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# OSCILLATION OF SOLUTIONS OF THE DELAY DIFFERENTIAL EQUATION

$$y^{(2n)}(t) + \sum_{i=1}^{m} p_i(t) f_i(y[h_i(t)]) = 0, \quad n \geqq 1$$

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Our purpose in this paper is to give some ocsillation criteria for the nonlinear delay differential equation

$$y^{(2n)}(t) + \sum_{i=1}^{m} p_i(t) f_i[y_{h_i}(t)] = 0\,, \quad n \geqq 1\,, \tag{1}$$

where $y_{h_i}(t) = y[h_i(t)]$ $i = 1, \ldots, m$

$$p_i \in C[R_+ \equiv [0, \infty), R_+] \quad (i = 1, \ldots, m) \tag{2}$$

$$f_i \in C[R, R]\,, \quad z f_i(z) > 0 \quad \text{for} \quad z \neq 0,\ f_i(z) \quad \text{is nondecreasing on} \quad R (i = 1, \ldots, m) \tag{3}$$

$$h_i \in C[R_+, R]\,, \quad h_i(t) \leqq t \quad \text{for} \quad t \in R_+ \quad (i = 1, \ldots, m)\,. \tag{4}$$

We shall assume the under the initial conditions $y(t) = \varphi(t)$, $t \leqq t_0$, $y^{(k)}(t_0) = y_0^{(k)}$, $k = 1, \ldots, n - 1$, the equation (1) has a solution which exists for all $t \geqq t_0 > 0$.

A solution $y(t)$ of (1) is called *oscillatory* if the set of zeros of $y(t)$ is not bounded from the right. A solution $y(t)$ of (1) is called *nonoscillatory* if it is of constant sign for sufficiently large $t$. The equation (1) is called *oscillatory* if every solution is oscillatory.

Burkowski [2], Gollwitzer [3], Odarič-Ševelo [9, 10] have given necessary and sufficient conditions for second order nonlinear delay differential equations to be oscillatory. Ladas [4], Marušiak [8] have given oscillation criteria for the differential equation

$$y^{(n)}(t) + F(t, y(t), y[h(t)]) = 0\,.$$

Recently, KUSANO and ONOSE [7], ŠEVELO and VARECH [11] and STAIKOS and SFICAS [12] (these papers appeared while my article was being reviewed) have proved sufficient conditions for the oscillation of certain nonlinear delay differential equations of arbitrary order.

In the next part we shall need the following lemma due to KIGURADZE [5, Lemma 2].

**Lemma 1.** *Let $u(t), \ldots, u^{(m-1)}(t)$ be absolutely continuous and of cosnstant sign in the interval $(t_0, \infty)$. If $u(t) \geqq 0$, $u^{(m)}(t) \leqq 0$ for every $t \geqq t_0$, then there exists an integer $k$ with $0 \leqq k < m$, $m + k$ is odd and*

$$\text{(a)} \qquad u^{(i)}(t) \geqq 0\,, \quad i = 1, \ldots, k\,, \quad t \geqq t_0\,,$$

$$\text{(b)} \qquad (-1)^{m+i-1}\, u^{(i)}(t) \geqq 0\,, \quad i = k+1, \ldots, m\,, \quad t \geqq t_0\,,$$

$$(5) \quad \text{(c)} \qquad u^{(k)}(t) \leqq \frac{i!}{(t - t_0)^i}\, u^{(k-i)}(t)\,, \quad i = 1, \ldots, k, \quad t \geqq t_0\,.$$

Analogous statement can be made if $u(t) \leqq 0$, $u^{(m)}(t) \geqq 0$ in the interval $(t_0, \infty)$.

**Lemma 2.** *If $u(t), \ldots, u^{(m-1)}(t)$ are absolutely continuous and of constant sign in the interval $(t_0, \infty)$ and $u(t)\, u^{(m)}(t) \leqq 0$, then there exists an integer $k$ with $0 \leqq k < m$, $m + k$ is odd and*

$$(6) \qquad u^{(i)}(t)\, u(t) \geqq 0\,, \quad i = 0, 1, \ldots, k \quad \textit{and}$$
$$(-1)^{m+i-1}\, u^{(i)}(t)\, u(t) \geqq 0\,, \quad i = k+1, \ldots, m\,, \quad t \geqq t_0\,,$$

$$(7) \qquad |u^{(k)}(t)| \geqq t^{m-k-1}\, u^{(m-1)}(2^{m-k-1} t)\,, \quad t \geqq t_0\,,$$

$$(8) \qquad |u^{(k-i)}(t)| \geqq B_i t^{m-k+i-1} |u^{(m-1)}(t)|\,, \quad i = 1, \ldots, k\,, \quad t \geqq 2^{m-k} t_0\,,$$

where

$$B_i = \frac{2^{-(m+k+i)^3}}{(m-k)\ldots(m-k+i-1)}\,.$$

Proof. The correctness of (6), (7) follows from Kiguradze's lemma 1 [6] and its proof. Integrating (7) $i$ times ($i \in \{1, \ldots, k\}$) from $t_0$ to $t$ and using (6), we obtain

$$|u^{(k-i)}(t)| \geqq \frac{(t - t_0)^{m-k+i-1}}{(m-k)\ldots(m-k+i-1)} |u^{(m-1)}(2^{m-k-1} t)|\,, \quad t \geqq t_0\,.$$

If we put $t$ instead of $2^{m-k-1} t$ into the last inequality and then use $u(t)\, u^{(k-i+1)}(t) \geqq$ $\geqq 0$, we get

$$(9) \qquad |u^{(k-i)}(t)| \geqq |u^{(k-i)}(2^{-m+k+1} t)| \geqq$$
$$\geqq \frac{2^{-(m-k+i-1)^2}(t - 2^{m-k-1} t_0)^{m-k+i-1}}{(m-k)\ldots(m-k+i-1)} |u^{(m-1)}(t)|\,, \quad t \geqq 2^{m-k-1} t_0\,.$$

Let $t \geqq t_1 \geqq 2 \cdot 2^{m-k-1} t_0$, then $t - 2^{m-k-1} t_0 \geqq t/2$ and from (9) with regard to the last inequalities we get (8).

**Lemma 3.** *Let $u(t), \ldots, u^{(m)}(t)$ be continuous functions in the interval $(t_0, \infty)$ and $u^{(k)}(t)\, u(t) > 0$, $(k = 0, 1, \ldots, m)$, $u(t)\, u^{(m+1)}(t) \leqq 0$ (m is an integer and let $A$ be a nonnegative real number. Then*

$$\lim_{t\to\infty} \frac{u(t)}{u(t+A)} = 1 .$$

Proof.

$$1 \geqq \lim_{t\to\infty} \frac{u(t)}{u(t+A)} \geqq \frac{1}{1 + A \lim\limits_{t\to\infty} \dfrac{u'(t_1)}{u(t)}} = 1\,, \quad t_1 = \begin{Bmatrix} t; & m = 1 \\ t + A; & m > 1 \end{Bmatrix},$$

because

$$\lim_{t\to\infty} \frac{u'(t_1)}{u(t)} = \lim_{t=\infty} \frac{u^{(m)}(t_1)}{u^{(m-1)}(t)} = 0 .$$

**Theorem 1.** *Let functions $p_i, f_i, h_i$ satisfy (2), (3), (4) and, in addition, suppose that*

$$\sum_{i=1}^{m} \int^{\infty} t^{2n-1}\, p_i(t)\, \mathrm{d}t < \infty . \tag{10}$$

*Then the equation (1) has at least one nonoscillatory solution.*

Proof. Let us consider the following system

$$y_0(t) = \begin{cases} 1, & t \leqq t_0 \\ 1, & t \geqq t_0 \end{cases} \tag{11}$$

$$y_{j+1}(t) = \begin{cases} 1, \; t \leqq t_0 \\ 1 + \sum\limits_{i=1}^{m} \left\{ \displaystyle\int_{t_0}^{t} \frac{(s - t_0)^{2n-1}}{(2n-1)!}\, p_i(s)\, f_i(y_j[h_i(s)])\, \mathrm{d}s \; + \right. \end{cases}$$

$$+ \int_t^{\infty} \frac{(s - t_0)^{2n-1} - (s - t)^{2n-1}}{(2n-1)!}\, p_i(s)\, f_i(y_j[h_i(s)])\, \mathrm{d}s\,,$$

where $t_0$ is chosen such that

$$\max_{1\leqq i\leqq m} f_i(2) \sum_{i=1}^{m} \left\{ \int_{t_0}^{t} \frac{(s - t_0)^{2n-1}}{(2n-1)!}\, p_i(s)\, \mathrm{d}s \; + \right. \tag{12}$$

$$\left. + \int_t^{\infty} \frac{(s - t_0)^{2n-1} - (s - t)^{2n-1}}{(2n-1)!}\, p_i(s)\, \mathrm{d}s \right\} \leqq 1 .$$

That we can do because (10) holds.

By mathematical induction, with regard to (11), (12) and (3), it is easy to show that $1 \leqq y_j(t) \leqq y_{j+1}(t) \leqq 2$, $j = 0, 1, \ldots$, $t \geqq t_0$ holds. From the last inequalities it follows that the sequence $\{y_j(t)\}_{j=0}^{\infty}$ of continuous functions is nondecreasing and uniformly bounded on $[t_0, \infty)$ and therefore uniformly convergent on every finite interval. Let $y(t) = \lim_{j\to\infty} y_j(t)$. Then $1 \leqq y(t) \leqq 2$, $t \geqq t_0$ and $y(t)$ is the solution of the equation

$$y(t) = \begin{cases} 1, & t \leqq t_0 \\ 1 + \sum_{i=1}^{m} \left\{ \int_{t_0}^{t} \frac{(s - t_0)^{2n-1}}{(2n-1)!} p_i(s) f_i(y[h_i(s)]) \, ds + \right. \\ \left. + \int_t^{\infty} \frac{(s - t_0)^{2n-1} - (s - t)^{2n-1}}{(2n-1)!} p_i(s) f_i(y[h_i(s)]) \, ds \right\} . \end{cases}$$

However, it means that $y(t)$ is a nonoscillatory solution of the equation (1). The proof is therefore complete.

**Theorem 2.** *Let functions $p, f, h$, satisfy* (2), (3), (4) *and, in addition, suppose that*

(13) (i) $h(t) = t - g(t)$, $0 \leqq g(t) \leqq M$, $t \in R_+$

(ii) *there exists a number $\beta$, $1 < \beta$ such that*

$$\liminf_{|z|\to\infty} \frac{f(z)}{|z|^{\beta}} \neq 0$$

(14) (iii)
$$\int^{\infty} t^{2n-1} p(t) \, dt = \infty .$$

*Then the differential inequality*

$$y^{(2n)}(t) + p(t) f(y[h(t)]) \leqq 0 , \quad t \in R_+ \tag{A}$$

$$[y^{(2n)}(t) + p(t) f(y[h(t)]) \geqq 0 , \quad t \in R_+] \tag{B}$$

*has no positive [negative] solution on $[t_0, \infty)$ for every $t_0 \in R_+$.*

Proof. Suppose that the conclusion of Theorem 2 is false. Assume that there exists a positive solution $y(t)$ of (A) for $t \geqq t_0 \in R_+$. (The case of the differential inequality (B) is treated similarly.) Since $\lim h(t) = \infty$ as $t \to \infty$ there exists a $t_1 \geqq t_0$ such that $y[h(t)] > 0$ for $t \geqq t_1$. (A) with regard to (2) and (3) implies

$$y^{(2n)}(t) \leqq - p(t) f(y[h(t)]) < 0 , \quad t \geqq t_1 . \tag{15}$$

From $y^{(2n)}(t) < 0$, $y(t) > 0$ it follows that there exists $t_2 \geqq t_1$ such that $y(t), y'(t), \ldots$ $\ldots, y^{(2n-1)}(t)$ have constant sign for $t \geqq t_2$. Then by Lemma 2 for $y(t)$ and its derivatives (6)–(8) hold, where $k \in \{1, 3, \ldots, 2n-1\}$. By (6), $y^{(2n-1)}(t)$ is decreasing and $y^{(2n-1)}(\infty) = c \geqq 0$ holds.

Integrating (A) from $t$ ($t \geqq t_2$) to $\infty$ and neglecting $y^{(2n-1)}(\infty)$, we get

$$(16) \qquad y^{(2n-1)}(t) \geqq \int_t^\infty p(s) f(y_h(s))\, ds\,, \quad t \geqq t_2$$

and then in view of the monotonicity of $y^{(2n-1)}(t)$ and (4) we obtain

$$(17) \qquad y_h^{(2n-1)}(t) \geqq \int_t^\infty p(s) f(y_h(s))\, ds\,, \quad t \geqq t_2\,.$$

I. From (7), for $k = 1$ we get

$$(18) \qquad y'(t) \geqq t^{2n-2}\, y^{(2n-1)}(2^{2n-2}t)\,, \quad t \geqq t_2\,.$$

If $k = 1$ then, with regard to (6), $y''(t) \leqq 0$ for $t \geqq t_2$, $y^{(2n-1)}(t)$ is decreasing and so from (18) we have

$$y'(t-M) \geqq [t-M)^{2n-2}\, y^{(2n-1)}[2^{2n-2}(t-M]$$
$$\geqq [t-M]^{2n-2}\, y^{(2n-1)}(2^{2n-2)}t)\,, \quad t \geqq t_3 \geqq t_2 + M\,.$$

From (16) using the last inequality we get

$$(19) \qquad y'(t-M) \geqq [t-M]^{2n-2} \int_{2^{2n-2}t}^\infty p(s) f[y_h(s)]\, ds\,, \quad t \geqq t_3\,.$$

Integrating (19) from $t_3$ to $t$, $t \geqq t_3$ ,we obtain

$$(20) \quad y(t-M) - y(t_3 - M) \geqq \int_{2^{2n-2}t_3}^{2^{2n-2}t} \frac{[2^{2-2n}s - M]^{2n-1} - [t_3 - M]^{2n-1}}{2n-1} \times$$
$$\times\, p(s) f[y_h(s)]\, ds + \frac{[t-M]^{2n-1} - [t_3 - M]^{2n-1}}{2n-1} \int_{2^{2n-2}t}^\infty p(s) f[y_h(s)]\, ds\,.$$

From (20), with regard to the monotonicity of $y(t)$, $f(z)$ and $t - M \leqq h(t)$, we get

$$y(t-M) \geqq \int_{t_3}^t \frac{[s-M]^{2n-1} - [t_3 - M]^{2n-1}}{2n-1}\, p(2^{2n-2}s) f[y(s-M)]\, ds\,.$$

In the sequel we shall use the method due to Atkinson [1].

If we raise the last inequality by $-\beta$ ($\beta > 1$), then multiply by $\{[t-M]^{2n-1} - [t_3 - M]^{2n-1}\}\, p(2^{2n-2}t) f[y(t-M)]$, ($t \geqq t_3$) and integrate the resulting inequality from $t_4$ to $t_5$ $(t_3 < t_4 < t < t_5)$, we have

$$(21)\quad \int_{t_4}^{t_5} \{[s - M]^{2n-1} - [t_3 - M]^{2n-1}\}\, p(2^{2n-2}s) f[y(s - M)]\, [y(s - M)]^{-\beta}\, ds \leqq$$

$$\leqq \frac{(2n-1)^\beta}{\beta - 1} \left[ \left\{ \int_{t_3}^{t} ([s - M]^{2n-1} - [t_3 - M]^{2n-1})\, p(2^{2n-2}s) f[y(s - M)]\, ds \right\}^{1-\beta} \right]_{t_4}^{t_5}.$$

For $t_5 \to \infty$ the right hand side of (21) is bounded and therefore the integral

$$\int_{t_4}^{\infty} \{[s - M]^{2n-1} - [t_3 - M]^{2n-1}\}\, p(2^{2n-2}s) f[y(s - M)]\, [y(s - M)]^{-\beta}\, ds$$

is convergent. If we choose $t_4 \geqq 2M$, we can show easily that

$$(22)\qquad J(t_4) = \int_{t_4}^{\infty} s^{2n-1} p(2^{2n-2}s) f[y(s - M)]\, [y(s - M)]^{-\beta}\, ds < \infty\,.$$

By virtue of the assumption $y(t) > 0$, $t \geqq t_0$ and Lemma 2 either $y(\infty) = b > 0$ or $y(\infty) = \infty$. In either case, with regard to the continuity and the monotonicity of $f(z)$ and the assumption (ii) of Theorem 2, there exists $T \geqq t_4$ such that

$$\frac{f[y(t - M)]}{[y(t - M)]^\beta} \geqq d > 0, \quad t \geqq T\,.$$

Then, from (22) we get

$$\infty > J(t_4) \geqq J(T) \geqq d \int_T^{\infty} s^{2n-1}\, p(2^{2n-2}s)\, ds = d\, (2^{2-2n})^{2n-1} \int_{2^{2n-2}T}^{\infty} t^{2n-1}\, p(t)\, dt\,,$$

which contradicts (14).

II. Let $k \in \{3, \ldots, 2n - 1\}$. Frome (8), for $i = k - 1$) we obtain,

$$y'(t) \geqq K t^{2n-2}\, y^{(2n-1)}(t)\,, \quad t \geqq 2^{(n-k}t_2 = \bar{t}_3\,,$$

where $K = B_{k-1}$.

Then, with regard to (6) and (13) we have

$$y'(t) \geqq y'(t - M) \geqq K[t - M]^{2n-2}\, y^{(2n-1)}(t - M)\,, \quad t \geqq \bar{t}_4 \geqq \bar{t}_3 + M\,.$$

From (17), by means of the last inequality it follows

$$y'(t) \geqq K[t - M]^{2n-2} \int_t^{\infty} p(s) f[y_h(s)]\, ds\,, \quad t \geqq \bar{t}_4\,,$$

Further, exactly as in the case I we obtain

$$(23)\qquad J(\bar{t}_5) = \int_{\bar{t}_5}^{\infty} s^{2n-1} p(s) f[y(s - M)]\, [y(s)]^{-\beta}\, ds < \infty\,.$$

(6) implies $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ and therefore $y(\infty) = \infty$. Then, by virtue of the assumption (ii) and Lemma 3

$$\liminf_{t\to\infty} \frac{f[y(t-M)]}{[y(t)]^\beta} = \liminf_{t\to\infty} \frac{f[y(t)]}{[y(t+M)]^\beta} = \liminf_{t\to\infty} \frac{f[y(t)]}{[y(t)]^\beta} > 0$$

holds. In view of the last inequality there exitss $\bar{T} \geqq \bar{t}_5$ such that

$$\frac{f[y(t-M)]}{[y(t)]^\beta} \geqq \bar{d} \geqq 0\,, \quad t \geqq \bar{T}\,.$$

Then we get from (23)

$$\infty > J(\bar{t}_5) \geqq J(\bar{T}) \geqq \bar{d} \int_{\bar{T}}^{\infty} s^{2n-1}\, p(2^{2n-2}s)\, ds = \bar{d}(2^{2-2n})^{2n-1} \int_{2^{2n-2}\bar{T}}^{\infty} t^{2n-1} p(t)\, dt\,,$$

which contradicts (14).

This completes the proof of Theorem 2.

We shall now apply Theorem 2 to obtain the oscillatory character for the equation (1).

**Theorem 3.** *Let functions* $p_i, f_i, h_i$ *satisfy* (2), (3), (4) *and, in addition, suppose*

(i) $h_i(t) = t - g_i(t)$, $0 \leqq g_i(t) \leqq M$, $t \in R_+$, $(i = 1, \ldots, m)$

(ii) *there exists a number* $\beta$, $\beta > 1$ *such that*

$$\liminf_{|z|\to\infty} \frac{|f_i(z)|}{|z|^\beta} > 0\,, \quad (i = 1, \ldots, m)\,.$$

*Then the equation* (1) *is oscillatory if and only if*

$$\int^{\infty} t^{2n-1}\, p_j(t)\, dt = \infty \tag{24}$$

*at least for one* $j \in \{1, \ldots, m\}$.

Proof. I. The necessity follows immediately from Theorem 1.

II. The sufficient condition. Let us suppose that the conclusion of Theorem is false. Let $y(t)$ be a nonoscillatory solution of the equation (1). We may assume to be specific that $y[h_i(t)] > 0$ $(i = 1, \ldots, m)$ for $t \geqq t_1 \geqq t_0 \in R_+$. Then from the equation (1), in view of (2), (3) we have

$$y^{(2n)}(t) + p_j(t) f_j(y[h_j(t)]) \leqq 0\,, \quad t \geqq t_1 \tag{25}$$

and $y(t)$ is a solution of (25). By virtue of Theorem 2, the inequality (25) has no positive solution and this contradicts the fact that $y(t)$ is a positive solution of the equation (1). The proof of Theorem is complete.

**Theorem 4.** *Let $p$ satisfy* (2) *and, in addition,*

(26) (a) $h \in C^1[R_+, R]$, $h'(t) \geqq 0$ *for* $t \geqq T \in R_+$, $h(t) \leqq t$, $t \in R_+$, $\lim h(t) = \infty$ as $t \to \infty$,

(b) $f \in C^1[R, R]$, $zf(z) > 0$ for $z \neq 0$, $f'(z) \geqq 0$, $z \in R$,

(c) *for every* $\varepsilon > 0$

$$\int_\varepsilon^\infty \frac{dz}{f(z)} < \infty \quad \left[\int_{-\varepsilon}^{-\infty} \frac{dz}{f(z)} < \infty\right]$$

(27) (d)
$$\int^\infty [h(t)]^{2n-1}\, p(t)\, dt = \infty .$$

*Then the differential inequality* (A) [(B)] *has no positive* [negative] *solutions on* $[t_0, \infty)$ *for every* $t_0 \in R_+$.

Proof. Suppose that the conclusion of Theorem 4 is false. Assume that there exists a positive solution $y(t)$ of (A) for $t \geqq t_0 \in R_+$. [The case of (B) is treated similarly.] It follows from (26) that there exists $t_1 \geqq t_0$ such that $y[h(t)] > 0$ for $t \geqq t_1$. From (A), in view of (2) and (b) of Theorem 4 we get $y^{(2n)}(t) \leqq 0$ for $t \geqq t_1$. From the last inequality, by virtue of $y[h(t)] > 0$, $t \geqq t_1$, we can assert that the assumptions of Lemma 1 are fulfilled. Then (5), for $k = 2v + 1$, $i = 2v$ ($v \in \{0, 1, \ldots, n - 1\}$) implies

$$0 \leqq y^{(2v+1)}(t) \leqq \frac{(2v)!}{(t - t_1)^{2v}}\, y'(t), \quad t > t_1 .$$

By virtue of the last inequality there exists a constant $K$, $0 < K < 1$ and a number $t_2 > t_1$ such that

(28)
$$0 \leqq t^{2v}\, y^{(2v+1)}(t) \leqq K(2v)!\, y'(t), \quad t \geqq t_2, \quad v \in \{0, 1, \ldots, n - 1\} .$$

If we multiply (A) by $[h(t)]^{2n-1} f^{-1}[y_h(t)]$, integrate the resulting inequality from $a(\geqq \max\{t_2, T\})$ to $t$, use Lemma 1, the assumption (b) and omit negative numbers, we obtain

(29)
$$\int_a^t [h(s)]^{2n-1}\, p(s)\, ds \leqq c_1 + (2n - 1) \int_a^t y^{(2n-1)}(s)\, [h(s)]^{2n-2}\, h'(s) \times$$
$$\times f^{-1}[y_h(s)]\, ds \leqq c_1 + (2n - 1) \int_a^t y_h^{(2n-1)}(s)\, [h(s)]^{2n-2}\, h'(s) \times$$
$$\times f^{-1}[y_h(s)]\, ds \leqq c_1 + (2n - 1) \int_{h(a)}^t y^{(2n-1)}(x)\, x^{2n-2} f^{-1}(y(x))\, dx ,$$

where $c_1 = y^{(2n-1)}(a)[h(a)]^{2n-1} f^{-1}(y_h(a)) \geqq 0$.

If we integrate the last integral in (29) by parts $2(n - v - 1)$ times and neglect negative numbers, we obtain

(30)

$$\int_a^t [h(s)]^{2n-1} p(s)\,ds \leqq C + (2n-1)\dots(2v+1)\int_{h(a)}^t y^{(2v+1)}(x)\,x^{2v} f^{-1}(y(x))\,dx\,,$$

where $C$ is a positive constant.

From (30), in view of (28), we get

$$\int_a^t [h(s)]^{2n-1} p(s)\,ds \leqq C + K(2n-1)!\int_{h(a)}^t y'(x) f^{-1}(y(x))\,dx$$

$$\leqq C + K(2n-1)!\int_{y[h(a)]}^t dz / f(z) < \infty \quad \text{for} \quad t \to \infty\,.$$

It means that $\int_a^\infty [h(s)]^{2n-1} p(s)\,ds < \infty$, but this contradicts (27). This completes the proof of Theorem 4.

**Corollary 1.** *Let* $p_i$, $i = 1, \dots, m$ *satisfy* (2) *and, in addition,*

(31) (a) $h_i \in C^1[R_+, R]$, $h_i(t) \leqq t$ for $t \in R_+$, $h_i'(t) \geqq 0$ *for* $t \geqq T \in R_+$, $\lim h_i(t) = \infty$ *as* $t \to \infty$ $(i = 1, \dots, m)$,

(32) (b) $f_i$, $i = 1, \dots, m$ *satisfy the assumptions* (b), (c) *of Theorem* 4. *Then the equation* (1) *is oscillatory if*

$$\int^\infty [h_j(t)]^{2n-1} p_j(t)\,dt = \infty \tag{33}$$

*at least for one* $j \in \{1, \dots, m\}$.

Proof. Let us suppose that the conclusion of Corollary is false. Let $y(t)$ be a non-oscillatory solution of the equation (1) and let $y[h_i(t)] > 0$ $(i = 1, \dots, m)$ for $t \geqq t_1 \geqq t_0 \in R_+$. [The case $y(t) < 0$ is treated similarly.] Then from the equation (1), in view of (2), (32) we have (25) and $y(t)$ is a positive solution of (25). This contradics Theorem 4.

The proof of Corollary is complete.

Finally, we shall study the oscillatory properties of the differential equation

$$y^{(2n)}(t) + F(t, y_{h_1}(t), \dots, y_{h_m}(t)) = 0\,. \tag{34}$$

With regard to the equation (34) we assume that the following conditions are satisfied:

(35) $$F(t, x_1, \ldots, x_m) \begin{cases} \geqq \sum_{i=1}^{m} p_i(t)\, \varphi_i(x)\,, & x_i > 0\,, \quad i = 1, \ldots, m \\ \leqq \sum_{i=1}^{m} p_i(t)\, \psi_i(x)\,, & x_i < 0\,, \quad i = 1, \ldots, m \end{cases}$$

$$F(t, 0, \ldots, 0) \equiv 0\,,$$

where (a) $p_i(t)$, $i = 1, \ldots, m$, satisfy (2),

(b) $\varphi_i \in C[(0, \infty), (0, \infty)]$, $\psi_i \in C[(-\infty, 0), (-\infty, 0)]$, $i = 1, \ldots, m$,

**Theorem 5.** *Let the equation* (34) *satisfy* (35) *and, in addition,*

(i) $h_i$, $i = 1, \ldots, m$, *satisfy* (4), (13),

(ii) $\varphi_i(z)\, \psi_i(z)$, $i = 1, \ldots, m$, *are nondecreasing functions,*

(iii) *there exists* $\beta > 1$ *such that*

$$\liminf_{z \to \infty} \frac{|\varphi_i(z)|}{|z|^\beta} > 0\,, \quad \liminf_{z \to -\infty} \frac{|\psi_i(z)|}{|z|^\beta} > 0\,, \quad i = 1, \ldots, m\,.$$

*Then the equation* (34) *is oscillatory if* (24) *holds at least for one* $j \in \{1, \ldots, m\}$.

Proof. The proof of this Theorem is very similar to that of Theorem 2 and hence we omit it.

**Theorem 6.** *Let the equation* (34) *satisfy* (35) *and, in addition,*

(i) $h_i$, $i = 1, \ldots, m$, *satisfy* (31),

(ii) *there exist* $\varphi_i'(u)$, $\psi_i'(v)$ *and* $\varphi_i'(u) \geqq 0$ *for* $u > 0$, $\psi_i'(v) \geqq 0$ *for* $v < 0$, $i = 1, \ldots$ $\ldots, m$,

(iii) *for every* $\varepsilon > 0$

$$\int_\varepsilon^\infty \frac{du}{\varphi_i(u)} < \infty\,, \quad \int_{-\varepsilon}^{-\infty} \frac{dv}{\psi_i(v)} < \infty\,, \quad i = 1, \ldots, m\,.$$

*Then the equation* (34) *is oscillatory if* (33) *holds at least for one* $j \in \{1, \ldots, m\}$.

Proof. The proof of this Theorem is very similar to that of Theorem 4 and hence we omit it.

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