0(z_1^0))$ is $[N(\Phi, y_0)/(k + \alpha)]$-null. Thus, the set $\varphi(B \cap V(x_0))$ is $[N(\Phi, y_0)/(k + \alpha)]$-null, where $V(y_0) = U_0(z_1^0) \times U(z_2^0)$.

**Corollary 2.3.** *Let Banach spaces $Y_1, Y_2$ satisfy the condition (Y), let the space $Y_1$ be separable, let $\Omega$ be an open set in $Y_1$. Let $\varphi$ be a functional, $\varphi \in C^{k,\alpha}(\Omega)$. Suppose that $\Phi$ is a mapping of $\Omega$ into $Y_2$, $\Phi \in C^{k,\alpha}(\Omega)$, the condition ($\Phi$) is satisfied. Set*

$$B_n = \{y \in B : \Phi \text{ satisfies Fredholm condition in } y,\ N(\Phi, y) \leqq n\}\,.$$

*Then the set $\varphi(B_n)$ is $[n/(k + \alpha)]$-null.*

Proof. Analogously as Corollary 2.1 but by using Theorem 2.2.

**Corollary 2.4.** *Let the assumptions of Corollary* 2.3 *be fulfilled and let* $\varphi \in C^\infty(\Omega)$, $\Phi \in C^\infty(\Omega)$. *Set* $B_F = \bigcup_{n=1}^{\infty} B_n$.

*Then the set* $\varphi(B_F)$ *is s-null for each* $s > 0$.

(It follows immediately from Corollary 2.3.)

## 3. INVESTIGATION OF THE SET OF ALL CRITICAL LEVELS

Let $X$ be a Banach space, let $f, g$ be real functionals on $X$. For a given number $r > 0$ define

$$M_r(f) = \{x \in X : f(x) = r\}.$$

This Section deals with the eigenvalue problem

$$\lambda f'(x) = g'(x), \quad x \in M_r(f). \tag{3.1}$$

If $x_0 \in X$ is a solution of the problem (3.1) with a certain number $\lambda = \lambda_0$, then $x_0$ is said to be a *critical point of the functional g with respect to the manifold* $M_r(f)$ and the corresponding number $\lambda_0$ is said to be an *eigenvalue* of the problem (3.1), the number $g(x_0)$ is said to be a *critical level* of $g$. We shall denote the set of all critical levels by $\Gamma$ and the set of all critical points by $S$, i.e.,

$$S = \{x \in M_r(f): \textit{there exists } \lambda,\ \lambda f'(x) = g'(x)\}, \quad \Gamma = g(S).$$

**Remark 3.1.** Suppose that $f$ is $(a + 1)$-homogeneous, $g$ is $(b + 1)$-homogeneous with $a > 0$, $b > 0$ (i.e, $f(tx) = t^{a+1} f(x)$, $g(tx) = t^{b+1} g(x)$ for each $t > 0$, $x \in X$). It is easy to see that $f'$ is $a$-homogeneous, $g'$ is $b$-homogeneous (as the mappings of $X$ into $X^*$) and

$$f(x) = (a + 1)^{-1} (x, f'(x)), \quad g(x) = (b + 1)^{-1} (x, g'(x))$$

for each $x \in X$ (the brackets $(x, x^*)$ denote the value of the functional $x^* \in X^*$ at the point $x \in X$). Let $x_0$ be an arbitrary critical point of the functional $g$ with respect to the manifold $M_r(f)$, $\lambda_0$ a corresponding eigenvalues (i.e., (3.1) holds with $x = x_0$, $\lambda = \lambda_0$). Then we obtain (under assumption $(x_0, f'(x_0)) \neq 0$) that

$$\lambda_0 = \frac{(x_0, g'(x_0))}{(x_0, f'(x_0))} = \frac{b + 1}{a + 1} \frac{g(x_0)}{f(x_0)} = \frac{b + 1}{r(a + 1)} g(x_0).$$

Hence, if we obtain that there the set $\Gamma$ is $s$-null for some $s > 0$, then the same is true for the set of all eigenvalues.

The reason for the formulation of Theorem 3.1 is the same as in the case of Theorem 2.1. Theorem 3.1 is a special case of Theorem 3.2, but it can be proved also directly from Theorem 2.1. Theorem 3.1 gives a possibility to obtain information about the set of critical levels (or eigenvalues) in certain special applications (see the proof of Theorem 4.1 for the case $p = 2$). Theorem 3.2 is applicable in more general setting, namely, in the case of differential operators with higher growths (see the proof of Theorem 4.1 for $p > 2$).

**Theorem 3.1.** *Let $f, g$ be two functionals defined on a real Hilbert space $H$. Suppose $f, g \in C^{k+1,\alpha}(H)$ and let $x_0 \in S$ and let $\lambda_0$ be the corresponding eigenvalue.*

*Then under assumption $f'(x_0) \neq 0$ and $\lambda_0 f - g$ is a Fredholm functional at $x_0$ there exists a neighborhood $V(x_0)$ of the point $x_0$ such that the set $g(S \cap V(x_0))$ is $[(N(\lambda_0 f - g, x_0) + 1)/(k + \alpha)]$-null.*

**Corollary 3.1.** *Let $f, g$ be two functionals defined on $H$, $f, g \in C^{k+1,\alpha}(H)$. Suppose $f'(x) \neq 0$ for each $x \in S$ and denote by $S_n$ the set of all $y \in S$ such that the functional*

$$x \mapsto \frac{(y, g'(y))}{(y, f'(y))} f(x) - g(x)$$

*is a Fredholm functional at the point $y$ and*

$$N\left(\frac{(y, g'(y))}{(y, f'(y))} f - g, y\right) \leqq n .$$

*Then the set $g(S_n)$ is $[(n + 1)/(k + \alpha)]$-null.*

**Corollary 3.2.** *Suppose that the assumptions of Corollary 3.1 are fulfilled with $f, g \in C^\infty(H)$. Then the set $g(S_F)$ is s-null for each $s > 0$, where $S_F = \bigcup_{n=1}^{\infty} S_n$.*

**Theorem 3.2.** *Let $X, X_1, X_2$ be three real Banach spaces, $X_1 \subset X$. Suppose $X_1, X_2$ satisfy the condition (Y) (see Section 2). Let $f, g$ be functionals on $X$, $f, g \in C^1(X) \cap \cap C^{k+1,\alpha}(X_1)$. Suppose for each $x \in X_1$ there exist $F(x) \in X_2$, $G(x) \in X_2$ (under our assumptions there exist uniquely) such that*

$$(\mathrm{f}_1) \qquad f'(x)(h) = \langle h, F(x)\rangle , \quad g'(x)(h) = \langle h, G(x)\rangle$$

*for each $x, h \in X_1$.*

*Suppose $F, G \in C^{k,\alpha}(X_1)$. Let $x_0 \in S \cap X_1$ and let $\lambda_0$ be the corresponding eigenvalue. Assume that the mapping $\lambda_0 F - G : X_1 \to X_2$ is Fredholmian at the point $x_0$ and,*

$$(\mathrm{f}_2) \qquad \text{there exists } h_0 \in X_1 \text{ such that } f'(x_0)(h_0) \neq 0 .$$

*Then there exists a neighborhood $V(x_0) \subset X_1$ of $x_0$ such that the set $g(S \cap V(x_0))$ is $[(N(\lambda_0 F - G, x_0) + 1)/(k + \alpha)]$-null.*

**Remark 3.2.** Setting $X_1 = X_2 = H$, $\langle .,. \rangle$ the inner product in $H$ and $F = f'$, $G = g'$ we obtain that Theorem 3.2 implies Theorem 3.1.

**Proof of Theorem 3.2.** Denote

$$Y_1 = \{y \in X_1 : f'(x_0)(y) = 0\}.$$

Then $X_1 = Y_1 \oplus \{h_0\}$, hence for each $x \in X_1$ there exist $\xi \in E_1$ and $y \in Y_1$ such that $x = \xi h_0 + y$. Consider $\xi_0 \in E_1$, $y_0 \in Y_1$ such that $x_0 = \xi_0 h_0 + y_0$. Define $\bar{f}(\xi, y) = f(\xi h_0 + y)$.

Then $\bar{f}$ is a functional defined on $E_1 \times Y_1$, $\bar{f} \in C^{k+1,\alpha}(E_1 \times Y_1)$,

$$\partial_\xi \bar{f}(\xi_0, y_0) = f'(x_0)(h_0) \neq 0$$

and

$$\bar{f}(\xi, y) = r,$$

for $(\xi h_0 + y) \in M_r(f)$.

Implicit function theorem (see Proposition 1.1) implies there exist neighborhoods $U(\xi_0) \subset E_1$ (of the point $\xi_0$), $U(y_0) \subset Y_1$ (of the point $y_0$) and only one mapping $\eta$ which maps $U(y_0)$ into $U(\xi_0)$ and such that

$$\bar{f}(\eta(y), y) = r$$

for each $y \in U(y_0)$.

Moreover, $\eta \in C^{k+1,\alpha}(U(y_0))$. Define

$$\varphi(y) = g(\eta(y)\, h_0 + y)$$

for $y \in U(y_0)$.

For $y \in U(y_0)$, $v \in Y_1$ we have

$$\eta'(y)(v) = -\frac{\partial_y \bar{f}(\eta(y), y)(v)}{\partial_\xi \bar{f}(\eta(y), y)(1)} = -\frac{f'(\eta(y)\, h_0 + y)(v)}{f'(\eta(y)\, h_0 + y)(h_0)} \tag{3.2}$$

(see the proof of Proposition 1.1). From here

$$\varphi'(y)(v) = -g'(\eta(y)\, h_0 + y)(h_0)\frac{f'(\eta(y)\, h_0 + y)(v)}{f'(\eta(y)\, h_0 + y)(h_0)} + \tag{3.3}$$

$$+ g'(\eta(y)\, h_0 + y)(v).$$

Denote

$$V(x_0) = \{x \in X_1 : x = \xi h_0 + y,\ \xi \in U(\xi_0),\ y \in U(y_0)\},$$
$$B = \{y \in U(y_0) : \varphi'(y) = 0\}.$$

From (3.3) we obtain: if $x \in S \cap V(x_0)$, then $y \in B$. Hence, $g(S \cap V(x_0)) \subset \varphi(B)$. It is easy to see that it is sufficient to prove there exists a neighborhood $U_0(y_0) \subset U(y_0)$ of the point $y_0$ such that the set $\varphi(B \cap U_0(y_0))$ is $[(N(\lambda_0 F - G, x_0) + 1) : (k + \alpha)]$-null.

We shall prove that the functional $\varphi$ satisfies the assumptions of Theorem 2.2. Define

$$Y_2 = \{y \in X_2 : \langle h_0, y \rangle = 0\} .$$

It is easy to see the spaces $Y_1, Y_2$ satisfy the condition $(Y)$ with the restriction of the form $\langle .,. \rangle$ on $Y_1 \times Y_2$. Define

$$\Phi(y) = -\frac{\langle h_0, G(\eta(y)\, h_0 + y)\rangle}{\langle h_0, F(\eta(y)\, h_0 + y)\rangle} F(\eta(y)\, h_0 + y) + G(\eta(y)\, h_0 + y) \tag{3.4}$$

for $y \in U(y_0)$.

Obviously, $\Phi$ maps $U(y_0)$ into $Y_2$ and, $\Phi \in C^{k,\alpha}(U(y_0))$. From (3.3), (3.4) and the assumption $(f_1)$ the validity of the assumption $(\Phi)$ in Theorem 2.2 follows. Now, we shall show that $\Phi$ is Fredholmian at the point $y_0$.

By calculation we obtain

$$\begin{aligned}\Phi'(y_0)(v) = &-\lambda_0 F'(x_0)(v) + G'(x_0)(v) - \\ &- \frac{\langle h_0, -\lambda_0 F'(x_0)(v) + G'(x_0)(v)\rangle}{\langle h_0, F(y_0)\rangle} F(y_0)\end{aligned} \tag{3.5}$$

for each $v \in Y_1$.

Denote

$$\begin{aligned} M &= \{v \in Y_1 : \Phi'(y_0)(v) = 0\} , \\ K &= \{v \in X_1 : \lambda_0 F'(x_0)(v) - G'(x_0)(v) = 0\} . \end{aligned}$$

If $v \in M$ and at the same time

$$\langle h_0, \lambda_0 F'(x_0)(v) - G'(x_0)(v)\rangle = 0 , \tag{3.6}$$

then clearly (from (3.5)) it is $v \in K$.

Thus, if the relation (3.6) holds for each $v \in M$, then $M \subset K$. In the oposite case we can write $M = M_1 \oplus \{v_0\}$, where

$$\langle h_0, \lambda_0 F'(x_0)(v_0) - G'(x_0)(v_0)\rangle \neq 0$$

and

$$\langle h_0, \lambda_0 F'(x_0)(v) - G'(x_0)(v)\rangle = 0$$

for all $v \in M_1$. Now we obtain as the above, that $M_1 \subset K$, hence

$$M \subset K \oplus \{v_0\} .$$

In all cases, we have

$$\dim M \leqq \dim K + 1 ,$$

i.e.,

$$N(\Phi, y_0) \leqq N(\lambda_0 F - G, x_0) + 1 .$$

Further, the range $R = (\lambda_0 F'(x_0) - G'(x_0))(X_1)$ is a closed subspace of $X_2$ of finite codimension, the same is true also for the subspace

$$R' = (\lambda_0 F'(x_0) - G'(x_0))(Y_1)$$

of the space $Y_2$, for:

(1) if $\lambda_0 F'(x_0)(h_0) - G'(x_0)(h_0) \in R'$, then clearly $R = R'$;

(2) if $\lambda_0 F'(x_0)(h_0) - G'(x_0)(h_0) \notin R'$, then we know that $(\lambda_0 F'(x_0) - G'(x_0))$ maps $X_1$ onto $R$ and $X_1 = Y_1 \oplus \{h_0\}$, where $Y_1$ is the closed subspace of $X_1$. Now it follows immediately from Banach open mapping theorem that

$$R' = (\lambda_0 F'(x_0) - G'(x_0))(Y_1)$$

is also a closed subspace of $X_2$.

Since

$$R = R' \oplus \{\lambda_0 F'(x_0)(h_0) - G'(x_0)(h_0)\}$$

it is clear that $R'$ has a finite codimension. Now, if we define the projection $P : X_2 \to$ $\to Y_2$ by

$$P : x \mapsto x - \frac{\langle h_0, x\rangle}{\langle h_0, F(x_0)\rangle} F(x_0),$$

then clearly

$$\Phi'(y_0)(Y_1) = P(R')$$

and such projection of closed subspace of finite codimension is again closed subspace of finite codimension.

Hence, the assumptions of Theorem 2.2 are verfied and thus there exists a neighborhood $U_0(y_0) \subset U(y_0)$ of the point $y_0$ such that the set $\varphi(B \cap U_0(y_0))$ is $[(N(\lambda_0 F - G, x_0) + 1)/(k + \alpha)]$-null.

Therefore the set $g(S \cap V_0(x_0))$ is $[(N(\lambda_0 F - G, x_0) + 1)/(k + \alpha)]$-null, where

$$V_0(x_0) = \{x \in X_1 : x = \xi h_0 + y,\ \xi \in U(\xi_0),\ y \in U_0(y_0)\}.$$

**Corollary 3.3.** *Let the assumptions of Theorem 3.2 be fulfilled with $X_1$ separable. Moreover, suppose that for each $x \in X_1$ there exists $h \in X_1$ such that $f'(x)(h) \neq 0$ and let for $y \in S \cap X_1$ be $(y, f'(y)) \neq 0$. Denote by $S_n$ the set of all $y \in S \cap X_1$ such that the mapping*

$$x \mapsto \frac{(y, g'(y))}{(y, f'(y))} F(x) - G(x)$$

*is Fredholmian at the point $y$ and*

$$N\left(\frac{(y, g'(y))}{(y, f'(y))} F - G, y\right) \leqq n.$$

*Then the set $g(S_n)$ is $[(n + 1)/(k + \alpha)]$-null.*

(The proof is similar to that of Corollary 2.1.)

**Corollary 3.4.** *Let the assumptions of Corollary 3.3 be fulfilled with $F, G \in C^\infty(X_1)$, $f, g \in C^\infty(X_1) \cap C^1(X)$.*

*Then the set $g(S_F)$ is s-null for each $s > 0$, where*

$$S_F = \bigcup_{n=1}^{\infty} S_n .$$

(This Corollary follows immediately from Corollary 3.3.)

**Corollary 3.5.** *Suppose the assumptions of Corollary 3.3 (3.4, respectively) are satisfied. Let $f$ be $(a + 1)$-homogeneous and $g$ be $(b + 1)$-homogeneous $(a, b > 0)$. Denote by $\Lambda_n$ ($\Lambda_F$, respectively) the set of all eigenvalues corresponding to the set $S_n$ ($S_F$, respectively).*

*Then the set $\Lambda_n$ is $[(n + 1)/(k + \alpha)]$-null (the set $\Lambda_F$ is s-null for each $s > 0$, respectively).*

(This follows from Corollary 3.3 (3.4, respectively) and from Remark 3.1.)

## 4. APPLICATION TO THE BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

Let $m$ be a positive integer, $p$ a real number, $p \geqq 2$. Denote by $W_p^m(\langle 0, 1\rangle)$ the well-known Sobolev space with the norm

$$\|u\|_{p,m} = \left(\sum_{i=0}^{m} \int_0^1 |u^{(i)}(x)|^p \, dx\right)^{1/p},$$

i.e., $W_p^m(\langle 0, 1\rangle)$ is the space of all functions $u$ with the absolute continuous derivatives $u^{(i)}$ on the interval $\langle 0, 1\rangle$ $(i = 0, 1, \ldots, m - 1)$ and such that for the derivative of the order $m$ (which exists almost everywhere on $\langle 0, 1\rangle$) it is

$$\int_0^1 |u^{(m)}(x)|^p \, dx < \infty .$$

If $\zeta = [\zeta_0, \zeta_1, \ldots, \zeta_m] \in E_{m+1}$, then we shall denote $\eta = [\zeta_0, \zeta_1, \ldots, \zeta_{m-1}] \in E_m$. For each $u \in W_p^m(\langle 0, 1\rangle)$ define

$$\zeta(u) = [u, u^{(1)}, \ldots, u^{(m)}] \in [L_p]^{m+1} ,$$
$$\eta(u) = [u, u^{(1)}, \ldots, u^{(m-1)}] \in [L_p]^m .$$

Set

$$\mathring{W}_p^m(\langle 0, 1\rangle) = \{u \in W_p^m(\langle 0, 1\rangle) : u(0) = u(1) = \ldots = u^{(m-1)}(0) = u^{(m-1)}(1) = 0\} .$$

Further, let $V$ be a subspace of $W_p^m(\langle 0, 1\rangle)$ which is determined by the conditions

$$\sum_{i=0}^{m-1} c_{ij}^0 \, u^{(i)}(0) = 0 \,, \quad j = 1, \ldots, r \,, \tag{4.1a}$$

$$\sum_{i=0}^{m-1} c_{ij}^1 \, u^{(i)}(1) = 0 \,, \quad j = 1, \ldots, s \,, \tag{4.1b}$$

where $r$, $s$ are given numbers, $0 \leqq r \leqq m$, $0 \leqq s \leqq m$ and the rank of the matrix $(c_{ij}^0)$ is $r$, the rank of the matrix $(c_{ij}^1)$ is $s$. (If $r = 0$, then no condition (4.1a) is prescribed.) Obviously,

$$\mathring{W}_p^m(\langle 0, 1\rangle) \subset V \subset W_p^m(\langle 0, 1\rangle) \,.$$

Let us consider two real functions

$$A(x, \zeta_0, \ldots, \zeta_m) \in C^2(\langle 0, 1\rangle \times E_{m+1}) \,,$$

$$B(x, \eta_0, \ldots, \eta_{m-1}) \in C^2(\langle 0, 1\rangle \times E_m) \,.$$

Suppose that the following growth conditions hold for each $\zeta \in E_{m+1}$, $x \in \langle 0, 1\rangle$ ($\mu$ is a positive function defined on $E_m$):

$$\left|\frac{\partial A}{\partial \zeta_i}(x, \zeta)\right| \leqq \mu(\eta)\,(1 + |\zeta_m|)^p \,, \quad i = 0, 1, \ldots, m - 1 \,; \tag{4.2a}$$

$$\left|\frac{\partial A}{\partial \zeta_m}(x, \zeta)\right| \leqq \mu(\eta)\,(1 + |\zeta_m|)^{p-1} \,; \tag{4.2b}$$

$$\left|\frac{\partial^2 A}{\partial \zeta_i \, \partial \zeta_j}(x, \zeta)\right| \leqq \mu(\eta)\,(1 + |\zeta_m|)^p \,, \quad i, j = 0, 1, \ldots, m - 1 \,; \tag{4.2c}$$

$$\left|\frac{\partial^2 A}{\partial \zeta_m \, \partial \zeta_j}(x, \zeta)\right| \leqq \mu(\eta)\,(1 + |\zeta_m|)^{p-1} \,, \quad j = 0, 1, \ldots, m - 1 \,; \tag{4.2d}$$

$$\left|\frac{\partial^2 A}{\partial \zeta_m^2}(x, \zeta)\right| \leqq \mu(\eta)\,(1 + |\zeta_m|)^{p-2} \,. \tag{4.2e}$$

Assume there exist $c_1 > 0$, $c_2 \geqq 0$ and in the case $V \neq \mathring{W}_p^m(\langle 0, 1\rangle)$ also $c_2 > 0$ such that for each $\zeta, \zeta^0 \in E_{m+1}$, $x \in \langle 0, 1\rangle$

$$\sum_{i,j=0}^{m} \frac{\partial^2 A}{\partial \zeta_i \, \partial \zeta_j}(x, \zeta^0)\, \zeta_i \zeta_j \geqq c_1 |\zeta_m|^2 + c_2 |\eta|^2 \,, \tag{4.3}$$

where $|\cdot|$ denotes the norm in $E_m$ and the absolute value in $E_1$.

Let us consider functions $H_0, H_1, N_0, N_1 \in C^2(E_m)$ such that

$$(4.4) \qquad \sum_{i,j=0}^{m-1} \frac{\partial^2 H_k}{\partial \eta_i \, \partial \eta_j} (\eta^0) \, \eta_i \eta_j \geqq 0 \quad (k = 0, 1)$$

for each $\eta^0, \eta \in E_m$.

Now, we define two functionals $f, g$ on $V$:

$$(4.5) \qquad f(u) = \int_0^1 A(x, \zeta(u)(x)) \, dx + H_0(\eta(u)(0)) + H_1(\eta(u)(1)) ,$$

$$g(u) = \int_0^1 B(x, \eta(u)(x)) \, dx + N_0(\eta(u)(0)) + N_1(\eta(u)(1)) .$$

We shall consider the eigenvalue problem

$$(4.6) \qquad \lambda f'(u) = g'(u) , \quad u \in M_r(f) = \{u \in V : f(u) = r\} ,$$

where $r > 0$ is a prescribed number. An element $u \in V$ is a solution of the problem (4.6) if $f(u) = r$ and

$$(4.7) \qquad \lambda \int_0^1 \sum_{j=0}^{m} \frac{\partial A}{\partial \zeta_j} (x, \zeta(u)(x)) \, h^{(j)}(x) \, dx +$$

$$+ \lambda \sum_{j=0}^{m-1} \left[ \frac{\partial H_0}{\partial \eta_j} (\eta(u)(0)) \, h^{(j)}(0) + \frac{\partial H_1}{\partial \eta_j} (\eta(u)(1)) \, h^{(j)}(1) \right] -$$

$$- \int_0^1 \sum_{j=0}^{m-1} \frac{\partial B}{\partial \eta_j} (x, \eta(u)(x)) \, h^{(j)}(x) \, dx - \sum_{j=0}^{m-1} \left[ \frac{\partial N_0}{\partial \eta_j} (\eta(u)(0)) \, h^{(j)}(0) + \right.$$

$$\left. + \frac{\partial N_1}{\partial \eta_j} (\eta(u)(1)) \, h^{(j)}(1) \right] = 0$$

for each $h \in V$.

**Lemma 4.1.** *Let the conditions* (4.2a, b) *and* (4.3) *be fulfilled. If $u \in V$ is a solution of the problem* (4.6) *with $\lambda \neq 0$, then $u \in C^m(\langle 0, 1 \rangle)$.*

Proof. The equation (4.7) holds for each $h \in V$. If $h \in \mathring{W}_p^m(\langle 0, 1 \rangle) \subset V$, then (4.7) can be written as follows:

$$(4.8) \quad \int_0^1 \left\{ \lambda \frac{\partial A}{\partial \zeta_m} (x, \zeta(u)(x)) + \sum_{j=0}^{m-1} (-1)^{m-j} \int_0^x \frac{(x - t)^{m-j-1}}{(m - j - 1)!} \left[ \lambda \frac{\partial A}{\partial \zeta_j} (t, \zeta(u)(t)) - \right.\right.$$

$$\left.\left. - \frac{\partial B}{\partial \eta_j} (t, \eta(u)(t)) \right] dt \right\} h^{(m)}(x) \, dx = 0 .$$

Hence, for each $h \in \mathring{W}_p^m(\langle 0, 1 \rangle)$ we have the equation of the type

$$\int_0^1 R(x)\, h^{(m)}(x)\, \mathrm{d}x = 0\,, \tag{4.9}$$

where $R$ is a function of the class $L_{p^*}(\langle 0, 1 \rangle)$, $1/p + 1/p^* = 1$ (this follows from the growth conditions (4.2a, b)). Let us show that the following assertion (∗) holds: if $R \in L_{p^*}(\langle 0, 1 \rangle)$ and (4.9) holds for each $h \in \mathring{W}_p^m(\langle 0, 1 \rangle)$, then there exist constants $a_0, \ldots, a_{m-1}$ such that

$$R(x) = a_0 + a_1 x + \ldots + a_{m-1} x^{m-1}\,.$$

For the proof of the assertion (∗) denote by $a_0, a_1, \ldots, a_{m-1}$ such constants that

$$\int_0^1 (R(x) + a_0 + a_1 x + \ldots + a_{m-1} x^{m-1})\, x^j\, \mathrm{d}x = 0$$

for each $j = 0, 1, \ldots, m - 1$.

The last relation implies

$$\int_0^1 (R(x) + a_0 + a_1 x + \ldots + a_{m-1} x^{m-1})\, h^{(m)}(x)\, \mathrm{d}x = 0$$

for each $h \in \mathring{W}_p^m(\langle 0, 1 \rangle)$. Suppose $f \in L_p(\langle 0, 1 \rangle)$ and set

$$h(x) = \int_0^x \frac{(x - t)^{m-1}}{(m - 1)!} (f(t) + b_0 + b_1 t + \ldots + b_{m-1} t^{m-1})\, \mathrm{d}t\,,$$

where $b_j$ $(j = 0, \ldots, m - 1)$ are choosen such that $h \in \mathring{W}_p^m(\langle 0, 1 \rangle)$. Substituting the function $h$ into (4.9) we have

$$0 = \int_0^1 (R(x) + a_0 + \ldots + a_{m-1} x^{m-1})(f(x) + b_0 + \ldots + b_{m-1} x^{m-1})\, \mathrm{d}x =$$
$$= \int_0^1 (R(x) + a_0 + \ldots + a_{m-1} x^{m-1}) f(x)\, \mathrm{d}x\,.$$

Thus $R(x) + a_0 + \ldots + a_{m-1} x^{m-1} = 0$, for the function $f \in L_p(\langle 0, 1 \rangle)$ was arbitrary. Hence, the assertion (∗) is proved.

In our case we have

$$F(x, \zeta(u)(x)) = \frac{\partial A}{\partial \zeta_m}(x, \zeta(u)(x)) = \tag{4.10}$$
$$= \frac{1}{\lambda} \left( \sum_{j=0}^{m-1} (-1)^{m-j} \int_0^x \frac{(x - t)^{m-j-1}}{(m - j - 1)!} \left[ \lambda \frac{\partial A}{\partial \zeta_j}(t, \zeta(u)(t)) - \frac{\partial B}{\partial \eta_j}(t, \eta(u)(t)) \right] \mathrm{d}t + \right.$$
$$\left. + a_0 + \ldots + a_{m-1} x^{m-1} \right) = g(x) \in C(\langle 0, 1 \rangle)\,.$$

Since

$$\frac{\partial F}{\partial \zeta_m}(x, \eta, \zeta_m) > 0$$

for each $x \in \langle 0, 1 \rangle$ and all $[\eta, \zeta_m] \in E_{m+1}$ (see (4.3)), there exists on some neighborhood $U$ of the point $[x_0, \eta(u)(x_0)]$ only one function (according to Implicit function theorem) $\zeta_m(x, \eta)$ such that

$$F(x, \eta, \zeta_m(x, \eta)) = g(x)$$

for each $[x, \eta] \in U$. Moreover, $\zeta_m$ is continuous on $U$. For sufficiently small $|x - x_0|$ it is $[x, \eta(u)(x)] \in U$ and

$$F(x, \eta(u)(x), \zeta_m(x, \eta(u)(x))) = g(x)$$

and $\zeta_m(x, \eta(u)(x))$ is continuous, for $\eta(u)(x)$ is continuous. From (4.10) follows that $u(x)$ is a solution of the equation

$$F(x, \zeta(u)(x)) = g(x),$$

too, and the uniqueness of the implicit function implies

$$u^{(m)}(x) = \zeta_m(x, \eta(u)(x))$$

and thus $u^{(m)}$ is continuous on some neighborhood of arbitrary point $x_0 \in \langle 0, 1 \rangle$, which proves our lemma.

**Lemma 4.2.** *Let the conditions* (4.2) *be fulfilled. Let* $u_0 \in V$, $\lambda \neq 0$ *and*

$$D = \{v \in V : \lambda f''(u_0)(v, h) = g''(u_0)(v, h) \text{ for each } h \in V\}.$$

*Then* $\dim D \leqq m$.

Proof. Let $v \in D$ and $h \in V$. Then

$$0 = \lambda f''(u_0)(v, h) - g''(u_0)(v, h) = \lambda \int_0^1 \sum_{i,j=0}^{m} \frac{\partial^2 A}{\partial \zeta_i \, \partial \zeta_j}(x, \zeta(u_0)(x)) v^{(i)}(x) \,.$$

$$. \, h^{(j)}(x) \, dx + \lambda \sum_{i,j=0}^{m-1} \left[ \frac{\partial^2 H_0}{\partial \eta_i \, \partial \eta_j}(\eta(u_0)(0)) \, v^{(i)}(0) \, h^{(j)}(0) + \right.$$

$$\left. + \frac{\partial^2 H_1}{\partial \eta_i \, \partial \eta_j}(\eta(u_0)(1)) \, v^{(i)}(1) \, h^{(j)}(1) \right] - \int_0^1 \sum_{i,j=0}^{m-1} \frac{\partial^2 B}{\partial \eta_i \, \partial \eta_j}(x, \eta(u_0)(x)) \, v^{(i)}(x) \, h^{(j)}(x) \, dx -$$

$$- \sum_{i,j=0}^{m-1} \left[ \frac{\partial^2 N_0}{\partial \eta_i \, \partial \eta_j}(\eta(u_0)(0)) \, v^{(i)}(0) \, h^{(j)}(0) + \frac{\partial^2 N_1}{\partial \eta_i \, \partial \eta_j}(\eta(u_0)(1)) \, v^{(i)}(1) \, h^{(j)}(1) \right]$$

(with respect to the conditions (4.2c–e)).

At first, let us consider $H_0 \equiv H_1 \equiv N_0 \equiv N_1 \equiv 0$. Set

$$V_1 = \{h \in V : h(1) = h'(1) = \ldots = h^{(m-1)}(1) = 0\} .$$

By using the formula

$$v^{(i)}(x) = \int_0^x \frac{(x-t)^{m-i-1}}{(m-i-1)!} v^{(m)}(t)\,\mathrm{d}t + v^{(i)}(0) + x v^{(i+1)}(0) + \ldots$$
$$\ldots + \frac{x^{m-i-1}}{(m-i-1)!} v^{(m-1)}(0)$$

and integration by parts we obtain for $v \in D$, $h \in V_1$

$$(4.11) \qquad 0 = \lambda f''(u_0)(v, h) - g''(u_0)(v, h) =$$
$$= \int_0^1 \Big\{ \lambda \frac{\partial^2 A}{\partial \zeta_m^2}(x, \zeta(u_0)(x))\, v^{(m)}(x) + \lambda \sum_{i=0}^{m-1} \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_m}(x, \zeta(u_0)(x)) \,.$$
$$\cdot \Big( \int_0^x \frac{(x-t)^{m-i-1}}{(m-i-1)!} v^{(m)}(t)\,\mathrm{d}t + P_i(v, x) \Big) + \lambda \sum_{j=0}^{m-1} (-1)^{m-j} \,.$$
$$\cdot \int_0^x \frac{(x-t)^{m-j-1}}{(m-j-1)!} \frac{\partial^2 A}{\partial \zeta_m\, \partial \zeta_j}(t, \zeta(u_0)(t))\, v^{(m)}(t)\,\mathrm{d}t + \sum_{i,j=0}^{m-1} (-1)^{m-j} \,.$$
$$\cdot \int_0^x \frac{(x-t)^{m-j-1}}{(m-j-1)!} \Big[ \lambda \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(t, \zeta(u_0)(t)) - \frac{\partial^2 B}{\partial \eta_i\, \partial \eta_j}(t, \eta(u_0)(t)) \Big] \,.$$
$$\cdot \int_0^t \frac{(t-\tau)^{m-i-1}}{(m-i-1)!} v^{(m)}(\tau)\,\mathrm{d}\tau + P_i(v, t))\,\mathrm{d}t \Big\}\, h^{(m)}(x)\,\mathrm{d}x \,,$$

where

$$P_i(v, x) = v^{(i)}(0) + x\, v^{(i+1)}(0) + \ldots + \frac{x^{m-i-1}}{(m-i-1)!} v^{(m-1)}(0) \,.$$

Analogously as the assertion (*) in the proof of Lemma 4.1 we can prove the following assertion (**): if $R \in L_{p^*}(\langle 0, 1 \rangle)$, $(1/p + 1/p^* = 1)$,

$$\int_0^1 R(x)\, h^{(m)}(x)\,\mathrm{d}x = 0$$

for each $h \in V_1$, then there exist constants $a_0, a_1, \ldots, a_{r-1}$ such that

$$R(x) = a_0 + a_1 x + \ldots + a_{r-1} x^{r-1} \,,$$

where $r$ is the integer from the condition (4.1a).

Thus, we have from (4.11)

$$\lambda \frac{\partial^2 A}{\partial \zeta_m^2}(x, \zeta(u_0)(x))\, v^{(m)}(x) + \lambda \sum_{i=0}^{m-1} \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_m}(x, \zeta(u_0)(x)) \,.$$

$$\cdot \left( \int_0^x \frac{(x-t)^{m-i-1}}{(m-i-1)!} v^{(m)}(t)\, dt \right) + \lambda \sum_{j=0}^{m-1} (-1)^{m-j} \int_0^x \frac{(x-t)^{m-j-1}}{(m-j-1)!} \,.$$

$$\frac{\partial^2 A}{\partial \zeta_m\, \partial \zeta_j}(t, \zeta(u_0)(t))\, v^{(m)}(t)\, dt + \sum_{i,j=0}^{m-1} (-1)^{m-j} \,.$$

$$\int_0^x \frac{(x-t)^{m-j-1}}{(m-j-1)!} \left[ \lambda \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(t, \zeta(u_0)(t)) - \frac{\partial^2 B}{\partial \eta_i\, \partial \eta_j}(t, \eta(u_0)(t)) \right] .$$

$$\cdot \left( \int_0^t \frac{(t-\tau)^{m-i-1}}{(m-i-1)!} v^{(m)}(\tau)\, d\tau \right) dt = a_0 + a_1 x + \ldots + a_{r-1} x^{r-1} -$$

$$- \lambda \sum_{i=0}^{m-1} \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_m}(x, \zeta(u_0)(x))\, P_i(v, x) - \sum_{i,j=0}^{m-1} (-1)^{m-j} \,.$$

$$\int_0^x \frac{(x-t)^{m-j-1}}{(m-j-1)!} \left[ \lambda \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(t, \zeta(u_0)(t)) - \frac{\partial^2 B}{\partial \eta_i\, \partial \eta_j}(t, \eta(u_0)(t)) \right] P_i(v, t)\, dt \,.$$

This can be written in the form

(4.12) $$v^{(m)}(x) + \int_0^x K(x, t)\, v^{(m)}(t)\, dt = a_0 + \ldots + a_{r-1} x^{r-1} + \sum_{i=0}^{m-1} v^{(i)}(0) f_i(x) \,,$$

where $K(x, t) \in C(\langle 0, 1\rangle \times \langle 0, 1\rangle)$, $f_i \in C(\langle 0, 1\rangle)$ $(i = 0, 1, \ldots, m-1)$.

Let us consider the mapping

(4.13) $$W : v \in V \mapsto v^{(m)}(x) + \int_0^x K(x, t)\, v^{(m)}(t)\, dt - \sum_{i=0}^{m-1} v^{(i)}(0) f_i(x) \,.$$

Because

$$v(x) = \int_0^x \frac{(x-t)^{m-1}}{(m-1)!} v^{(m)}(t)\, dt + P(x) \,,$$

where $P$ is a polynomial of the degree at most $(m-1)$, we obtain immediately from the fact that the Volterra's operator

$$w \longmapsto w(x) + \int_0^x K(x, t)\, w(t)\, dt$$

is continuously invertible in the space $C(\langle 0, 1\rangle)$, that the space $D$ of all solutions $v \in V$ of the equation (4.12) is finite-dimensional. So we can restrict the mapping $W$ to $D$. Thus we have

$$\dim D = \dim \operatorname{Ker} W + \dim \operatorname{Im} W \,.$$

Denote $w_i \in C(\langle 0, 1\rangle)$ such that

$$w_i(x) + \int_0^x K(x, t)\, w_i(t)\, dt = f_i(x)\,.$$

Then each $v \in \operatorname{Ker} W$ has the form

$$w(x) = \int_0^x \frac{(x - t)^{m-1}}{(m - 1)!} \left(\sum_{i=0}^{m-1} v^{(i)}(0) f_i(t)\right) dt + \sum_{i=0}^{m-1} \frac{v^{(i)}(0)}{i!} x^i \tag{4.14}$$

and with respect to condition (4.1a) we have

$$\dim \operatorname{Ker} W \leqq m - r\,.$$

Since $\dim \operatorname{Im} W = r$, we conclude

$$\dim D \leqq r + m - r = m\,.$$

This concludes the proof in the case $H_0 \equiv H_1 \equiv N_0 \equiv N_1 \equiv 0$.

Let us consider the general case. We can write

$$\frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j} (\eta(u_0)(0))\, v^{(i)}(0)\, h^{(j)}(0) + \frac{\partial^2 H_1}{\partial \eta_i\, \partial \eta_j} (\eta(u_0)(1))\, v^{(i)}(1)\, h^{(j)}(1) =$$
$$= - \frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j} (\eta(u_0)(0))\, v^{(i)}(0) \int_0^1 \frac{x^{m-j-1}}{(m - j - 1)!} h^{(m)}(x)\, dx$$

for each $h \in V_1$, $i, j = 0, 1, \ldots, m - 1$.

Hence, also in the case we can derived the equation of the type (4.12), where the functions

$$\frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j} (\eta(u_0)(0)) \frac{x^{m-j-1}}{(m - j - 1)!}$$

can be included in the functions $f_i(x)$. This completes the proof.

The following assumptions will be usefull for the main theorem of this Section:

$$\sum_{i=0}^{m-1} \frac{\partial B}{\partial \eta_i} (x, \eta)\, \eta_i > 0\,, \tag{4.15}$$

$$\sum_{i=0}^{m-1} \frac{\partial N_k}{\partial \eta_i} (\eta)\, \eta_i \geqq 0 \tag{4.16}$$

for each $x \in \langle 0, 1\rangle$, all $\eta \in E_m$ with $\eta \neq 0$ and $k = 0, 1$.

**Theorem 4.1.** *Let the conditions* (4.2)–(4.4), (4.15), (4.16) *be fulfilled. Suppose* $A \in C^{k+1,\alpha}(\langle 0, 1\rangle \times E_{m+1})$, $B \in C^{k+1,\alpha}(\langle 0, 1\rangle \times E_m)$, $H_0, H_1, N_0, N_1 \in C^{k+1,\alpha}(E_m)$ $(k \geqq 1, \alpha \in \langle 0, 1\rangle)$.

*Then the set of all critical levels of the problem* (4.6) (*where* $f$, $g$ *are defined by* (4.5)) *is* $[(m+1)/(k+\alpha)]$-*null.*

**Corollary 4.1.** *Let the assumptions of Theorem* 4.1 *be fulfilled, let* $a > 0$, $b > 0$. *Suppose that*

$$\begin{aligned} A(x, \tau\zeta) &= \tau^{a+1} A(x, \zeta), \\ H_j(\tau\eta) &= \tau^{a+1} H_j(\eta), \\ B(x, \tau\eta) &= \tau^{b+1} B(x, \eta), \\ N_j(\tau\eta) &= \tau^{b+1} N_j(\eta) \end{aligned}$$

*for* $x \in \langle 0, 1 \rangle$, $\zeta \in E_{m+1}$, $\eta \in E_m$, $\tau > 0$ *and* $j = 0, 1$.

*Then the set of all eigenvalues of the problem* (4.6) *is* $[(m+1)/(k+\alpha)]$-*null.* (This follows from Theorem 4.1 and Remark 3.1.)

**Remark 4.1.** The assumption $B \in C^{k+1,\alpha}(\langle 0, 1 \rangle \times E_m)$ implies $g \in C^{k+1,\alpha}(V)$. But it is not true that

$$A \in C^{k+1,\alpha}(\langle 0, 1 \rangle \times E_{m+1}) \Rightarrow f \in C^{k+1,\alpha}(V). \tag{4.17}$$

In general setting this is true for certain subspaces of $V$ of the smooth functions. The implication (4.17) holds under additional growth conditions on the derivatives of the function $A$ up to the order $(k+1)$.

**Proof of Theorem 4.1.** At first, let us show that this theorem in the case $p = 2$ and under assumption $f \in C^{k+1,\alpha}(V)$ follows easily from Theorem 3.1 and Corollary 3.1. In this case, $V$ is a Hilbert space with the inner product

$$(u, v)_{2,m} = \sum_{j=0}^{m} \int_0^1 u^{(j)}(x)\, v^{(j)}(x)\, dx .$$

We have $g \in C^{k+1,\alpha}(V)$ (see Remark 4.1) and

$$f''(u)(h, h) = \sum_{i,j=0}^{m} \int_0^1 \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(x, \zeta(u)(x))\, h^{(i)}(x)\, h^{(j)}(x)\, dx + \tag{4.18}$$

$$+ \sum_{i,j=0}^{m-1} \left[\frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j}(\eta(u)(0))\, h^{(i)}(0)\, h^{(j)}(0) + \frac{\partial^2 H_1}{\partial \eta_i\, \partial \eta_j}(\eta(u)(1))\, h^{(i)}(1)\, h^{(j)}(1)\right] \geqq c\|h\|_{2,m}^2$$

(see (4.3) and (4.4)), where $c > 0$. (We have in (4.3) $c_2 > 0$ in the case $V \neq \mathring{W}_p^m(\langle 0, 1 \rangle)$ and in the case $V = \mathring{W}_p^m(\langle 0, 1 \rangle)$ the norm $\|\cdot\|_{2,m}$ is equivalent with the norm defined by

$$\left(\int_0^1 |u^{(m)}(x)|^2\, dx\right)^{1/2}$$

only.) It follows from (4.18) that $f''(u)(h)$ (as a mapping of variable $h$ of $V$ into $V^* = V$) is an isomorphism of $V$ onto $V$. Further, we have

$$g''(u)(h, v) = \sum_{i,j=0}^{m-1} \int_0^1 \frac{\partial^2 B}{\partial \eta_i \, \partial \eta_j} (\eta(u)(x)) \, h^{(i)}(x) \, v^{(j)}(x) \, dx +$$

$$+ \sum_{i,j=0}^{m-1} \left[ \frac{\partial^2 N_0}{\partial \eta_i \, \partial \eta_j} (\eta(u)(0)) \, h^{(i)}(0) \, v^{(j)}(0) + \frac{\partial^2 N_1}{\partial \eta_i \, \partial \eta_j} (\eta(u)(1)) \, h^{(i)}(1) \, v^{(j)}(1) \right].$$

It is easy to see from here that for fixed $u \in V$ the mapping $g''(u)(h)$ of $V$ into $V$ is completely continuous. Properties of $f''$ and $g''$ imply that the functional $(\lambda f - g)$ is Fredholm in each point $u \in V$. Using Lemma 4.2, Theorem 3.1 and Corollary 3.1 we obtain our assertion.

Now, let us consider the more general case. We shall show the assumption of Theorem 3.2 are satisfied setting $X = V$, $X_1 = \{v \in C^m(\langle 0, 1 \rangle) : v$ satisfies (4.1a, b)$\}$. Further, we shall denote by $w = [w_0, w_1, \ldots, w_m]$ the elements of the space $[C(\langle 0, 1 \rangle)]^{m+1}$, the elements of $E_{2m}$ are denoted by $y = [y_0, \ldots, y_{2m-1}]$ and the elements of the space $[C(\langle 0, 1 \rangle)]^{m+1} \times E_{2m}$ are denoted by $[w, y]$, where $w \in [C(\langle 0, 1 \rangle)]^{m+1}$, $y \in E_{2m}$. Set

$$P = \{[w, y] \in [C(\langle 0, 1 \rangle)]^{m+1} \times E_{2m} : \sum_{i=0}^{m} \int_0^1 w_i(x) \, v^{(i)}(x) \, dx +$$

$$+ \sum_{i=0}^{m-1} (y_i \, v^{(i)}(0) + y_{m+i} \, v^{(i)}(1)) = 0 \quad \text{for each} \quad v \in X_1\}.$$

Set $X_2 = ([C(\langle 0, 1 \rangle)]^{m+1} \times E_{2m})/P$ with the usual norm of the factor space. If $[w, y] \in [C(\langle 0, 1 \rangle)]^{m+1} \times E_{2m}$, then we shall denote by $[\tilde{w}, \tilde{y}]$ an element of $X_2$ which is generated by $[w, y]$. For each $v \in X_1$, $[\tilde{w}, \tilde{y}] \in X_2$ define

$$\langle v, [\tilde{w}, \tilde{y}] \rangle = \sum_{i=0}^{m} \int_0^1 w_i(x) \, v^{(i)}(x) \, dx + \sum_{i=0}^{m-1} (y_i \, v^{(i)}(0) + y_{m+i} \, v^{(i)}(1)),$$

where $[w, y] \in [C(\langle 0, 1 \rangle)]^{m+1} \times E_{2m}$ is an element generatting the class $[\tilde{w}, \tilde{y}]$. It is easy to see that $X_1$, $X_2$ with the bilinear form $\langle .,. \rangle$ satisfy the condition $(Y)$. For each $u \in X_1$ define

$$F(u) = [\tilde{w}, \tilde{y}],$$

where

$$w_i(x) = \frac{\partial A}{\partial \zeta_i} (x, \zeta(u)(x)), \quad i = 0, \ldots, m;$$

$$y_i = \frac{\partial H_0}{\partial \eta_i} (\eta(u)(0)), \quad i = 0, \ldots, m-1;$$

$$y_{m+i} = \frac{\partial H_1}{\partial \eta_i} (\eta(u)(1)), \quad i = 0, \ldots, m-1.$$

Then $F$ is a mapping of $X_1$ into $X_2$ and for each $u, v, h \in X_1$ we have

$$\langle h, F(u)\rangle = \sum_{i=0}^{m} \int_0^1 \frac{\partial A}{\partial \zeta_i}(x, \zeta(u)(x))\, h^{(i)}(x)\, dx +$$
$$+ \sum_{i=0}^{m-1} \left( \frac{\partial H_0}{\partial \eta_i}(\eta(u)(0))\, h^{(i)}(0) + \frac{\partial H_1}{\partial \eta_i}(\eta(u)(1))\, h^{(i)}(1) \right),$$

$$(4.19) \qquad \langle h, F'(u)(v)\rangle = \sum_{i,j=0}^{m} \int_0^1 \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(x, \zeta(u)(x))\, v^{(i)}(x)\, h^{(j)}(x)\, dx +$$
$$+ \sum_{i,j=0}^{m-1} \left( \frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j}(\eta(u)(0))\, v^{(i)}(0)\, h^{(j)}(0) + \frac{\partial^2 H_1}{\partial \eta_i\, \partial \eta_j}(\eta(u)(1))\, v^{(j)}(1)\, h^{(i)}(1) \right) =$$
$$= f''(u)(h, v)\,.$$

Let us consider a fixed element $u_0 \in X_1$. Then $F'(u_0)(v)$ is a mapping from $X_1$ into $X_2$. Using (4.3), (4.4) it is

$$(4.20) \qquad \langle v, F'(u_0)(v)\rangle = \sum_{i,j=0}^{m} \int_0^1 \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(x, \zeta(u_0)(x))\, v^{(j)}(x)\, v^{(i)}(x)\, dx +$$
$$+ \sum_{i,j=0}^{m-1} \left( \frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j}(\eta(u_0)(0))\, v^{(j)}(0)\, v^{(i)}(0) + \frac{\partial^2 H_1}{\partial \eta_i\, \partial \eta_j}(\eta(u_0)(1))\, v^{(j)}(1)\, v^{(i)}(1) \right) \geqq$$
$$\geqq \int_0^1 \left( c_1 |v^{(m)}(x)|^2 + c_2 \sum_{i=0}^{m-1} |v^{(i)}(x)|^2 \right) dx\,,$$

where $c_1 > 0$, $c_2 \geqq 0$ and $c_2 > 0$ in the case $V \neq \mathring{W}_p^m(\langle 0, 1\rangle)$. Hence, if $F'(u_0)(v) = 0$, then $v = 0$. That means, the mapping $F'(u_0)(v)$ is one-to-one.

Let $[\tilde{w}, \tilde{y}] \in X_2$ be arbitrary. Let us show there exists $v \in X_1$ such that

$$F'(u_0)(v) = [\tilde{w}, \tilde{y}]\,.$$

This holds if and only if

$$(4.21) \qquad \int_0^1 \sum_{i,j=0}^{m} \frac{\partial^2 A}{\partial \zeta_i\, \partial \zeta_j}(x, \zeta(u_0)(x))\, v^{(j)}(x)\, h^{(i)}(x)\, dx +$$
$$+ \sum_{i,j=0}^{m-1} \left( \frac{\partial^2 H_0}{\partial \eta_i\, \partial \eta_j}(\eta(u_0)(0))\, v^{(j)}(0)\, h^{(i)}(0) + \frac{\partial^2 H_1}{\partial \eta_i\, \partial \eta_j}(\eta(u)(1))\, v^{(j)}(1)\, h^{(i)}(1) \right) =$$
$$= \sum_{i=0}^{m} \int_0^1 w_i(x)\, h^{(i)}(x)\, dx + \sum_{i=0}^{m-1} (y_i\, h^{(i)}(0) + y_{m+i}\, h^{(i)}(1))$$

for each $h \in X_1$.

Introduce a Hilbert space

$$V_2 = \{z \in W_2^m(\langle 0, 1\rangle) : z \text{ satisfies (4.1a, b)}\}$$

with the inner product $(.,.)_{2,m}$. Let us seek a function $v \in V_2$ such that the equation (4.21) holds for each $h \in V_2$. The right hand side in (4.21) can be considered as a linear functional on $V_2$, i.e., as an element of $V_2$. The left hand side in (4.21) can be considered as a bilinear form $((v, h))$ on $V_2$. By (4.18) we have

$$((v, v)) \geqq c\|v\|_{2,m},$$

where $c > 0$.

Thus, there exists $v$ satisfying (4.20) for each $h \in V_2$. Further, analogously as in the proof of Lemma 4.1 we can show $v \in X_1$ and

$$\|v\|_{X_1} \leqq c(u_0)\,\|[\tilde{w}, \tilde{y}]\|_{X_2}.$$

We have proved that the mapping $F'(u_0)(v)$ for each fixed $u_0 \in X_1$ is an isomorphism of $X_1$ onto $X_2$.

For $u \in X_1$ set

$$G(u) = [\tilde{w}, \tilde{y}] \in X_2,$$

where

$$w_i(x) = \frac{\partial B}{\partial \eta_i}(x, \eta(u)(x)),$$

$$y_i = \frac{\partial N_0}{\partial \eta_i}(\eta(u)(0)),$$

$$y_{m+i} = \frac{\partial N_1}{\partial \eta_i}(\eta(u)(1))$$

for $i = 0, \ldots, m - 1$ and $w_m(x) = 0$.

We have

$$\langle h, G(u)\rangle = \sum_{i=0}^{m-1} \int_0^1 \frac{\partial B}{\partial \eta_i}(x, \eta(u)(x))\,\mathrm{d}x + \sum_{i=0}^{m-1} \left(\frac{\partial N_0}{\partial \eta_i}(\eta(u)(0))\, h^{(i)}(0) + \right.$$

$$\left. + \frac{\partial N_1}{\partial \eta_i}(\eta(u)(1))\, h^{(i)}(1)\right) = g'(u)(h),$$

$$\langle h, G'(u)(v)\rangle = \int_0^1 \sum_{i,j=0}^{m-1} \frac{\partial^2 B}{\partial \eta_i \partial \eta_j}(x, \eta(u)(x))\, v^{(j)}(x)\, h^{(i)}(x)\,\mathrm{d}x + \tag{4.22}$$

$$+ \sum_{i,j=0}^{m-1} \left(\frac{\partial^2 N_0}{\partial \eta_i \partial \eta_j}(\eta(u)(0))\, v^{(j)}(0)\, h^{(i)}(0) + \frac{\partial^2 N_1}{\partial \eta_i \partial \eta_j}(\eta(u)(1))\, v^{(i)}(1)\, h^{(i)}(1)\right) = g''(u)(h, v)$$

for each $u, v, h \in X_1$.

It is easy to see that for each $u_0 \in X_1$ the mapping $G'(u_0)(v)$ of $X_1$ into $X_2$ is completely continuous. Suppose $\lambda_0 \neq 0$. Thus the mapping $\lambda_0 F - G$ is Fredholmian at arbitrary point $u_0 \in X_1$. Lemma 4.2 together with (4.19), (4.22) gives