Časopis pro pěstování matematiky, roč. 99 (1974), Praha

REMARK ON LINEAR EQUATIONS IN BANACH SPACE

Štefan Schwabik, Praha

(Received July 20, 1972).

In this note Fredholm theorems for linear equations in a Banach space are established without requiring the knowledge of the usual adjoint space.

The main result (Theorem 3) concerns the operator $A = I + T$ where T is a compact (completely continuous) operator in a Banach space. In this theorem the usual adjoint operator is replaced by the operator which is conjugate to A with respect to a total space of continuous linear functionals on the Banach space. The investigation is based on some results about the dimensional characteristic of linear operators in a Banach space [4]. Reformulating the results from [4] in terms of linear equations we obtain a generalization of the well known Fredholm theorems.

Let X be a Banach space (over the field of real or complex numbers). The set of all linear operators A mapping X into itself such that Ax is defined for all $x \in X$ $(D_A = X)$ let be denoted by $L_0(X)$. Let $B_0(X)$ be the set of all bounded operators belonging to $L_0(X)$.

We denote by $N(A) = \{x \in X;\ Ax = 0\}$ the kernel of $A \in L_0(X)$, by $R(A) = \{y \in X;\ y = Ax,\ x \in X\}$ the range of $A \in L_0(X)$ and define $\alpha_A = \dim N(A)$, $\beta_A = \dim X/R(A)$ [1]). The index of $A \in L_0(X)$ is the number

$$\operatorname{ind} A = \beta_A - \alpha_A .$$

Let X' be the space of all linear functionals on X. A space $\Xi \subset X'$ of linear functionals on X is said to be total if $\xi(x) = 0$ for all $\xi \in \Xi$ implies $x = 0 \in X$.

For a given $A \in L_0(X)$ and a total space $\Xi \subset X'$ we define on Ξ the conjugate operator A' with values in X':

$$A'\,\xi(x) = \xi(Ax) \quad \text{for all} \quad x \in X \quad \text{and} \quad \xi \in \Xi .$$

[1]) By dim the dimension of a linear set is denoted, $X/R(A)$ means the quotient space.

By X^+ the Banach space of all continuous linear functionals on X is denoted; X^+ is evidently total. The conjugate operator to $A \in L_0(X)$ with respect to the space X^+ is denoted by A^+. If $A \in B_0(X)$ then evidently A^+ is continuous; i.e. $A^+ \in B_0(X^+)$.

The space X can be embedded into Ξ' (the space of linear functionals on Ξ) via the usual embedding $\varkappa : X \to \Xi'$, i.e. $\varkappa\, x(\xi) = \xi(x)$ for $x \in X$. The image $\varkappa X$ of X in Ξ' is a total space.

Let now $\Xi \subset X'$ be a total space. As above, for a given $A' \in L_0(\Xi)$ we can define $A'' : \varkappa X \to \Xi'$ and $A'^+ : \Xi^+ \to \Xi'$. If $A' \in B_0(\Xi)$ then $A'^+ \in B_0(\Xi^+)$.

If for a given $A \in L_0(X)$ and a total space $\Xi \subset X'$ we have $R(A') \subset \Xi$ then we say that the space Ξ is preserved by the conjugate operator A'. If this is the case then also the space $\varkappa X \subset \Xi'$ is preserved by A'' ($\varkappa X$ is a total space in Ξ').

Further it can be shown that $A''\varkappa x = \varkappa A x$ for all $x \in X$, i.e. the operator $A'' :$ $: \varkappa X \to \varkappa X$ conjugate to $A' \in L_0(\Xi)$ is (up to the natural embedding $\varkappa$) identical with the operator A if Ξ is preserved by A'.

Let us suppose that $\Xi \subset X^+$ is a total space (Ξ is normed with respect to the norm in X^+). Any $x \in X$ is assigned the linear functinonal $\varkappa x \in \Xi'$; the natural embedding $\varkappa : X \to \Xi'$ is a monomorphism (cf. [2]). Since we have $|\xi(x)| \leqq \|\xi\| \cdot \|x\|$, the functional $\varkappa x$ is continuous, i.e. $\varkappa x \in \Xi^+$. Moreover, the image $\varkappa X$ of X in Ξ^+ is a total space (cf. [2]).

Theorem 1. *Let X be a Banach space, $\Xi \subset X^+$ a total subspace of continuous linear functionals on X.*

Let $A \in B_0(X)$ and let Ξ be preserved by the conjugate operator A'.

If ind $A = 0$ *then either*

I. *the equation*

$$(1) \qquad Ax = \tilde{x}$$

has in X only one solution for any $\tilde{x} \in X$

or

II. *the equation*

$$(2) \qquad Ax = 0$$

has r linearly independent solutions in X (r is an integer).

If moreover ind $A' = 0$ *then in the case* I. *the equation*

$$(3) \qquad A'\xi = \xi$$

has also a unique soltuion in Ξ for any $\xi \in \Xi$ and in the case II. *the equation*

$$A'\xi = 0 \tag{4}$$

has r linearly independent solutions in Ξ.

Proof. The first part of this theorem is almost trivial. Indeed, if ind $A = 0$ then dim $N(A) =$ dim $X/R(A) = r$, where $r = 0$ or $r > 0$ is an integer. The case I. corresponds to $r = 0$ and the case II. to $r > 0$. The second part of this theorem is a consequence of Theorem 3 in [4] which assures that if ind $A =$ ind $A' = 0$ then dim $(N(A) =$ dim $N(A') =$ dim $\Xi/R(A')$.

Remark 1. Theorem 1 has the form of the usual Fredholm theorems. The first part is the well known alternative and it is only a trivial reformulation of the assumption ind $A = 0$. As for the second part let us mention that A' is not the usual adjoint operator. The classical (second) Fredholm theorem is a special case of our Theorem 1 if we set $\Xi = X^+$.

If the case II. in Theorem 1 occurs then some solvability conditions for the equation (1) are needed. Such conditions for the classical case are given by the third Fredholm theorem. Our aim is to obtain such a condition in terms of the conjugate equation (3).

Theorem 2. *Let X be a Banach space, $\Xi \subset X^+$ a total subspace. Let $A \in B_0(X)$, $R(A)$ is closed in X and $N(A^+) \subset \Xi$ (A^+ is the conjugate operator to A with respect to X^+.) Then the equation* (1) *has a solution if and only if the relation*

$$\xi(\tilde{x}) = 0 \tag{5}$$

holds for any solution $\xi \in \Xi$ of the equation (4).

Proof. Since $R(A)$ is closed, we have $R(A)^\perp = N(A^+)$, where $R(A)^\perp$ is the orthogonal complement of $R(A)$ in X^+; this is a well known fact (see for example [1]). Further we have evidently $N(A') = N(A^+) \cap \Xi$ and by the assumption $N(A^+) \subset \Xi$ we have $N(A') = N(A^+)$. This proves our theorem.

In the sequel we will formulate Fredholm theorems for the case $A = I + T$ where I is the identical operator in X and $T \in B_0(X)$ is compact.

Theorem 3. *Let X be a Banach space, $\Xi \subset X^+$ a total space which is also a Banach space. Let $T \in L_0(X)$ be a compact operator and let Ξ be preserved by the conjugate operator T'. Then the following assertion holds:*

Either

I. *the equation*

$$x + Tx = \tilde{x} \tag{6}$$

has in X only one solution for any $\tilde{x} \in X$

or

II. *the equation*

(7) $$x + Tx = 0$$

has r linearly independent solutions in X (r is an integer).

In the case I. *the equation*

(8) $$\xi + T'\xi = \tilde{\xi}$$

has also only one solution in Ξ for any $\tilde{\xi} \in \Xi$ and in the case II. *the equation*

(9) $$\xi + T'\xi = 0$$

admits r linearly independent solutions in Ξ.

Moreover, the equation (6) *has a solution in X if and only if $\xi(\tilde{x}) = 0$ for any solution $\xi \in \Xi$ of the equation* (9) (*and symmetrically* (8) *has a solution in Ξ if and only if $\tilde{\xi}(x) = 0$ for any solution $x \in X$ of the equation* (7)).

Proof. First let us mention that this theorem is well known if we set $\Xi = X^+$. Further it is known that under the present assumptions $A = I + T \in B_0(X)$ and ind $A = 0$. Moreover the operator $T' \in L_0(\Xi)$ is also compact (cf. Theorem 7,4 from C III in [3]). Hence ind $A' = 0$ where $A' = I + T'$ and all assumptions of Theorem 1 are fulfilled. This yields our theorem except the last part concerning the solvability conditions for the equation (6) and (8).

The proof of this part we obtain from Theorem 2. For the case of a compact $T \in L_0(X)$ it is known that $R(A)$ is closed, $A = I + T$ and similarly $R(A')$ is closed in Ξ, $A' = I + T' \in L_0(\Xi)$. It remains to prove that $N(A^+) \subset \Xi$ and $N(A'^+) \subset \varkappa X$. By definition we have

(10) $$N(A^+) \cap \Xi = N(A')$$

and therefore

(11) $$\dim N(A') \leqq \dim N(A^+) .$$

Similarly

(12) $$N(A'^+) \cap \varkappa X = N(A'')$$

and

(13) $$\dim N(A'') \leqq \dim N(A'^+) .$$

Since $\varkappa : X \to \Xi^+$ is a monomorphism we have $\dim N(A'') = \dim N(A)$. Hence the inequality (13) assumes the form

$$\dim N(A) \leqq \dim N(A'^+) . \tag{14}$$

Using (14), (11) and the equalities $\dim N(A) = \dim N(A^+)$, $\dim N(A') = \dim N(A'^+)$ which are consequences of the compactness of $T \in L_0(X)$, $T' \in L_0(\Xi)$ respectively we obtain

$$\dim N(A) \leqq \dim N(A'^+) = \dim N(A') \leqq \dim N(A^+) = \dim N(A)$$

and therefore

$$\dim N(A) = \dim N(A') = \dim N(A'^+) = \dim N(A^+).$$

These equalities together with (10) yields

$$\dim (N(A^+) \cap \Xi) = \dim N(A') = \dim N(A^+).$$

Hence $N(A^+) \subset \Xi$. Using (12) we obtain in the same way

$$\dim (N(A'^+) \cap \varkappa X) = \dim N(A'') = \dim N(A) = \dim N(A'^+)$$

and also $N(A'^+) \subset \varkappa X$.

Remark 2. Theorem 3 is a complete collection of Fredholm theorems for a compact operator $T \in L_0(X)$, the only difference between it and the usual Fredholm theorems being that it is sufficient to know a smaller total space of functionals $\Xi \subset X^+$ and the conjugate operator acting in this space.

We conclude this note by an example in which Theorem 2 and 1 is used.

Let BV be the usual linear space of all real functions defined on $[0, 1]$ with bounded variation. If we set

$$\|x\|_{BV} = |x(0)| + \operatorname{var}_0^1 x$$

for $x \in BV$ then $\|\,.\,\|_{BV}$ is a norm and BV is a Banach space. A satisfactory description of the conjugate space BV^+ of all continuous linear functionals on BV is not available.

We denote by S the set of all break functions $w(t)$ from BV for which we have $\lim_{\tau \to t+} w(\tau) = \lim_{\tau \to t-} w(\tau)$ for all $t \in (0, 1)$. The set S is closed in BV.

Let us form the quotient space BV/S. The elements of BV/S are denoted by capitals and they are classes of functions such that their difference belongs to S. The canonical mapping of BV into BV/S is denoted by ψ, i.e. for $\varphi \in BV$ we have $\psi(\varphi) = \varphi + S = \Phi \in BV/S$. The space BV/S forms a Banach space with the norm

$$\|\Phi\|_{BV/S} = \inf_{\psi(\varphi)=\Phi} \|\varphi\|_{BV} = \inf_{\psi(\varphi)=\Phi} \operatorname{var}_0^1 \varphi . \tag{15}$$

Let now $\Phi \in BV/S$. We define for $x \in BV$

$$\Phi(x) = \int_0^1 x(t) \, d\varphi(t) \tag{16}$$

where $\psi(\varphi) = \Phi$. The integral in (16) is the Perron-Stieltjes integral. All integrals occurring in the sequel are also Perron-Stieltjes integrals.

The relation (16) is independent of the choice of $\varphi \in BV$ with the property $\psi(\varphi) = \Phi$ (see [3], p. 326) and $\Phi(x)$ from (16) is evidently a linear functional on BV. Since $\Phi(x)$ from (16) is independent of the choice of the representant of the class Φ and the inequality

$$\left| \int_0^1 x(t) \, d\varphi(t) \right| \leqq \sup_{t \in [0,1]} |x(t)| \, . \, \operatorname{var}_0^1 \varphi$$

holds we have

$$|\Phi(x)| \leqq \|x\|_{BV} \cdot \|\Phi\|_{BV/S}$$

and the functional $\Phi(x)$ from (16) is continuous. The Banach space BV/S can be identified with a subspace in BV^+ which will be also denoted by BV/S ($BV/S \subset BV^+$).

If $x \in BV$, $x \neq 0$ then there is a $\Phi \in BV/S$ such that $\Phi(x) \neq 0$ (see Lemma 5,1 in [3]). Hence BV/S is a total space in BV^+.

For a given real function $k(s, t)$ defined on $[0, 1] \times [0, 1]$ ($k : [0, 1] \times [0, 1] \to R$) and a nondegenerate interval $J = [a, b] \times [c, d] \subset [0, 1] \times [0, 1]$ we set

$$m(J) = k(b, d) - k(b, c) - k(a, d) + k(a, c)$$

and define

$$v(k) = \sup \sum_i |m(J_i)|$$

where the supremum is taken over all finite systems of nonoverlapping intervals J_i in $[0, 1] \times [0, 1]$ (i.e. $J_i^0 \cap J_j^0 = \emptyset$ when $i \neq j$). The number $v(k)$ is a kind of two-dimensional variation (the so called Vitali variation) of the function k.

Let us suppose that $k : [0, 1] \times [0,]1 \to R$ is such a function that $v(k) < +\infty$ and $\operatorname{var}_0^1 k(0, \cdot) < +\infty$. We define the operator

$$Tx = \int_0^1 x(t) \, d_t k(s, t)$$

on BV. We have evidently $T \in L_0(BV)$ and by Theorem 3,1 from [3] the operator T is compact. Hence $\operatorname{ind}(I + T) = 0$.

If moreover $\operatorname{var}_0^1 k(\cdot, 0) < +\infty$ then for $\Phi \in BV/S$; $\psi(\varphi) = \Phi$ we have (cf. Lemma 2,2 in [3])

$$\Phi(Tx) = \int_0^1 \left(\int_0^1 x(t) \, d_t k(s, t) \right) d\varphi(s) = \int_0^1 x(t) \, d_t \left(\int_0^1 k(s, t) \, d\varphi(s) \right) = T' \, \Phi(x)$$

where

$$T'\Phi = \psi\left(\int_0^1 k(s, t)\, \mathrm{d}\varphi(s)\right), \quad \Phi = \psi(\varphi).$$

The operator T' is the conjugate of T and evidently preserves the conjugate space BV/S. By theorem 5,1 from [3] $T' \in B_0(\Xi)$ is compact. Hence $\operatorname{ind}(I + T') = 0$.

All the assumptions of Theorem 2 being satisfied we obtain easily the following

Theorem 4. *Let $k : [0, 1] \times [0, 1] \to R$ be such a function that $v(k) < +\infty$, $\operatorname{var}_0^1 k(0, .) < +\infty$, $\operatorname{var}_0^1 k(., 0) < +\infty$. Then either the equation*

$$x(s) + \int_0^1 x(t)\, \mathrm{d}_t k(s, t) = \tilde{x}(s) \tag{17}$$

has in BV only one solution for any $x \in BV$ or the homogeneous equation

$$x(s) + \int_0^1 x(t)\, \mathrm{d}_t k(s, t) = 0 \tag{18}$$

has r linearly independent solutions (r is an integer).

In the first case the equation

$$\varphi(t) + \int_0^1 k(s, t)\, \mathrm{d}\varphi(s) = \tilde{\varphi}(t) \tag{19}$$

has a solution (not unique) for any $\tilde{\varphi} \in BV$ and in the second case the equation

$$\varphi(t) + \int_0^1 k(s, t)\, \mathrm{d}\varphi(s) = 0 \tag{20}$$

admits r solutions in BV which are independent over the subspace S in BV[2]).

Moreover the equation (17) has a solution in BV iff

$$\int_0^1 \tilde{x}(t)\, \mathrm{d}\varphi(t) = 0$$

for any solution $\varphi \in BV$ of the equation (20) *and the equation (19) has a solution in BV iff*

$$\int_0^1 x(t)\, \mathrm{d}\tilde{\varphi}(t) = 0$$

for any solution $x \in BV$ of the equation (18).

---

[2]) The functions $\varphi_1, \ldots, \varphi_r \in BV$ are linearly independent over the subspace S if the relation $c_1\varphi_1 + \ldots + c_r\varphi_r \in S$($c_1, \ldots, c_r$ are real numbers) yields $c_1 = c_2 = \ldots = c_r = 0$.