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BOUNDARY VALUE PROBLEMS FOR GENERALIZED LINEAR INTEGRODIFFERENTIAL EQUATIONS WITH LEFT-CONTINUOUS SOLUTIONS

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Introduction. In [11] the boundary value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}\left[A(t)\,x + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s) + f(t)\right],$$

$$\int_a^b [\mathrm{d}L(s)]\,x(s) = l \quad (-\infty < a < b < \infty)$$

was treated. In particular, theorems analogous to the well-known Fredholm theorems from the theory of integral equations were proved. The first of the above equations is a generalized ordinary differential equation in the sense of J. KURZWEIL [6]. An n-vector function $x(t)$ is said to be its solution on $[a, b]$ if for any $t \in [a, b]$

$$x(t) = x(a) + \int_a^t [\mathrm{d}A(s)]\,x(s) + \int_a^b [\mathrm{d}_s(G(t, s) - G(a, s))]\,x(s) + f(t) - f(a)\,.$$

The principal assumption was that of the regularity of $A(t)$ on $[a, b]$ (i.e. $A(a+) = A(a)$, $A(b-) = A(b)$ and $A(t+) + A(t-) = 2A(t)$ for all $a < t < b$). In this note the analogous investigation of the case of left-continuous $A(t)$ is carried out. Before treating the general problem we also generalize the results of [11] concerning the two-point problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A(t)\,x + f(t)]\,, \quad M\,x(a) + N\,x(b) = l\,.$$

For motivations of the study of such problems and for more detailed bibliography see [11]. Let us note furthermore that in [10] J. TAUFER developed numerical methods for solving some boundary value problems (taken from the technical practise) which are very close to those we are going to study here. Some of them (or at least some important special cases) even can be reduced to boundary value problems for generalized ordinary differential equations. For example, the following (interface) problem occurs in [10]. To find a vector function $x(t)$ piecewise absolutely continuous on $[a, b]$ such that

$$\dot{x} = A(t)\, x + f(t) \quad \text{a.e. on} \quad [a, b]\,,$$
$$x(t_i+) = W_i\, x(t_i-) + w_i \quad (i = 1, 2, \ldots, p)\,,$$
$$M\, x(a) + N\, x(b) = l\,.$$

(A and f are L-integrable on $[a, b]$, M, N and W_i are constant matrices, w_i and l are constant vectors, $a \leqq t_1 < t_2 < \ldots < t_p \leqq b$.) Putting

$$B(t) = \int_a^t A(s)\, \mathrm{d}s \quad \text{for} \quad a \leqq t \leqq t_1\,,$$

$$B(t) = \int_a^t A(s)\, \mathrm{d}s + (W_1 - I) \quad \text{for} \quad t_1 < t \leqq t_2\,, \ldots$$

$$\ldots, \ B(t) = \int_a^t A(s)\, \mathrm{d}s + (W_1 - I) + \ldots + (W_p - I) \quad \text{for} \quad t_p < t \leqq b$$

and

$$g(t) = \int_a^t f(s)\, \mathrm{d}s \quad \text{for} \quad a \leqq t \leqq t_1\,,$$

$$g(t) = \int_a^t f(s)\, \mathrm{d}s + w_1 \quad \text{for} \quad t_1 < t \leqq t_2, \ldots$$

$$\ldots, \ g(t) = \int_a^t f(s)\, \mathrm{d}s + w_1 + \ldots + w_p \quad \text{for} \quad t_p < t \leqq b\,,$$

we get an equivalent two-point boundary value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[B(t)\, x + g(t)]\,, \quad M\, x(a) + N\, x(b) = l\,.$$

Moreover, in [10] the transformation of problems with integral additional conditions to equivalent two-point boundary value problems was found independently of W. R. Jones [5].

1. Preliminaries. Let $-\infty < a < b < \infty$. The closed interval $a \leqq t \leqq b$ is denoted by $[a, b]$, its interior $a < t < b$ by (a, b) and the corresponding half-open intervals by $[a, b)$ and $(a, b]$. Given a matrix M, M' denotes its transpose, I is the identity matrix, O is the zero matrix. Vectors are generally considered as columns. Row vectors are denoted as transpositions of column vectors. The space of n-vector functions with bounded variation on $[a, b]$ is denoted by BV_n and the n-dimensional Euclidean space is denoted by R_n. Given a matrix function $F(t)$ of bounded variation on $[a, b]$, we put $F(a-) = F(a)$, $F(b+) = F(b)$, $\Delta^- F(t) = F(t) - F(t-)$, $\Delta^+ F(t) = F(t+) - F(t)$ and $\Delta F(t) = F(t+) - F(t-)$ for any $t \in [a, b]$. The space of all functions $F(t)$ of bounded variation on $[a, b]$ such that $F(t+) = F(t-) = F(a)$ for all $t \in [a, b]$ is denoted by N. Integrals are always the Perron-Stieltjes ones. For our purpose the notion of the σ-Young integral (Y-integral), which is on the space of functions with bounded variation equivalent to the Perron-Stieltjes integral, is fully sufficient. (For the definition and basic properties of the Y-integral see [4] II 19,3. An exhaustive survey was given in [12], too.)

Without any special quotation we shall make use of the fundamental results concerning generalized linear differential equations

$$\frac{dx}{d\tau} = D[A(t)\, x + f(t)]\,,$$

which were established by T. H. Hildebrandt [3] and Št. Schwabik [7]. (In particular, in [7] the case of left-continuous $A(t)$ was intensively studied. For some details see also [9]. A detailed survey of these matters was given also in [11] and [12].)

2. Two-point problem. The subject of this section is the two-point boundary value problem (p)

$$\frac{dx}{d\tau} = D[A(t)\, x + f(t)]\,, \tag{2,1}$$

$$M\, x(a) + N\, x(b) = l\,, \tag{2,2}$$

where $A(t)$ is an $n \times n$-matrix function of bounded variation on $[a, b]$, $f \in BV_n$, $l \in R_m$ and M, N are constant $m \times n$-matrices. We are seeking a function $x \in BV_n$ which is on $[a, b]$ a solution to the generalized linear differential equation (2,1) and fulfils the condition (2,2). (Let us notice that the equation (2,1) possesses only solutions with bounded variation on $[a, b]$, cf. [9].)

Any problem (p*) to find $y \in BV_n$ and $\lambda \in R_m$ such that $y'(t)$ is on $[a, b]$ a solution to the generalized linear differential equation

$$\frac{dy'}{d\tau} = D[-y'\, B(t)] \tag{2,3}$$

and

$$y'(a) = -\lambda' M\,, \quad y'(b) = \lambda' N \tag{2,4}$$

with $A - B \in N$ is said to be the adjoint of the problem (p).

Throughout the section we suppose that

$$A - B \in N\,, \quad \Delta^+ B(t)\, \Delta^+ A(t) = \Delta^- B(t)\, \Delta^- A(t) \quad \text{on} \quad [a, b] \tag{2,5}$$

and that at least one of the following conditions is satisfied

(2,6) $\det(I - \Delta^- A(t)) \det(I - \Delta^+ B(t)) \det(I + \Delta^+ A(t)) \neq 0$ on $[a, b]$,

(2,7) $\det(I - \Delta^- A(t)) \det(I - \Delta^+ B(t)) \det(I + \Delta^- B(t)) \neq 0$ on $[a, b]$,

(2,8) $\det(I + \Delta^+ A(t)) \det(I + \Delta^- B(t)) \det(I - \Delta^+ B(t)) \neq 0$ on $[a, b]$,

(2,9) $\det(I + \Delta^+ A(t)) \det(I + \Delta^- B(t)) \det(I - \Delta^- A(t)) \neq 0$ on $[a, b]$.

Without any loss of generality we may also assume $A(a) = B(a) = O$.

Let $U(t, s)$ and $V(t, s)$ denote the fundamental matrix solutions on $[a, b]$ to the equations (2,1) and (2,3), respectively. It means that

$$U(t, s) = I + \int_s^t [\mathrm{d}A(\tau)]\, U(\tau, s) \tag{2,10}$$

for $t, s \in [a, b]$, $t \geqq s$ if $\det(I - \Delta^- A(t)) \neq 0$ on $[a, b]$ and for $t \leqq s$ if $\det(I + \Delta^+ A(t)) \neq 0$ on $[a, b]$. Furthermore,

$$V(t, s) = I + \int_s^t V(t, \sigma)\, \mathrm{d}B(\sigma) \tag{2,11}$$

for $t, s \in [a, b]$, $t \geqq s$ if $\det(I - \Delta^+ B(t)) \neq 0$ on $[a, b]$ and for $t \leqq s$ if $\det(I + \Delta^- B(t)) \neq 0$ on $[a, b]$. The following lemma establishes the relation between the functions U and V.

2,1. Lemma. *Let $A - B \in N$. If (2,6) or (2,7) holds, then for $t, s \in [a, b]$, $t \geqq s$*

$$\begin{aligned} V(t, s) = {} & U(t, s) + V(t, s)(A(s) - B(s)) - (A(t) - B(t))\, U(t, s) + \\ & + V(t, s)\, \Delta^+ B(s)\, \Delta^+ A(s) - \Delta^- B(t)\, \Delta^- A(t)\, U(t, s) + \\ & + \sum_{s<\tau<t} (V(t, \tau)(\Delta^+ B(\tau)\, \Delta^+ A(\tau) - \Delta^- B(\tau)\, \Delta^- A(\tau))\, U(\tau, s)). \end{aligned} \tag{2,12}$$

If (2,8) or (2,9) holds, then for $t, s \in [a, b]$, $t \leqq s$

$$\begin{aligned} V(t, s) = {} & U(t, s) + V(t, s)(A(s) - B(s)) - (A(t) - B(t))\, U(t, s) + \\ & + V(t, s)\, \Delta^- B(s)\, \Delta^- A(s) - \Delta^+ B(t)\, \Delta^+ A(t)\, U(t, s) + \\ & + \sum_{t<\tau<s} (V(t, \tau)(\Delta^- B(\tau)\, \Delta^- A(\tau) - \Delta^+ B(\tau)\, \Delta^+ A(\tau))\, U(\tau, s)). \end{aligned} \tag{2,13}$$

Proof. Let e.g. (2,6) be satisfied. Let $t, s \in [a, b]$, $t \geq s$. By the substitution theorem for Y-integrals and by (2,10) and (2,11)

$$\begin{aligned} Q = {} & \int_s^t [\mathrm{d}_\tau V(t, \tau)]\, U(\tau, t) + \int_s^t V(t, \tau)\, \mathrm{d}_\tau U(\tau, t) = \int_s^t V(t, \tau)\, [\mathrm{d}(A(\tau) - B(\tau))]\, U(\tau, t) = \\ & = V(t, s)(\Delta^+ A(s) - \Delta^+ B(s))\, U(s,t) + (\Delta^- A(t) - \Delta^- B(t)) = \\ & = -V(t, s)(A(s) - B(s)) + (A(t) - B(t)). \end{aligned}$$

On the other hand, according to the integration-by-parts theorem

$$Q = I - V(t, s)\, U(s, t) - \Delta_2^+ V(t, s)\, \Delta_1^+ U(s, t) + \Delta_2^- V(t, t)\, \Delta_1^- U(t, t) + \\ + \sum_{s<\tau<t} (\Delta_2^- V(t, \tau)\, \Delta_1^- U(\tau, t) - \Delta_2^+ V(t, \tau)\, \Delta_1^+ U(\tau, t)),$$

where $\Delta_1^+ U(t, s) = U(t+, s) - U(t, s)$, $\Delta_2^+ U(t, s) = U(t, s+) - U(t, s), \ldots$ It follows readily from (2,10) and (2,11) and from the properties of the Y-integral as a special kind of the Kurzweil integral ([6], Theorem 1, 3, 6; cf. also [4], [7], [9] or [12]) that

$$\Delta_2^+ V(t, s) = -V(t, s)\, \Delta^+ B(s)\,, \quad \Delta_1^+ U(t, s) = \Delta^+ A(t)\, U(t, s)\,,$$

$$\Delta_2^- V(t, s) = -V(t, s)\, \Delta^- B(s)\,, \quad \Delta_1^- U(t, s) = \Delta^- A(t)\, U(t, s)\,.$$

Hence the relation (2,12) immediately follows.

The other cases can be treated similarly. If (2,8) or (2,9) holds, then instead of the expression Q we should handle the expression

$$\int_s^t [\mathrm{d}_\tau V(s, \tau)]\, U(\tau, s) + \int_s^t V(s, \tau)\, \mathrm{d}_\tau U(\tau, s)\,.$$

The variation-of-constants formula yields the following

2,2. Lemma. *Let (2,5) hold. If (2,6) or (2,7) holds then an n-vector function $x(t)$ is a solution of* (p) *iff it is on $[a, b]$ given by*

$$x(t) = U(t, a)\, c + f(t) - f(a) - \int_a^t [\mathrm{d}_s U(t, s)]\, (f(s) - f(a))\,, \tag{2,14}$$

where $c \in \mathrm{R}_n$ is a solution to the linear equation

$$[M + NV(b, a)]\, c = l + N\left\{V(b, a) f(a) - f(b) - \int_a^b [\mathrm{d}_s V(b, s)]\, f(s)\right\}. \tag{2,15}$$

If (2,8) or (2,9) holds, then any solution $x(t)$ to (p) *is of the form*

$$x(t) = U(t, b)\, c + f(t) - f(b) + \int_t^b [\mathrm{d}_s U(t, s)]\, (f(s) - f(b))\,, \tag{2,16}$$

where

$$[MV(a, b) + N]\, c = l + M\left\{-f(a) + V(a, b) f(b) - \int_a^b [\mathrm{d}_s V(a, s)]\, f(s)\right\}. \tag{2,17}$$

Proof. Putting the variation-of-constants formula (2,14) (valid if (2,6) or (2,7) holds) into (2,2) we get that $x(t)$ is a solution to (p) iff

$$[M + NU(b, a)]\, c = l + N\left\{U(b, a) f(a) - f(b) + \int_a^b [\mathrm{d}_s U(b, s)]\, f(s)\right\}.$$

Since by (2,12)

$$V(b, s) = U(b, s) + V(b, s)(A(s) - B(s)) + V(b, s)\Delta^+B(s)\Delta^+A(s)$$

for $s \in [a, b]$, $W(s) = V(b, s) - U(b, s) \in N$ and $V(b, a) = U(b, a)$. Thus

$$\int_a^b [d_s V(b, s)]\, u(s) = \int_a^b [d_s U(b, s)]\, u(s)$$

for any $u \in BV_n$. (In fact, $W(a) = W(a+) = W(t+) = W(t-) = W(b-) = W(b) = O$ for all $t \in (a, b)$. The continuous part W_c of W vanishes everywhere on $[a, b]$. Hence by the definition of the Y-integral

$$\int_a^b [dW(s)]\, u(s) = \int_a^b [dW_c(s)]\, u(s) + \Delta^+W(a)\, u(a) + \Delta^-W(b)\, u(b) +$$
$$+ \sum_{a<s<b} \Delta W(s)\, u(s) = O.)$$

Hence our assertion follows. The cases (2,8) and (2,9) could be treated analogously.

2,3. Lemma. *Let* $\det(I - \Delta^+B(t)) \neq 0$ *on* $[a, b]$ (*or* $\det(I + \Delta^-B(t)) \neq 0$ *on* $[a, b]$). *Then a couple* $(y(t), \lambda)$ *is a solution to* (p*) *iff*

$$y'(t) = \lambda' N\, V(b, t) \ (\text{or } y'(t) = -\lambda' M\, V(a, t)) \quad \text{on} \quad [a, b],$$

where $\lambda \in R_m$ *is such that*

$$\lambda'[M + N\, V(b, a)] = O \ (\textit{or } \lambda'[M\, V(a, b) + N] = O). \tag{2,18}$$

Proof. In the former case, a couple $(y(t), \lambda)$ is a solution to (p*) iff $y'(t) = \gamma' V(b, t)$ on $[a, b]$, where $y'(a) = \gamma' V(b, a) = -\lambda' M$ and $y'(b) = \gamma' = \lambda' N$.

2,4. Theorem. *Let* (2,5) *and at least one of the conditions* (2,6)–(2,9) *be satisfied. Then the problem* (p) *possesses a solution iff*

$$y'(b) f(b) - y'(a) f(a) - \int_a^b [dy'(s)]\, f(s) = \lambda' l$$

for any solution $(y(t), \lambda)$ *of* (p*).

Proof follows immediately from 2,2 and 2,3.

2,5. Remark. The assumptions (2,5) are fulfilled e.g. if

(i) $B(t) = A(t)$ on $[a, b]$ and $(\Delta^+A(t))^2 = (\Delta^-A(t))^2$ on $[a, b]$ (this case was studied in [11]),

(ii) $B(a) = A(a)$, $B(t) = A(t+)$ for $t \in (a, b)$, $B(b) = A(b)$ $(B = A^*)$, $(\Delta^+A(a))^2 = (\Delta^-A(b))^2 = O$ and $\Delta^+A(t)\Delta^-A(t) = \Delta^-A(t)\Delta^+A(t)$ on (a, b).

In particular, the assumptions of this section are fulfilled if e.g.

a) A, f are left continuous on $(a, b]$ and right-continuous at a, $\det(I + \Delta^+ A(t)) \neq 0$ on $[a, b]$ and $B = A^*$, or

b) A, f are regular on $[a, b]$, $\det(I - (\Delta^+ A(t))^2) \neq 0$ on $[a, b]$ and $B = A$, or

c) A, f are regular on $[a, b]$, $(\Delta^+ A(t))^2 = O$ on $[a, b]$ and $B = A$, or

d) A, f are right-continuous on $[a, b)$ and left-continuous at b, $\det(I - \Delta^- A(t)) \neq 0$ on $[a, b]$ and $B = A^*$.

(Only the cases b) and c) are included in [11].)

2,6. Remark. Of course, also Theorems 2,1, 2,3 and 2,4 of [11] are valid under the assumptions of this section.

3. General problem. The subject of this section is the boundary value problem

(P) to find $x \in \mathrm{BV}_n$ which is on $[a, b]$ a solution to the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}\left[A(t)\,x + C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s) + f(t)\right] \tag{3,1}$$

and fulfils the condition

$$M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L(s)]\,x(s) = l\,. \tag{3,2}$$

We assume that

(3,3) $A(t), C(t), D(t)$ are $n \times n$-matrix functions of bounded variation on $[a, b]$, $f \in \mathrm{BV}_n$, $G(t, s)$ is an $n \times n$-matrix function of strongly bounded variation on $[a, b] \times [a, b]$, $L(t)$ is an $m \times n$-matrix function of bounded variation on $[a, b]$, M and N are constant $m \times n$-matrices and $l \in \mathrm{R}_m$

and

(3,4) A, C, D, f and $G(., s)$ are for an arbitrary $s \in [a, b]$ left-continuous on $(a, b]$ and right-continuous at a, while $\det(I + \Delta^+ A(t)) \neq 0$ on $[a, b]$.

(A matrix function $K(t, s)$ is said to be of strongly bounded variation on $[a, b] \times \times [a, b]$ if it is of bounded two-dimensional Vitali variation $v(K)$ on $[a, b] \times [a, b]$ and $\mathrm{var}_a^b K(a, .) + \mathrm{var}_a^b K(., a) < \infty$. For details see [4] III,4 and also [8] or [11].)

Without any loss of generality we may assume further that

(3,5) $G(t, .)$ and L are for any $t \in [a, b]$ right-continuous on $[a, b)$ and left-continuous at b, while $G(a, .) = O$ on $[a, b]$, $C(a) = D(a) = O$, $L(a) = O$ and $f(a) = O$.

Let us put

(3,6) $B(t) = A(t+)$ on $[a, b]$ $(B(b) = A(b+) = A(b))$.

Then B is right-continuous on $[a, b)$ and left-continuous at b, while $\Delta^- B(t) = \Delta^+ A(t)$ for $t \in [a, b]$ ($\Delta^+ B(a) = \Delta^- B(b) = O$). Evidently $A - B \in N$. Let U and V be again the fundamental matrix solutions to the equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A(t)\, x] \quad \text{and} \quad \frac{\mathrm{d}y'}{\mathrm{d}\tau} = \mathrm{D}[-y'\, B(t)],$$

respectively.

3,1. Remark. Let $t, s \in [a, b]$. Then by 2,1 $V(t, s) = U(t, s) - V(t, s)\, \Delta^+ A(s) + \Delta^+ A(t)\, U(t, s)$ or $(I + \Delta^+ A(t))^{-1}\, V(t, s) = U(t, s)\, (I + \Delta^+ A(s))^{-1}$. Since $U(t, s) \,.\, (I + \Delta^+ A(s))^{-1} = U(t, s+)$ (cf. Theorem 2 in [9]) and analogously $(I + \Delta^- B(t))^{-1} \,.\, V(t, s) = V(t-, s)$, we have

$$V(t-, s) = U(t, s+) \quad \text{for all} \quad t, s \in [a, b]\,. \tag{3,7}$$

Furthermore,

$$V(b-, s) = V(b, s) \quad \text{and} \quad U(t, a+) = U(t, a) \quad \text{for all} \quad t, s \in [a, b]\,. \tag{3,8}$$

(Hence $V(b, a) = U(b, a)$.)

3,2. Lemma. *Given $g \in \mathrm{BV}_n$ right-continuous on $[a, b)$ and $c \in R_n$, the unique solution $y(t)$ of*

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = \mathrm{D}[-y'\, B(t) + g'(t)]$$

on $[a, b]$ such that $y(b) = c$ is on $[a, b]$ given by

$$y'(t) = c'V(b, t) - \int_t^b [\mathrm{d}g'(s)]\, V(s-, t)\,.$$

(The proof is similar to that of the analogous assertion for the equation (2,1) given in [7].)

3,3. Definition. The problem (P*) of finding a couple $(y(t), \lambda) \in \mathrm{BV}_n \times \mathrm{R}_m$ such that $y(t)$ is on $[a, b]$ a solution to

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = \mathrm{D}\left[-y'\, B(t) - \lambda'\, L(t) - \int_a^b y'(s)\, \mathrm{d}_s G(s, t)\right] \tag{3,9}$$

and

$$y'(a) + \lambda' M + \int_a^b y'(s)\, \mathrm{d}C(s) = O\,, \tag{3,10}$$
$$y'(b) - \lambda' N - \int_a^b y'(s)\, \mathrm{d}D(s) = O$$

is called the *adjoint boundary value problem* to the problem (P).

3,4. Remark. Any solution $x(t)$ of (3,1) is left-continuous on $(a, b]$ and right-continuous at a, while any solution $y(t)$ of (3,9) is right-continuous on $[a, b)$ and left-continuous at b.

3,5. Theorem. *Let* (3,3)–(3,6) *hold. The given problem* (P) *has a solution iff*

$$\int_a^b y'(s)\,\mathrm{d}f(s) = \lambda' l$$

for any solution $(y(t), \lambda)$ *of its adjoint* (P*).

Proof. Sufficiency. An n-vector function $x \in \mathrm{BV}_n$ is a solution of (3,1) on $[a, b]$ such that $x(a) = c \in \mathrm{R}_n$ iff it is on $[a, b]$ given by

$$x(t) = U(t, a)\,c + \int_a^t U(t, s+)\,\mathrm{d}h(s) + \int_D^t U(t, s+)\,\mathrm{d}f(s)\,, \tag{3,11}$$

where

$$h(t) = C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s) \tag{3,12}$$

has bounded variation on $[a, b]$, is left-continuous on $(a, b]$ and right-continuous at a and vanishes at a ($h \in \mathrm{V}_n$). Putting (3,11) into (3,12) and (3,2) and making use of the Dirichlet formula we find that the given problem (P) is equivalent to the system of equations for $(h(t), c) \in \mathrm{V}_n \times \mathrm{R}_n$

$$h(t) + P(t)\,c + \int_a^b K(t, s)\,\mathrm{d}h(s) = u(t) \quad \text{on} \quad [a, b]\,, \tag{3,13}$$

$$Qc + \int_a^b R(s)\,\mathrm{d}h(s) = v\,,$$

where for $t, s \in [a, b]$

$$P(t) = -\left\{C(t) + D(t)\,U(b, a) + \int_a^b [\mathrm{d}_\sigma G(t, \sigma)]\,U(\sigma, a)\right\}, \tag{3,14}$$

$$R(t) = NU(b, t+) + \int_t^b [\mathrm{d}L(\sigma)]\,U(\sigma, t+)\,,$$

$$K(t, s) = -\left\{D(t)\,U(b, s+) + \int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\,U(\sigma, s+)\right\},$$

$$u(t) = -\int_a^b K(t, \sigma)\,\mathrm{d}f(\sigma)\,, \quad v = l - \int_a^b R(\sigma)\,\mathrm{d}f(\sigma)\,.$$

(If a couple $(h(t), c)$ is a solution to (3,13) and if $x(t)$ is given by (3,11), then $x(t)$ is a solution to (P). Conversely, if $x(t)$ is a solution to (P) and $c = x(a)$ and $h(t)$ is given by (3,12), then a couple $(h(t), c)$ is a solution to (3,13).) Obviously, P, R and u have bounded variation on $[a, b]$, while P, u and $K(., s)$ are for any $s \in [a, b]$ left-continuous on $(a, b]$ and right-continuous at a and vanishing at a. Moreover, by Lemma 5,1 of [11], $K(t, s)$ is of strongly bounded variation on $[a, b] \times [a, b]$. Analogously as in Lemma 4,1 in [11] we can show that the system (3,13) possesses a solution iff

$$\int_a^b \chi'(s)\, \mathrm{d}u(s) + \gamma' v = O \tag{3,15}$$

for any solution $(\chi(t), \gamma) \in \mathrm{BV}_n \times \mathrm{R}_m$ of the system adjoint to (3,13)

$$\chi'(t) + \gamma' R(t) + \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) = O \quad \text{on} \quad [a, b]\,, \tag{3,16}$$

$$\gamma' Q + \int_a^b \chi'(s)\, \mathrm{d}P(s) = O\,.$$

Inserting (3,14) into (3,15), we get by Proposition 2,4 of [8]

$$O = \gamma' l - \int_a^b \left\{\gamma' R(t) + \int_a^b \chi'(s) \left[\mathrm{d}_s K(s, t)\right]\right\} \mathrm{d}f(t)$$

or by $(3,16)_1$

$$O = \gamma' l + \int_a^b \chi'(t)\, \mathrm{d}f(t)\,.$$

Let $(\chi(t), \gamma)$ be an arbitrary solution of (3,16). We shall show that then the couple $(\chi(t), -\gamma)$ is a solution to (P*) and thus we shall complete the first part of the proof.

Since by Prop. 2,4 of [8]

$$\int_a^b \chi'(s) \left[\mathrm{d}_s \int_t^b \left[\mathrm{d}_\sigma G(s, \sigma)\right] U(\sigma, t+)\right] = \int_t^b \left[\mathrm{d}_s \int_a^b \chi'(\sigma) \left[\mathrm{d}_\sigma G(\sigma, s)\right]\right] U(s, t+)\,,$$

we have according to (3,7) and (3,8)

$$\chi'(t) = -\gamma' R(t) - \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) =$$

$$= \left\{-\gamma' N + \int_a^b \chi'(s)\, \mathrm{d}D(s)\right\} V(b, t) - \int_t^b \left[\mathrm{d}_s \left\{\gamma' L(s) - \int_a^b \chi'(\sigma) \left[\mathrm{d}_\sigma G(\sigma, s)\right]\right\}\right] V(s-, t)$$

for $t \in [a, b]$. Hence by 3,2 $\chi(t)$ is a solution of (3,9) on $[a, b]$ such that

$$\chi'(b) = -\gamma' N + \int_a^b \chi'(s)\, \mathrm{d}D(s)\,.$$

Finally, by $(3,16)_2$

$$\chi'(a) = -\gamma' N U(b, a) + \int_a^b \chi'(s)\, \mathrm{d}D(s)\, U(b, a) - \gamma' \int_a^b \left[\mathrm{d}L(s)\right] U(s, a) +$$