Časopis pro pěstování matematiky, roč. 99 (1974), Praha

ON CUBES AND DICHOTOMIC TREES

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(Received December 20, 1972)

The notion of the *n-cube* Q_n (and other notions not defined here) can be found in BEHZAD and CHARTRAND [1] or in HARARY [2]. The *complete dichotomic tree* D_n can be defined as follows: if $n = 1$, then D_n is the complete bigraph $K(1, 2)$; if $n \geqq 2$, then D_n is the tree obtained from two disjoint copies T and T' of D_{n-1} and from a new vertex v in such a way that v is joined by one edge to the only vertex of degree 2 of T and by another edge to the analogous vertex of T'. Thus D_n has 2^n vertices of degree 1, one vertex of degree 2, and $2^n - 2$ vertices of degree 3. The vertex of degree 2 of D_n will be referred to as its root. HAVEL and LIEBL [3] have proved that if $n \geqq 2$, then D_n is a subgraph of Q_{n+2} but D_n is not a subgraph of Q_{n+1}. Obviously, D_1 is a subgraph of Q_2.

If $n \geqq 1$, then we denote by $\approx{D}_n$ the tree obtained from two disjoint copies of D_n in such a way that their roots are joined by an edge; this edge will be referred to as the axial edge of $\overset{\approx}{D}_n$. Obviously, $\overset{\approx}{D}_n$ has $2^{n+2} - 2$ vertices. Havel and Liebl [4] conjectured that $\overset{\approx}{D}_n$ is a subgraph of Q_{n+2}, for $n \geqq 1$. In the present paper, this conjecture will be verified.

We introduce the graphs $Q_n^\triangledown$ and Q_n' which are certain local modifications of Q_n. Let $n \geqq 2$; by $Q_n^\triangledown$ we denote the graph $Q_n + rt - s$, where r, s and t are such vertices of Q_n that rs and st are distinct edges of Q_n; by Q_n' we denote the graph $Q_n - u - v$, where u and v are such vertices of Q_n that uv is an edge of Q_n. The first two theorems which will be proved in the present paper are:

Theorem 1. *D_n is a spanning subgraph of $Q_{n+1}^\triangledown$, for $n \geqq 1$.*

Theorem 2. *$\overset{\approx}{D}_n$ is a spanning subgraph of Q_{n+2}', for $n \geqq 1$.*

Both theorems will be easily obtained from the following lemma. An edge of a tree T incident with an end-vertex of T will be referred to as an end-edge. Let $n \geqq 1$. By $\hat{D}_n$ or $\check{D}_n$ we denote the tree obtained from D_n by inserting two new vertices of degree 2 into the axial edge or into one end-edge, respectively. The path of $\hat{D}_n$ obtained from the axial edge of $\approx{D}_n$ is referred to as the axial path of $\hat{D}_n$.

Lemma. *$\hat{D}_n$ and $\check{D}_n$ are spanning subgraphs of Q_{n+2}, for $n \geqq 1$.*

Proof. Obviously, the graphs $\hat{D}_n$, $\check{D}_n$ and Q_{n+2} have the same number of vertices. Hence it is sufficient to prove that both $\hat{D}_n$ and $\check{D}_n$ are subgraphs of Q_{n+2}.

Let m be a positive integer. We shall say that a tree T is m-valued if each edge of T is assigned a positive integer not exceeding m. As follows from the work of Havel and Morávek [5], a tree T is a subgraph of Q_m if and only if T can be m-valued so that

(1) for each path P of T, there exists k such that precisely an odd number of edges belonging to P is assigned k.

(Cf. also Hlavička [6].)

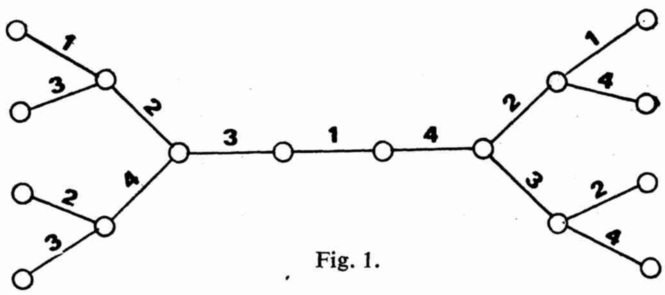

Fig. 1.

(A) We shall prove that $\hat{D}_n$ can be $(n + 2)$-valued so that (1) holds and that the edges of the axial path are assigned the integers 1, $n + 1$, and $n + 2$ (in some order). The case $n = 1$ is obvious. The case $n = 2$ is given in Fig. 1.

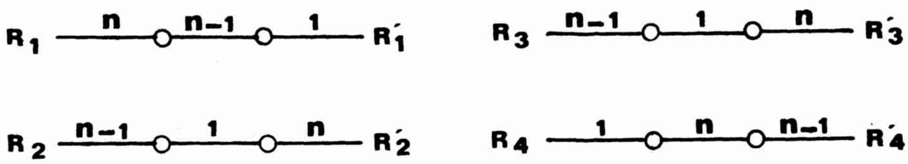

Fig. 2.

Let $n = m \geqq 3$. Assume that for $n = m - 2$, the statement is proved. Consider four disjoint copies of $\hat{D}_{n-2}$ which are n-valued so that (1) holds and that they can be expressed as in Fig. 2, where R_i and R_i' are n-valued copies of D_{n-2}. If we identify the root of each of the n-valued trees R_i and R_i' with the vertex r_i and r_i', respectively, in Fig. 3, we obtain an $(n + 2)$-valued tree $\hat{D}_n$. Obviously, the edges of the axial path are assigned 1, $n+1$, and $n+2$. It is routine to prove that this valuation fulfils (1).

(B) Let $n \geqq 1$; by D_n^* we denote the tree obtained from D_n by inserting two new vertices of degree 2 into one end-edge of D_n; the vertex of D_n^* obtained from the root of D_n will be referred to as the root of D_n^*. We shall prove that $\check{D}_n$ can be $(n+2)$-valued so that (1) holds. The case $n = 1$ is obvious. Let $n = m \geqq 2$. Assume that

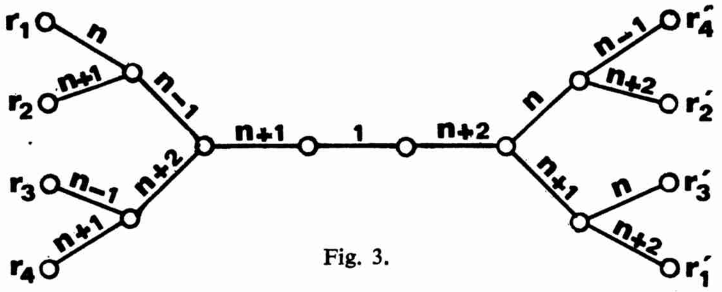

Fig. 3.

for $n = m - 1$, the statement is proved. Consider disjoint $\hat{D}_{n-1}$ and $\check{D}_{n-1}$ which are $(n+1)$-valued so that (1) holds and that they can be expressed as in Fig. 4, where T_1, T_1' and T_2 are $(n+1)$-valued copies of D_{n-1}, and T_2' is an $(n+1)$-valued copy of D_{n-1}^*. Join the root of T_2 by an edge assigned $n+2$ to the vertex t_2 and the root of T_2' by an edge assigned $n+2$ to the vertex t_2' (see Fig. 5). Thus we obtain $\check{D}_n$ which is $(n+2)$-valued such that (1) holds. Hence the lemma follows.

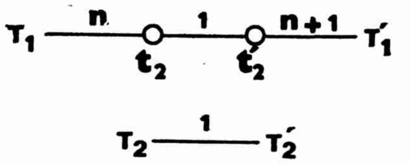

Fig. 4.

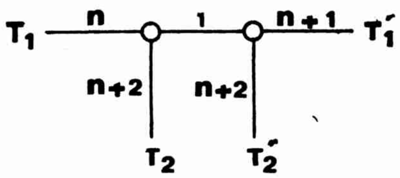

Fig. 5.

Proof of Theorem 1. The case $n = 1$ is obvious. Let $n \geqq 2$ and let t, u, v and w be such vertices of $\hat{D}_{n-1}$ that tu, uv and vw are the edges of the axial path. Then $D_n = \hat{D}_{n-1} + uw - v$. Thus the lemma implies the theorem.

Proof of Theorem 2 directly follows from the lemma.

Corollary. *$\overset{\approx}{D}_n$ is a subgraph of Q_{n+2}, for $n \geqq 1$.*

Let $n \geqq 2$. By $\tilde{D}_n$ we denote the tree obtained from disjoint D_{n-1} and D_n by joining their roots by an edge. As $\tilde{D}_n$ is a subgraph of $\overset{\approx}{D}_n$, it is also a subgraph of Q_{n+2}.