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THE NONEXISTENCE OF FREE COMPLETE VECTOR LATTICES

MÁRIA JAKUBÍKOVÁ, Košice

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Free vector lattices were investigated in [1], [3], [9], [11] (cf. also [2], Chap. XV, § 5). Since the class of all vector lattices is an equational one, for each cardinal $\mathfrak{m}$ there exists a free vector lattice $X_{\mathfrak{m}}$ with a set A of free generators such that card $A = \mathfrak{m}$. HALES [4] proved that there does not exist a free complete Boolean algebra with an infinite set of free complete generators (this solved the problem proposed by RIEGER [8]). Using the result of Hales we show that there does not exist a free complete vector lattice with an infinite set of free complete generators. An analogous result concerning complete l-groups was proved in [5]. Further, we examine the existence of free (α, ∞)-distributive vector lattices where α is an infinite regular cardinal.

For the terminology, cf. [2], Chap. XV. Lattice unions and intersections are denoted by $\vee$ and $\wedge$, respectively. Set unions, set intersections and the inclusion are denoted by $\cup$, $\cap$ and $\subset$, respectively. A sublattice L_1 of a lattice L is said to be a closed sublattice of L, if, whenever $\{x_i\}$ $(i \in I)$ is a subset of L_1 such that $\bigvee x_i$ exists in L, then $\bigvee x_i \in L_1$, and if the dual condition also holds. A mapping φ of a lattice L into a lattice L' is said to be a complete homomorphism if it fulfils the following condition (c_1) and also the condition (c_2) that is dual to (c_1): If $\{x_i\} \subset L$ and if $\bigvee x_i$ exists in L, then

$$\bigvee \varphi(x_i) \text{ exists in } L' \text{ and } \varphi(\bigvee x_i) = \bigvee \varphi(x_i).$$

Let us recall the definition of a vector lattice (cf. [2]).

A real linear space L with elements $f, g, \ldots,$ is called a vector lattice if L is lattice ordered in such a manner that the partial ordering is compatible with the algebraic structure of L, i.e.,

(i) $f \leqq g$ implies $f + h \leqq g + h$ for every $f, g, h \in L$,

(ii) $f \geqq 0$ implies $\alpha f \geqq 0$ for every $f \in L$ and every real number $\alpha \geqq 0$.

Thus $(L; +, \wedge, \vee)$ is an Abelian lattice ordered group; hence $(L; \wedge, \vee)$ is a distributive lattice and

$$f + (g \vee h) = (f + g) \vee (f + h),$$
$$f + (g \wedge h) = (f + g) \wedge (f + h)$$

is valid for every $f, g, h \in L$.

Let A be a subset of a complete Boolean algebra B. We say that A completely generates B if $B_1 = B$ for each closed subalgebra B_1 of B with $A \subset B_1$. The set A is said to be a set of free generators of B, if it satisfies the following conditions: (a) A completely generates B; (b) if B' is a complete Boolean algebra and if f is a mapping of the set A into B' such that the set $f(A)$ completely generates B', then there exists a complete homomorphism ψ of B onto B' such that $\psi(a) = f(a)$ for each $a \in A$. (Cf. [4].)

Now we introduce analogous notions for complete vector lattices. For any vector lattice X the corresponding lattice will be denoted by $\bar{X}$. A vector sublattice X_1 of a vector lattice X is said to be a closed vector sublattice of X, if $\bar{X}_1$ is a closed sublattice of $\bar{X}$. Let A be a subset of a complete vector lattice X. We say that A completely generates X if $X_1 = X$ for each closed vector sublattice X_1 of X with $A \subset X_1$. A homomorphism φ of a complete vector lattice X into a complete vector lattice X' is called a complete homomorphism if φ is a complete homomorphism of the lattice $\bar{X}$ into the lattice $\bar{X}'$. Let A be a subset of a complete vector lattice X. Then A is said to be a set of free complete generators of X if it fulfils the following conditions: (a) A completely generates X, and (b) for each complete vector lattice X' and each mapping $f: A \to X'$ such that $f(A)$ completely generates X' there is a complete homomorphism ψ of X onto X' such that $\psi(a) = f(a)$ for each $a \in A$. If A is a set of free complete generators of a complete vector lattice X and card $A = \gamma$, then X is called a free complete vector lattice on γ free complete generators.

Let X be a complete vector lattice, $0 < e \in X$. The element e is called a weak unit of X if $e \wedge x > 0$ for each $0 < x \in X$. The element e is a strong unit of X if for each $0 < x \in X$ there is a positive integer $n(x)$ such that $x \leqq n(x)\, e$. Each strong unit of X is a weak unit of X. Let e be a weak unit of X and let $B(e)$ be the set of all elements $e_i \in X$ such that $e_i \geqq 0$ and $e_i \wedge (e - e_i) = 0$. The set $B(e)$ is said to be a basis of X.

We need the following results:

Theorem A. (Cf. [6], p. 92.) *Let e be a weak unit of a complete vector lattice X. Then the basis $B(e)$ is a closed sublattice of $\bar{X}$ and $B(e)$ is a Boolean algebra.*

Theorem B. (Cf. [6], p. 131, Thm. 1.53.) *Let B be a complete Boolean algebra. Then there is a complete vector lattice X and a weak unit e of X such that the basis $B(e)$ is isomorphic to B.*

Theorem C. (Cf. [4], § 4, Thm. 3.) *Let $\mathfrak{m}$ be an infinite cardinal. There exists a complete Boolean algebra $B_\mathfrak{m}$ and a subset $A \subset B_\mathfrak{m}$ such that A completely generates $B_\mathfrak{m}$, card $A = \aleph_0$ and card $B_\mathfrak{m} = \mathfrak{m}$.*

Theorem 1. *Let α be an infinite cardinal. There does not exist a free complete vector lattice on α free complete generators.*

Proof. Suppose that a set A_0 is the set of free complete generators of a complete lattice X_0, card $A_0 = \alpha$. Let $\mathfrak{m}$ be a cardinal, $\mathfrak{m} >$ card X_0. Let $B = B_\mathfrak{m}$ be a Boolean algebra fulfilling the assertion of Thm. C. Further let X be a complete vector lattice satisfying the assertion of Thm. B. Since the Boolean algebras $B_\mathfrak{m}$ and $B(e)$ are isomorphic we may put $B(e) = B_\mathfrak{m}$. Choose $a_0, a_1 \in A_0$ and $A_1 \subset A_0 \setminus \{a_0, a_1\}$, card $A_1 = \aleph_0$. Let $f_1 : A_1 \to A$ be a bijection and let f be a mapping of the set A_0 into X such that $f(a_0) = 0, f(a_1) = e, f(a) = f_1(a)$ for each $a \in A_1$ and $f(a) = 0$ for each $a \in A_0 \setminus (A_1 \cup \{a_0, a_1\})$. Let Y be the intersection of all closed vector sublattices Y_i of X with $f(A_0) \subset Y_i$. Then Y is a closed vector sublattice of X, hence Y is a complete lattice and Y is completely generated by the set $f(A_0)$.

According to the definition of a free complete vector lattice, there is a complete homomorphism ψ of X_0 onto Y such that $\psi(a) = f(a)$ for each $a \in A_0$. Since e is a weak unit of X, e is a weak unit of Y. By Thm. A, $B(e) = B$ is a closed sublattice of X and hence the set $B \cap Y = B_0$ is a closed sublattice of Y. Thus, since $0, e \in B_0$, the set B_0 is a complete lattice. Obviously B_0 is distributive. Let $b_0 \in B_0$. Then $b_0 \in B(e)$, hence $b_0 \wedge (e - b_0) = 0$. This implies $e - b_0 \in B(e)$ and so $e - b_0 \in B_0$. Further we have $b_0 \vee (e - b_0) = b_0 + (e - b_0) = e$, hence $e - b_0$ is the complement of b_0 in the Boolean algebra B. This implies that B_0 is a closed subalgebra of B. Since $A \subset B_0$ we obtain (because B is completely generated by A) that $B_0 = B$. Therefore $\mathfrak{m} =$ card $B \leqq$ card $Y =$ card $\psi(X_0)$. This implies card $X_0 \geqq \mathfrak{m}$, which is a contradiction.

Let α, β be cardinals. Let us consider the following condition on a lattice L (cf. [4]):
(d_1) L satisfies the identity

$$\bigwedge_{s\in S} \bigvee_{t\in T} x_{s,t} = \bigvee_{\varphi \in T^S} \bigwedge_{s\in S} x_{s,\varphi(s)}$$

whenever card $S \leqq \alpha$, card $T \leqq \beta$ and all joins and meets do exist in L.

If L satisfies (d_1) and the condition dual to (d_1) then L is called (α, β)-distributive. If L is (α, β)-distributive for each cardinal β, then it is said to be (α, ∞)-distributive. It is easy to verify that a vector lattice is (α, β)-distributive if it fulfils the condition (d_1).

A complete (α, ∞)-distributive Boolean algebra B is said to be a free complete (α, ∞)-distributive Boolean algebra on γ free complete generators if there is a subset $A \subset B$ with card $A = \gamma$ such that A is a set of free complete generators of B and every mapping f of A onto a subset A' of a complete (α, ∞)-distributive Boolean algebra B' which completely generates B' can be extended to a complete homomorphism of B onto B'.

Replacing "Boolean algebra" by "vector lattice" everywhere in the above definition, we obtain the definition of a free complete (α, ∞)-distributive vector lattice on γ complete generators.

Theorem C′. (Cf. [4], p. 62.) *Let γ be an infinite regular cardinal. Let $\mathfrak{m}$ be a cardinal, $\mathfrak{m} \geqq \gamma$. There exists a complete (γ, ∞)-distributive Boolean algebra $B^0_{\mathfrak{m}}$ and a subset $A \subset B^0_{\mathfrak{m}}$ such that A completely generates $B^0_{\mathfrak{m}}$, card $A = \gamma$, and card $B^0_{\mathfrak{m}} = \mathfrak{m}$.*

Theorem D. (Cf. [7].) *Let B be a Boolean algebra and let M be the Stone space of B. Then the lattice $C(M)$ fo all real continuous functions on M is (α, β)-distributive if and only if B is (α, β)-distributive.*

Theorem E. (Cf. [10], Thm. v. 3.1.) *Let e be a strong unit of a complete vector lattice Y. Let M be the Stone space of the Boolean algebra $B(e) = B$. Then Y is isomorphic with the vector lattice $B(M)$ consisting of all bounded continuous functions on M.*

A subset P of a vector lattice Q is said to be convex if $p_1, p_2 \in P$, $q \in Q$, $p_1 \leqq q \leqq p_2$ implies $q \in P$.

Lemma. *Let P be a vector sublattice of a vector lattice Q. Assume that P is a convex subset of Q and that for each $0 < q \in Q$ there exists $0 < p \in P$ with $p \wedge q > 0$. Then P is (α, β)-distributive.*

Proof. If $\{f_i\}$ is a subset of P and if $f \in P$ is the least upper bound of $\{f_i\}$ in P, then f is also the least upper bound of the set $\{f_i\}$ in Q (since P is convex in Q). A similar assertion holds for greatest lower bounds of subsets of P. Thus if P is not (α, β)-distributive, then Q fails to be (α, β)-distributive. Assume that Q is not (α, β)-distributive. Then there exists a system $\{x_{s,t}\} \subset Q$ with card $S \leqq \alpha$, card $T \leqq \beta$ such that all joins and meets standing in (d_1) do exist in Q and

$$v = \bigwedge_{s \in S} \bigvee_{t \in T} x_{s,t} > \bigvee_{\varphi \in T^S} \bigwedge_{s \in S} x_{s,\varphi(s)} = u .$$

There exists $0 < f_1 \in P$ with $f_1 \wedge (v - u) > 0$. Denote

$$(x_{s,t} \wedge v) \vee u = \bar{x}_{s,t} ,$$

$$(\bar{x}_{s,t} - u) \wedge f_1 = y_{s,t} .$$

Then we have

$$0 < f_1 \wedge (v - u) = \bigwedge_{s \in S} \bigvee_{t \in T} y_{s,t} \neq \bigvee_{\varphi \in T^S} \bigwedge_{s \in S} y_{s,\varphi(s)} = 0 ;$$

hence P is not (α, β)-distributive.