ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

Vydává Matematický ústav ČSAV, Praha

SVAZEK 99 * PRAHA 10. 5. 1974 * ČÍSLO 2

ON QUASICONTINUOUS AND CLIQUISH FUNCTIONS

ANNA NEUBRUNNOVÁ, Bratislava

(Received December 30, 1971)

The notion of quasicontinuous and cliquish function will be used in the sense as it was introduced in [1] or [2]. Thus a function defined on a topological space X and assuming values in a topological space is said to be quasicontinuous at the point $x \in X$ if for any neighbourhood U of the point x and any neighbourhood V of $f(x)$ there is an open set $\emptyset \neq G \subset U$ such that $f(G) \subset V$. Further, a function f defined on a topological space X and assuming values in a metric space Y with the metric ϱ is said to be cliquish at a point $x \in X$ if to any positive ε and any neighbourhood U of the point x there is an open set $\emptyset \neq G \subset U$ such that $\varrho[f(x_1), f(x_2)] < \varepsilon$ for any two $x_1, x_2 \in G$. A function f is said to be quasicontinuous (cliquish) if it is quasicontinuous (cliquish) at each point $x \in X$.

The present paper consists of three parts. The first one concerns the pointwise limits of transfinite sequences of quasicontinuous and cliquish functions. In the second part, the mutual connections between the Denjoy property and the properties $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and $\boldsymbol{L}$ (the definitions see belov) are studied. The last part contains some assertions connected with the results of S. MARCUS ([1], [4]) and H. THIELMANN ([2]).

I.

In this part (X, ϱ) denotes a separable metric space and (Y, ϱ') any metric space. The functions which are dealt with are defined on X and assume values in Y. Let Ω be the first uncountable ordinal number. The transfinite sequence $\{a_\xi\}_{\xi<\Omega}$ of elements of a metric space Y with the metric ϱ' is said to be convergent and have a limit $a \in Y$ if for each $\varepsilon > 0$ there exists an ordinal number $\mu < \Omega$ such that for each ξ, $\mu \leqq \xi < \Omega$ the inequality $\varrho'(a_\xi, a) < \varepsilon$ holds.

A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ defined on a set T with the values in a metric space M is said to be (pointwise) convergent to a function f defined on T if $\{f_\xi(t)\}_{\xi<\Omega}$ is convergent to $f_\xi(t)$ for any $t \in T$.

Instead of the notation $\varrho'(a, b)$ for $a, b \in Y$ the notation $|a - b|$ will be used.

Theorem 1. *Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of quasicontinuous functions pointwise converging to a function f. Then f is quasicontinuous.*

Proof. Let f be not quasicontinuous at $x_0 \in X$. Then there is an $\varepsilon > 0$ and a $\delta > 0$ such that for any nonempty open set $G \subset K(x_0, \delta)$, ($K(x_0, \delta)$ denotes the sphere with the centre x_0 and the radius δ) there exists $t \in G$ with

(1) $$|f(t) - f(x_0)| \geqq \varepsilon \ .$$

Hence the set T of all t for which (1) is true is dense in $K(x_0, \delta)$. Let S be a countable dense subset of T. There is $\mu < \Omega$ such that for $\xi > \mu$

(2) $$f_\xi(x) = f(x)$$

for any $x \in S \cup \{x_0\}$. The last fact easily follows from the definition of transfinite convergence and the fact that $S \cup \{x_0\}$ is countable. (See e.g. [8], Lemma 1.)

Let $\xi_0 > \mu$ be any fixed ordinal number. The quasicontinuity of f_{ξ_0} at x_0 implies the existence of a nonempty open set $\emptyset \neq U \subset K(x_0, \delta)$ such that $|f_{\xi_0}(x) - f_{\xi_0}(x_0)| < \varepsilon$ for $x \in U$. Evidently $U \cap S \neq \emptyset$. For any $t \in U \cap S$ we have $|f_{\xi_0}(x_0) - f_{\xi_0}(t)| < \varepsilon$. But $f_{\xi_0}(x_0) = f(x_0)$, $f_{\xi_0}(t) = f(t)$ (in view of (2)), hence $|f(x_0) - f(t)| < \varepsilon$, which is a contradiction to (1).

Note. It is clear from the above proof that the separability of X assumed in the theorem may be substituded by local separability.

Theorem 2. *Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of cliquish functions pointwise converging to a function f. Then f is cliquish.*

Proof. Let f be not cliquish at a point x_0. Then there are $\varepsilon > 0$, $\delta > 0$ such that in any open set G, $\emptyset \neq G \subset K(x_0, \delta)$ there is at least one pair y, z such that

(1) $$|f(y) - f(z)| \geqq \varepsilon .$$

Let $\{G_n\}_{n=1}^{\infty}$ be a countable basis of open sets in $K(x_0, \delta)$. In each G_n there is a pair y_n, z_n such that

(2) $$|f(y_n) - f(z_n)| \geqq \varepsilon .$$

Consider the set S of all y_n and z_n $(n = 1, 2, 3, \ldots)$.

The set S is countable, hence an ordinal number $\mu < \Omega$ exists such that

(3) $$f_\xi(x) = f(x) \quad \text{for all} \quad x \in S$$

whenever $\xi > \mu$. Let $\xi_0 > \mu$. Since f_{ξ_0} is cliquish, there exists an open set U, $\emptyset \neq U \subset K(x_0, \delta)$ such that $|f_{\xi_0}(y) - f_{\xi_0}(z)| < \varepsilon$ for any pair $y, z \in U$. There exists $G_{n_0} \neq \emptyset$ such that $G_{n_0} \subset U$. Hence $|f_{\xi_0}(y_{n_0}) - f_{\xi_0}(z_{n_0})| < \varepsilon$ in contradiction to (2).

II.

The definitions of the properties $\boldsymbol{D}_0, \boldsymbol{D}_1, \boldsymbol{D}_2$ will be used as introduced in [3]. Let I_0 denote any interval on the real line. The function $f : I_0 \to R$ $(R = (-\infty, \infty))$ is said to have the Denjoy property $\boldsymbol{D}_0$ if for any $a, b \in R, a < b$ the set $\{x \in I_0 : a < f(x) < b\}$ is either empty or of positive Lebesgue measure (see [6]). Further, the function $f : I_0 \to R$ is said to have the property $\boldsymbol{D}_1(\boldsymbol{D}_2)$ if it has the property $\boldsymbol{D}_0$ for any interval $I \subset I_0$ which is closed (open) in I_0, i.e., if for any such interval and any $a, b \in R$, $a < b$ the set $\{x \in I : a < f(x) < b\}$ is either empty or of positive Lebesgue measure. The $\boldsymbol{L}$-continuity (T. Šalát) is defined as follows. The function f defined on an interval I is said to be $\boldsymbol{L}$-continuous at a point $x_0 \in I$ if for any $\varepsilon > 0$, $\delta > 0$, the set $\{x : x \in (x_0 - \delta,\ x_0 + \delta) \cap I;\ |f(x) - f(x_0)| < \varepsilon\}$ is of positive Lebesgue measure. In what follows the phrase „a function f has the property $\boldsymbol{L}$ at a point x_0" means that f is at x_0 $\boldsymbol{L}$-continuous. The connections between the properties $\boldsymbol{D}_i$ $(i = 0, 1, 2)$ and $\boldsymbol{L}$ will be studied.

Theorem 3. *Let f be defined on the interval I. Then the following implications are true*: $\boldsymbol{D}_1 \Rightarrow \boldsymbol{L} \Leftrightarrow \boldsymbol{D}_2 \Rightarrow \boldsymbol{D}_0$ *while the implications* $\boldsymbol{L} \Rightarrow \boldsymbol{D}_1$, $\boldsymbol{D}_0 \Rightarrow \boldsymbol{D}_2$ *do not hold.*

Proof. Let f have the property $\boldsymbol{D}_1$. Let $x_0 \in I$, $\varepsilon > 0$, $\delta > 0$. Consider the set $\{x : x \in I \cap (x_0 - \delta,\ x_0 + \delta);\ |f(x) - f(x_0)| < \varepsilon\}$.

There exists a closed interval K such that $x_0 \in K$, $K \subset (x_0 - \delta, x_0 + \delta)$ and $K \subset I$.

Since $\{x : x \in I \cap (x_0 - \delta,\ x_0 + \delta);\ |f(x) - f(x_0)| < \varepsilon\} \supset K \cap \{x : |f(x) - f(x_0)| < \varepsilon\} \ni x_0$ the property $\boldsymbol{D}_1$ yields $|\{x : x \in I \cap (x_0 - \delta,\ x_0 + \delta);\ |f(x) - f(x_0)| < \varepsilon| \geqq |K \cap \{x : |f(x) - f(x_0)| < \varepsilon| > 0$ (where $|E|$ denotes the outer Lebesgue measure if E is a set). Hence f is a $\boldsymbol{L}$-continuous at x_0.

Let f have the $\boldsymbol{L}$-property at any point $x \in I$. Let $a < b$ and $U \subset I$ be any open interval. If there is not $x_0 \in U$ such that $a < f(x_0) < b$ then $\{x : a < f(x) < b\} \cap U = \emptyset$, hence $|\{x : a < f(x) < b\} \cap U| = 0$. If there is such a point x_0 then choose $\delta > 0$ and $\varepsilon > 0$ such that $(x_0 - \delta,\ x_0 + \delta) \subset U$ and $a < f(x_0) - \varepsilon < f(x_0) + \varepsilon < b$. Considering the $\boldsymbol{L}$-continuity at x_0 we get

$$|\{x : a < f(x) < b\} \cap U| \geqq$$
$$\geqq |\{x : x \in (x_0 - \delta, x_0 + \delta);\ |f(x) - f(x_0)| < \varepsilon\}| > 0\,.$$

Thus f has the property $\boldsymbol{D}_2$.

Let f possess the property $\boldsymbol{D}_2$. Let $x_0 \in I$. We shall prove the $\boldsymbol{L}$-continuity at x_0. Let $\varepsilon > 0$, $\delta > 0$. Put $U = (x_0 - \delta,\ x_0 + \delta) \cap I$, $a = f(x_0) - \varepsilon$, $b = f(x_0) + \varepsilon$.

Under the assumption the set $\{x \in U : a < f(x) < b\}$ is of positive Lebesgue measure. Hence f possesses the property $\boldsymbol{L}$.

The implication $\boldsymbol{D}_2 \Rightarrow \boldsymbol{D}_0$ was proved in [3], where also an example showing that $\boldsymbol{D}_0 \Rightarrow \boldsymbol{D}_2$ is not true was given. In [3] we find also an example showing that, the implication $\boldsymbol{D}_2 \Rightarrow \boldsymbol{D}_1$ does not hold. The last fact together with what was proved above shows that $\boldsymbol{L} \Rightarrow \boldsymbol{D}_1$ is not true.

III.

The following theorem is proved in [4]:

Let a real function f be a derivative and let it be almost everywhere continuous on (a, b).

Then f is quasicontinuous on (a, b).

The proof of the quoted theorem does not use the fact that f is a derivative. It uses only the property D_0. Moreover, the proof works even if f is supposed only to be L-continuous and quasicontinuous on (a, b). Thus the following theorem holds.

Theorem 4. *Let f be almost everywhere continuous and L-continuous on (a, b). Then f is quasicontinuous on (a, b).*

Proof. Let $a < x_0 < b$, $\varepsilon > 0$. Put $A_\varepsilon = \{x : |f(x) - f(x_0)| < \frac{1}{2}\varepsilon\}$. Choose an open interval I such that $x_0 \in I \subset (a, b)$. The L-continuity implies $|I \cap A_\varepsilon| > 0$. Under the assumption f is almost everywhere continuous on (a, b), hence a number $\xi \in I \cap A_\varepsilon$ exists such that f is at ξ continuous. Hence an interval $J \subset I$ containing ξ as an interior point exists such that the oscillation of f on J is less then $\frac{1}{2}\varepsilon$. Thus for $x \in J$

$$|f(x) - f(x_0)| \leqq |f(x) - f(\xi)| + |f(\xi) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

The quasicontinuity of f at x_0 is proved.

Let us introduce a usefull example of a quasicontinuous and L-continuous function.

Example 1. Let C be the Cantor discontinuum. Define f on $\langle 0, 1\rangle$ as follows: $f(x) = 1$ if $x \in C$. If $a, b \in C$, $(a, b) \subset \langle 0, 1\rangle - C$, put $f(x) = 1$ for $x_1 = a + \frac{1}{3}(b - a) \leqq x \leqq a + \frac{2}{3}(b - a) = x_2$. In the intervals (a, x_1) and (x_2, b) let f be linear, $f(a+) = f(b-) = 0$ and such that it is continuous in (a, b).

It is clear that the quasicontinuity of f implies its L-continuity. A question arises whether the converse of Theorem 4 is true, i.e., whether the quasicontinuity implies the almost everywhere continuity. A slight modification of the above example shows that the answer is negative. It is sufficient to consider a function defined on $\langle 0, 1\rangle$ as in Example 1, where the set C is substituded by a nowhere dense set of positive measure which may be constructed in the usual way. An example of a quasicontinuous function which is discontinuous at each point of a set C with $|C| > 0$ is obtained.

Theorem 5. *Let f be defined on (a, b) and aproximately continuous at a point $x_0 \in (a, b)$. Then f is L-continuous at x_0.*

Proof. Let f be aproximately continuous at x_0. Then a set $H \subset (a, b)$ exists such that $x_0 \in H$, x_0 is a point of density of H and $\lim_{\substack{x \to x_0 \\ x \in H}} f(x) = f(x_0)$.

Thus to $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for $x \in H \cap$ $\cap (x_0 - \delta, x_0 + \delta) \cap (a, b)$. Since x_0 is a point of density for H the number δ can be chosen so that $|H \cap (x_0 - \delta', x_0 + \delta') \cap (a, b)| > 0$ for any $\delta' \leqq \delta$. Then for any $\delta'' > 0$

$$\{x : x \in (x_0 - \delta'', x_0 + \delta'') \cap (a, b);\ |f(x) - f(x_0)| < \varepsilon\} \supset$$
$$\supset \{x \in (x_0 - \delta^*, x_0 + \delta^*) \cap (a, b);\ |f(x) - f(x_0)| < \varepsilon\} \supset$$
$$\supset \{x \in (x_0 - \delta^*, x_0 + \delta^*) \cap H;\ |f(x) - f(x_0)| < \varepsilon\},$$

where $\delta^* = \min(\delta, \delta'')$. The last set is of positive measure.

The following theorem is a corollary of Theorems 4 and 5.

Theorem 6. *If f is aproximately continuous and almost everywhere continuous on (a, b), then it is quasicontinuous on (a, b).*

Note. A direct proof of Theorem 6 is given in [4] (Theorem 5). Now we shall give some assertions closely related to the results of [1] and [2].

In what follows, the symbols A_f, C_f, D_f denote the set of points of cliquishness, continuity and discontinuity of the function f, respectively. A result proved in [1] asserts that $A_f - C_f$ is of the first category in X. In [1], X is supposed to be a metric space and the function f is defined on X with values in a metric space Y. We give another proof of the mentioned theorem. We suppose X to be only a topological space. The oscillation $o_f(x)$ of the function f defined on a topological space and assuming the values in a metric space Y with the metric ϱ is defined by

$$o_f(x_0) = \inf_{O(x_0)} \{ \sup_{x, y \in O(x_0)} \varrho(f(x), f(y))\}$$

where $O(x_0)$ is any neighbourhood of x_0. Then f is continuous at x_0 if and only if $o_f(x_0) = 0$.

Theorem 7. *Let f be defined on a topological space X and assuming values in a metric space Y with the metric ϱ. Then $A_f - C_f$ is a G_δ set of the first category in X.*

Proof.

$$A_f - C_f = A_f \cap D_f = A_f \cap \bigcup_{k=1}^{\infty} \left\{x : o_f(x) \geqq \frac{1}{k}\right\} = \bigcup_{k=1}^{\infty} A_f \cap \left\{x : o_f(x) \geqq \frac{1}{k}\right\}.$$

Denote $M_k = A_f \cap \{x : o_f(x) \geqq 1/k\}$. It is sufficient to prove that M_k is nowhere dense. The set A_f is closed (see [7]). Choose $x \in X$. Let U be any neighbourhood of the point x. If $x \notin A_f$ then $x \in (X - A_f) \cap U \subset U$ where $(X - A_f) \cap U$ is open and has an empty intersection with M_k. If $x \in A_f$ then in view of the cliquishness of f, to any k and any neighbourhood U of the point x there exists an open set $V \subset U$ such that $\varrho(f(y_1), f(y_2)) < 1/2k$ for any $y_1, y_2 \in V$.

Hence $o_f(y) \leqq 1/2k \leqq 1/k$ for any $y \in V$. Hence $V \cap M_k = \emptyset$.