Časopis pro pěstování matematiky, roč. 99 (1974), Praha

AN INEQUALITY INVOLVING POSITIVE KERNELS

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(Received November 7, 1972)

1.

A classical result concerning finite series of positive numbers (see [3], Theorem 328, pp. 318–319) can be formulated as follows.

Let n be a positive integer and let $x = (\xi_1, \ldots, \xi_n)$, $\xi_j > 0$, $j = 1, \ldots, n$. Let P be a permutation matrix, i.e. let $Px = y \Leftrightarrow y_j = x_{i_j}$, $j = 1, \ldots, n$, where $(i_1, \ldots, i_n)$ is an ordered system of all of the integers $1, \ldots, n$. Then the relation)*

$$(Px, z) \geqq (Pe, e) \tag{1.1}$$

holds for every $z = (\zeta_1, \ldots, \zeta_n)$, $\zeta_j > 0, j = 1, \ldots, n$, for which

$$\zeta_j \xi_j = 1 \,, \quad 1, \ldots, n \,, \tag{1.2}$$

*where $x = (\xi_1, \ldots, \xi_n)$, $\xi_j > 0$, $e = (1, \ldots, 1)$. Furthermore, if P is indecomposable,**) then the equality sign in (1.1) takes place if and only if $x = z = ce$, c being a constant.*

Using a result of G. Birkhoff [1] saying that every doubly stochastic matrix $T = (t_{jk})$ can be expressed as a convex combination of permutation matrices P_k

$$T = \sum_{k=1}^{N} \lambda_k P_k \,, \quad 0 < \lambda_k < 1 \,, \quad \sum_{k=1}^{N} \lambda_k = 1 \,,$$

we deduce from (1.1) that the relation

$$(Tx, z) \geqq (Te, e) \tag{1.3}$$

*) Here we let $(x,z) = \sum_{j=1}^{n} \xi_j \zeta_j$.

**) See Remark 2 of the Appendix.

holds for every couple of vectors x and z for which (1.2) is fulfilled. If T is indecomposable then the equality sign in (1.3) takes place if and only if $x = ce$, c being a constant.

Let T be a matrix for which

$$\mu_j = \sum_{k=1}^{n} t_{jk} = \sum_{k=1}^{n} t_{kj}, \quad t_{jk} \geqq 0, \quad \mu_j > 0.$$

Then relation (1.3) remains to be valid also for this case.

Let T be an arbitrary nonnegative matrix and let $r(T)$ be its spectral radius. Let u_0 and v_0 be some nonnegative eigenvectors of T and its transposed matrix T' respectively corresponding to the spectral radius: $Tu_0 = r(T)\, u_0$, $T'v_0 = r(T)\, v_0$, $u_0 = (\eta_1, \ldots, \eta_n)$, $v_0 = (\nu_1, \ldots, \nu_n)$.

We easily verify that for the matrix $U = (u_{jk})$, where $u_{jk} = \nu_j t_{jk} \eta_k + \delta \delta_{jk}$, $k, j = 1, \ldots, n$, $\delta > 0$, the following relations

$$\sum_{k=1}^{n} u_{jk} = r(T)\, \nu_j \eta_j + \delta = \sum_{k=1}^{n} u_{kj}$$

hold. Thus we have that

$$(1.4) \qquad (Ux, z) \geqq (Ue, e)$$

holds for every couple x and z for which (1.2) is fulfilled. But (1.4) is equivalent to the relation

$$\sum_{j=1}^{n} \sum_{k=1}^{n} [\nu_j t_{jk} \eta_k \xi_j \zeta_k + \delta \delta_{jk} \xi_j \zeta_k] \geqq \sum_{j=1}^{n} \sum_{k=1}^{n} [\nu_j t_{jk} \eta_k + \delta \delta_{jk}]$$

and, since δ is arbitrary, we obtain

$$(1.5) \qquad \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_j t_{jk} \eta_k \xi_j \zeta_k \geqq \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_j t_{jk} \eta_k .$$

The procedure just shown is a slightly modified procedure used by M. Fiedler [2].

We summarize the previous results in the following theorem.

Theorem 1. *Let $T = (t_{jk})$ be an $n \times n$ matrix with nonnegative entries t_{jk}, $1 \leqq j$, $k \leqq n$. Let u_0 and v_0 be any nonnegative eigenvectors of T and its transposed T' respectively corresponding to the spectral radius $r(T)$. Let x be an arbitrary vector with positive coordinates and z let be such that (1.2) holds. Then the relation*

$$(1.6) \qquad (Vx, z) \geqq r(T)\, (u_0, v_0)$$

holds, where $V = (v_{jk})$ and

$$v_{jk} = \nu_j t_{jk} \eta_k, \quad u_0 = (\eta_1, \ldots, \eta_n), \quad v_0 = (\nu_1, \ldots, \nu_n).$$

If moreover T is indecomposable then the equality sign in (1.6) *takes place if and only if* $x = ce$, c being a constant.

We note that only the last assertion has to be proved. We shall not do this now because our aim is to prove a slightly more general result in Section 2.

Remark. Note that for $T = P$, where P is a permutation matrix, the relation (1.6) is identical (1.1) because $u_0 = v_0 = e$ in this case.

2.

Let μ be a nonnegative σ-additive regular measure on a σ-algebra $\mathfrak{M}$ of subsets of Ω, where Ω is a closed bounded subset of a Euclidean space $\mathscr{E}^n$. Let $\mathscr{Y} = \mathscr{L}^2(\Omega, \mu)$ be the Banach space of classes of μ-measurable μ-equivalent real-valued functions on Ω with the inner product

$$([u], [v]) = \int_\Omega u(s)\, v(s)\, \mathrm{d}\mu(s)$$

and the norm $\|[u]\|^2 = [u], [u])$, where u and v are any representatives for $[u]$ and $[v]$ in $\mathscr{L}^2(\Omega, \mu)$ respectively. In the following we shall not distinguish the notation for classes and their representatives.

Let $\mathscr{T} = \mathscr{T}(s, t)$ be a kernel on Ω. We set $Tx = y$ if $y(s) = \int_\Omega \mathscr{T}(s, t)\, x(t)\, \mathrm{d}\mu(t)$.

The following theorem is a consequence of a well known result due to M. G. Krein and M. A. Rutman [4].

Theorem 2. *Let T be a compact linear operator mapping $\mathscr{Y}$ into $\mathscr{Y}$ having the property that $x \in Y$, $x(s) \geqq 0$, μ-almost everywhere in Ω* (μ-a.e.) *implies that $y(s) \geqq 0$ μ-a.e. in Ω, where $y = Tx$. If* $\dim \mathscr{Y}$ *is infinite then let the spectral radius $r(T) =$* $= \max \{0, \sup [|\lambda| : \lambda$ *an eigenvalue of* $T]\}$ *be positive. Then there exist eigenfunctions u_0 and v_0 of T and its adjoint T^* respectively corresponding to $r(T)$ and we have that $u_0(s) \geqq 0$ and $v_0(s) \geqq 0$, μ-a.e. in Ω:*

$$Tu_0 = r(T)\, u_0\,, \quad T^* v_0 = r(T)\, v_0\,.$$

Definition. We call the kernel $\mathscr{T} = \mathscr{T}(s, t)$, $\mathscr{T}(s, t) \geqq 0$ $\mu \times \mu$-a.e. in $\Omega \times \Omega$ indecomposable if for every couple of nonnegative μ a.e. functions u and v, $u \not\equiv 0$, $v \not\equiv 0$, there is an iteration T^p such that $(T^p u, v) > 0$ [6]. We also call T indecomposable, or $\mathscr{K}$-indecomposable, where $\mathscr{K} = \{u \in \mathscr{Y} : u(s) \geqq 0$ μ-a.e.$\}$.

Remark. If T in Theorem 2 is an indecomposable integral operator on $\mathscr{L}^2(\Omega, \mu)$, then u_0 and v_0 are positive μ-a.e. in Ω and up to a multiple constant uniquely determined [7].

Definition. We say that kernel $\mathscr{T} = \mathscr{T}(s, t)$ has property (B) if $y = Tx$ is a bounded function whenever $x \in \mathscr{L}^2(\Omega, \mu)$.

Definition. We say the kernel $\mathscr{T} = \mathscr{T}(s, t)$, $s, t \in \Omega$, satisfies condition (C) if for every $\varepsilon > 0$ there is a continuous on $\Omega \times \Omega$ kernel $\mathscr{T}_\varepsilon = \mathscr{T}_\varepsilon(s, t)$ such that

$$\int_\Omega \left[\int_\Omega |\mathscr{T}(s, t) - \mathscr{T}_\varepsilon(s, t)|^2 \, \mathrm{d}\mu(t)\right] \mathrm{d}\mu(s) < \varepsilon^2 .$$

Our generalization of Theorem 1 is as follows.

Theorem 3. *Let $\mathscr{T} = \mathscr{T}(s, t)$ be a kernel having property (C). Let x be any μ-measurable μ-a.e. positive function on Ω. Then we have*

$$(2.1) \qquad \int_\Omega \int_\Omega \mathscr{T}(s, t) \, v_0(s) \, u_0(t) \, \frac{x(s)}{x(t)} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) \geqq r(T) \int_\Omega u_0(s) \, v_0(s) \, \mathrm{d}\mu(s) ,$$

where u_0 and v_0 satisfy

$$(2.2) \qquad \int_\Omega \mathscr{T}(s, t) \, u_0(t) \, \mathrm{d}\mu(t) = r(T) \, u_0(s) , \quad 0 \not\equiv u_0 \in \mathscr{L}^2(\Omega, \mu) \quad u_0(s) \geqq 0 \ \mu\text{-a.e. in } \Omega ,$$

$$\int_\Omega \mathscr{T}(s, t) \, v_0(s) \, \mathrm{d}\mu(s) = r(T) \, v_0(t) , \quad 0 \not\equiv v_0 \in \mathscr{L}^2(\Omega, \mu) , \quad v_0(s) \geqq 0 \ \mu\text{-a.e. in } \Omega ,$$

for $r(T) > 0$ and $u_0 \geqq 0$, $v_0 \geqq 0$ are quite arbitrary for $r(T) = 0$.

If moreover T is indecomposable and has property (B), then the equality sign in (2.1) takes place if and only if $x(s) =$ constant μ-a.e. in Ω.

Remarks. Because of our assumption (C) and because of the density of the set of all continuous functions on Ω in $\mathscr{L}^2(\Omega, \mu)$ it is easy to see that it is enough to prove the first part of Theorem 3 concerning the inequality (2.1) only for continuous kernels and continuous functions x's.

Obviously the relation (2.1) holds trivially whenever $r(T) = 0$ and thus there is nothing to be proved.

Since $r(T) = 1$, and $u_0 = v_0 = e$, where $e(s) = 1$ μ-a.e. in Ω for T being defined by a doubly stochastic kernel $\mathscr{T}$, i.e. by kernel $\mathscr{T}$ for which

$$\frac{1}{\mu(\Omega)} \int_\Omega \mathscr{T}(s, t) \, \mathrm{d}\mu(t) = \frac{1}{\mu(\Omega)} \int_\Omega \mathscr{T}(t, s) \, \mathrm{d}\mu(t) = e(s) ,$$

the relation (2.1) turns to be expressed as

$$(2.2) \qquad (Tx, z) \geqq (Te, e) ,$$

where

$$z(s) = \frac{1}{x(s)}, \quad \mu\text{-a.e. in } \Omega . \tag{2.3}$$

Proof of Theorem 3. Since the integrand in (2.1) is nonnegative there is nothing to prove if the integral on the left hand side diverges. Hence, let us assume the left hand side in (2.1) to be finite. According to the previous remark we may assume that $\mathcal{T}$ is a continuous in $\Omega \times \Omega$ kernel and x is a continuous function in Ω.

First let us assume that $\mathcal{T}$ is a doubly stochastic kernel. According to the mean value theorem we can find disjoint subsets $\Omega_j \subset \Omega$ in such a way to have

$$\int_\Omega \int_\Omega \mathcal{T}(s, t)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) = \sum_{j=1}^{N} \sum_{k=1}^{N} \mathcal{T}(s_j, t_k)\, \mu(\Omega_j)\, \mu(\Omega_k), \tag{2.4}$$

where $s_j \in \Omega_j$ and $t_k \in \Omega_k$, $j = 1, \ldots, N$, N being a positive integer.

Obviously we have

$$\sum_{k=1}^{N} \tau_{jk} = \sum_{k=1}^{N} \tau_{kj} = \mu_j, \tag{2.5}$$

where

$$\tau_{jk} = \mathcal{T}(s_j, t_k)\, \mu(\Omega_j)\, \mu(\Omega_k), \quad \mu_j = \mu(\Omega_j) > 0 .$$

According to (2.5) and (1.5) we have that

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{jk} \frac{\xi_j}{\xi_k} \geqq \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{jk} \tag{2.6}$$

holds for every vector $x = (\xi_1, \ldots, \xi_n)$, $\xi_j > 0$, $j = 1, \ldots, N$.

Let us choose $\varepsilon > 0$ arbitrary. Then we can find N large enough to have

$$\left| \int_\Omega \int_\Omega \mathcal{T}(s, t) \frac{x(s)}{x(t)}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) - \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{jk} \frac{x(s_j)}{x(t_k)} \right| < \varepsilon .$$

According to (2.6) it follows that

$$\int_\Omega \int_\Omega \mathcal{T}(s, t) \frac{x(s)}{x(t)}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) \geqq \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{jk} - \varepsilon =$$

$$= \int_\Omega \int_\Omega \mathcal{T}(s, t)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) - \varepsilon = \int_\Omega \int_\Omega \mathrm{d}\mu(s)\, \mathrm{d}\mu(t) - \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we get that

$$\int_\Omega \int_\Omega \mathcal{T}(s, t) \frac{x(s)}{x(t)}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) \geqq \int_\Omega \int_\Omega \mathrm{d}\mu(t)\, \mathrm{d}\mu(s)$$

and this is equivalent to (2.1).

Further, we assume that $\mathscr{T}$ satisfies the following conditions

$$\int_\Omega \mathscr{T}(s, t)\, \mathrm{d}\mu(t) = \int_\Omega \mathscr{T}(t, s)\, \mathrm{d}\mu(t) = \alpha(s)\,, \quad s \in \Omega\,, \tag{2.7}$$

where α is a nonnegative continuous function. It is easy to find a positive constant δ to make the following expression positive

$$\beta(s) = \delta - \int_\Omega \mathscr{T}(s, t)\, \mathrm{d}\mu(t) = \delta - \int_\Omega \mathscr{T}(t, s)\, \mathrm{d}\mu(t) = \delta - \alpha(s)\,, \quad s \in \Omega\,.$$

Note that $\beta(s) > 0$, or more precisely, $\inf \beta(s) > 0$. We define an operator Z by setting

$$(Zy)(s) = \frac{1}{\delta} \int_\Omega \mathscr{T}(s, t)\, y(t)\, \mathrm{d}\mu(t) + \frac{1}{\delta}\, \beta(s)\, y(s)\,, \quad s \in \Omega\,. \tag{2.8}$$

It is easy to verify that for $s \in \Omega$

$$(Ze)(s) = \frac{1}{\delta}\left[\beta(s) + \int_\Omega \mathscr{T}(s, t)\, \mathrm{d}\mu(t)\right] = \frac{1}{\delta}\left[\beta(s) + \int_\Omega \mathscr{T}(t, s)\, \mathrm{d}\mu(t)\right] = e(s)\,.$$

Similarly as in the case of doubly stochastic kernels we can show that the following relation holds

$$\int_\Omega x(s)\, [Zz](s)\, \mathrm{d}\mu(s) \geqq \int_\Omega e(s)\, [Ze](s)\, \mathrm{d}\mu(s)\,,$$

where $z(s) = 1/x(s)$, or else,

$$\int_\Omega \int_\Omega \mathscr{T}(s, t)\, \frac{x(s)}{x(t)}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) \geqq \int_\Omega \int_\Omega \mathscr{T}(s, t)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s)\,. \tag{2.9}$$

This is the required relation for the case considered.

Finally, let us consider a general continuous kernel $\mathscr{T}$. Let us set

$$U(s, t) = v_0(s)\, \mathscr{T}(s, t)\, u_0(t)\,, \quad s, t \in \Omega\,.$$

Then

$$\int_\Omega U(s, t)\, \mathrm{d}\mu(t) = r(T)\, u_0(s)\, v_0(s) = \int_\Omega U(t, s)\, \mathrm{d}\mu(t)\,, \quad s \in \Omega\,.$$

Thus the kernel U satisfies (2.7) with $\alpha(s) = r(T)\, u_0(s)\, v_0(s) \geqq 0$. By virtue of (2.9) we have that

$$\int_\Omega \int_\Omega U(s, t)\, \frac{x(s)}{x(t)}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) \geqq \int_\Omega \int_\Omega U(s, t)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s)$$

and this is equivalent to the required relation (2.1) which was to be proved.

To finish the proof of Theorem 3 we have to examine the case of an indecomposable operator T. We shall use the same machinery as before.

Let $\mathscr{T} = \mathscr{T}(s, t)$ be a doubly stochastic kernel. We assume that

$$(2.10)\quad \int_\Omega \int_\Omega \mathscr{T}(s, t) \frac{x(s)}{x(t)} \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) = \int_\Omega \int_\Omega \mathscr{T}(s, t)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) = \int_\Omega \int_\Omega \mathrm{d}\mu(t)\, \mathrm{d}\mu(s)\,,$$

or else

$$\frac{(V_x T V_x^{-1} e, e)}{(e, e)} = \frac{(Te, e)}{(e, e)} = 1\,,$$

where $e(s) = 1\mu$-a.e. in Ω and

$$(2.11)\qquad V_x u = v \Leftrightarrow v(s) = x(s)\, u(s)\,, \quad s \in \Omega\,, \quad x, v, u \in \mathscr{L}^2(\Omega, \mu)\,.$$

Since $x \in \mathscr{L}^2(\Omega, \mu)$ is up to positivity quite arbitrary, we also have that (assuming $\mu(\Omega) = 1$)

$$1 \leqq (V_x^{-1} T V_x e, e) = (V_x T^* V_x^{-1} e, e)\,,$$

where T^* is the adjoint of T. Obviously, $\frac{1}{2}[T + T^*]$ is stochastic and it follows that

$$\tfrac{1}{2}(V_x[T + T^*]\, V_x^{-1} e, e) \geqq \tfrac{1}{2}([T + T^*]\, e, e) = \tfrac{1}{2} r(T + T^*) = 1\,.$$

According to our assumptions $T + T^*$ is compact and symmetric. Thus,

$$r(T + T^*) = \max \left\{ \frac{((T + T^*)\, u, v)}{(u, v)} : u \in \mathscr{L}^2(\Omega, \mu),\ (u, v) \neq 0,\ v > 0 \right\}.$$

This fact together with $\mathscr{K}$-indecomposability of $T + T^*$ according to the definition of V_x implies that $y_0 = V_x^{-1} e$ being an eigenvector of $T + T^*$ corresponding to $r(T + T^*)$ is a multiple of $e : V_x^{-1} e = ce,\ c > 0$. In other words, $x(s) = \text{const.}$ μ-a.e. in Ω, and this was to be proved.

Further let $\mathscr{T}$ satisfy (2.7) with some positive function $\alpha = \alpha(s)$. Then for the operator V_x defined in (2.11) we have with appropriate β that

$$\int_\Omega x(s)\, [Zz]\,(s)\, \mathrm{d}\mu(s) = \int_\Omega e(s)\, [Zz]\,(s)\, \mathrm{d}\mu(s)\,,$$

where Z is defined by (2.8). We deduce that

$$\tfrac{1}{2}(V_x[Z + Z^*]\, V_x^{-1} e, e) \geqq \tfrac{1}{2}((Z + Z^*)\, e, e) = 1 = r(Z + Z^*) = \|Z + Z^*\|\,.$$

Since the null space $\mathfrak{N}(Z + Z^*) = \{v \in \mathscr{L}^2(\Omega, \mu) : (Z + Z^*)\, v - \|Z + Z^*\|\, v = 0\}$ is one dimensional (see [7] and also the Appendix), we conclude that $V_x^{-1} e$ being an eigenfunction of $(Z + Z^*)$ corresponding to the eigenvalue $\|Z + Z^*\|$ is a multiple of e: $V_x^{-1}e = ce$, $c > 0$. Thus, the assertion is proved in this case too.

We conclude the proof of Theorem 3 by observing that for a general indecomposable kernel $\mathscr{T} = \mathscr{T}(s, t)$ the kernel $\mathscr{T}_1 = \mathscr{T}_1(s, t) = v_0(s)\, \mathscr{T}(s, t)\, u_0(t)$ satisfies (2.7) and u_0 and v_0 are positive and uniquely determined up to a multiple factor. Thus from

$$(V_x T_1 V_x^{-1} e, e) = (T_1 e, e),$$

where

$$T_1 u = v \Leftrightarrow v(s) = \int_\Omega \mathscr{T}_1(s, t)\, u(t)\, \mathrm{d}\mu(t), \quad s \in \Omega,$$

the required relation $x(s) =$ constant μ-a.e. in Ω follows. This completes the proof of Theorem 3.

The relation (2.13) contained in the following Corollary is essentially used in some applications concerning cone preserving operators (see [2, 6]).

Corollary. *Let T be an integral operator whose kernel $\mathscr{T} = \mathscr{T}(s, t)$ satisfies property (C). Let u_0 and v_0 be some nonnegative eigenfunctions of T and its adjoint T^* respectively corresponding to the spectral radius $r(T)$. Then we have*

$$(2.13) \quad \int_\Omega \int_\Omega \mathscr{T}(s, t)\, v_0(t)\, u_0(s)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) \geqq \int_\Omega \int_\Omega \mathscr{T}(s, t)\, u_0(t)\, v_0(s)\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s).$$

If moreover $\mathscr{T}$ is indecomposable and satisfies condition (B) then equality sign in (2.13) takes place if and only if $u_0(s) = c\, v_0(s)$ μ-a.e. in Ω with some $c > 0$.

Proof. According to the indecomposability of $\mathscr{T}$ we know that u_0 and v_0 are positive μ-a.e. in Ω. We then put $x(s) = u_0(s)/v_0(s)$ and apply Theorem 3. This completes the proof.

3.

With some minor changes the results of Section 2 can be generalized to $\mathscr{L}^p(\Omega, \mu)$ spaces with $p \in (1, +\infty)$. We formulate a particular result in this direction concerning bounded kernels.

Let $p \in (1, +\infty)$ and $1/p + 1/p^* = 1$. Let $\mathscr{T} = \mathscr{T}(s, t)$ be a bounded nonnegative kernel on $\Omega \times \Omega$. Let u_0 be an eigenfunction of $\mathscr{T}$ and v_0 an eigenfunction of the transposed kernel $\mathscr{T}^*(s, t) = \mathscr{T}(t, s)$, $s, t \in \Omega$.

We call $\mathscr{T} = \mathscr{T}(s, t)$ to satisfy condition (C_p) if for every $\varepsilon > 0$ there is a continuous kernel $\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon(s, t)$ such that

$$\int_\Omega \left[\int_\Omega |\mathcal{T}(s, t) - \mathcal{T}_\varepsilon(s, t)|^{p^*} \mathrm{d}\mu(t)\right]^{p/p^*} \mathrm{d}\mu(s) < \varepsilon^p .$$

We say that a kernel $\mathcal{T} = \mathcal{T}(s, t)$ is indecomposable if for any couple $u \in \mathcal{L}^p(\Omega, \mu)$ and $v \in \mathcal{L}^p(\Omega, \mu)$, $u \not\equiv 0$, $v \not\equiv 0$, there is a positive integer $p = p(u, v)$ such that

$$0 < \int_\Omega \ldots \int_\Omega \mathcal{T}(s, t_1) \ldots \mathcal{T}(t_{p-1}, t_p)\, v(s)\, u(t_p)\, \mathrm{d}\mu(t_1) \ldots \mathrm{d}\mu(t_p)\, \mathrm{d}\mu(s) .$$

Theorem 4. *With the previous notation we have the following relation*

$$\int_\Omega \int_\Omega \mathcal{T}(s, t)\, v_0(s)\, u_0(t) \frac{x(s)}{x(t)}\, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) \geqq r(T) \int_\Omega u_0(s)\, v_0(s)\, \mathrm{d}\mu(s) , \tag{3.1}$$

where x is any μ-measurable positive function on Ω. If moreover, $\mathcal{T}$ is indecomposable and such that Tu is bounded for $u \in \mathcal{L}^p(\Omega, \mu)$ and x is bounded, then equality sign in (3.1) *takes place if and only if $x(s) =$ constant μ-a.e. in Ω.*

4. Appendix. We shall prove an assertion a corollary of which was already used in the proof of a part of the main result.

Let V be defined as follows.

$Vx = y \Leftrightarrow y(s) = f(s)\, x(s)$, $x \in \mathcal{L}^2(\Omega, \mu)$ and $f \in \mathcal{L}^\infty(\Omega, \mu)$, $f(s) \geqq 0$ μ-a.e. in Ω. Set $\mathcal{L}^2(\Omega)$ instead of $\mathcal{L}^2(\Omega, \mu)$.

Theorem 5. *Let U be a bounded operator on $\mathcal{L}^2(\Omega)$ mapping μ-a.e. nonnegative functions into μ-a.e. nonnegative ones. Let $x_0 \in \mathcal{L}^2(\Omega)$ be an eigenvector of U. Let x_0 have the property that $x_0(s) \geqq \beta(\varepsilon)\, \chi_{\Omega(\varepsilon)}(s)$ μ-a.e., where*

$$\Omega(\varepsilon) = \{t \in \Omega : f(t) > \sup \operatorname{ess} f - \varepsilon\}$$

for sufficiently small $\varepsilon > 0$ and where $\chi_{\Omega(\varepsilon)}$ is the characteristic function of $\Omega(\varepsilon)$ and $\beta(\varepsilon)$ is a positive constant. Furthermore, let for every μ-a.e. nonnegative $v \in \mathcal{L}^2(\Omega)$, $v \not\equiv 0$, there be an $\alpha(v) > 0$ such that

$$(Uv)(s) \geqq \alpha(v)\, x_0(s) \quad \mu\text{-a.e.} \tag{4.1}$$

Let $\mu(\Omega(\varepsilon)) > 0$ for all sufficiently small $\varepsilon > 0$. Then

$$r(T) > r(V) = \sup \operatorname{ess} f ,$$

where $T = U + V$.

Proof. We may assume that $\mu(\Omega(\varepsilon)) < +\infty$.

Let $\varrho > r(T)$. It is easy to see that for every $x \in \mathscr{L}^2(\Omega)$, $x \geqq 0$ μ-a.e. we have that

$$[R(\varrho, T)\,x](s) \geqq [U\,R(\varrho, U)\,x](s) + [R(\varrho, V)\,x](s) \quad \mu\text{-a.e.}$$

where $R(\varrho, A) = (\varrho I - A)^{-1}$ and A is a bounded linear operator on $\mathscr{L}^2(\Omega, \mu)$ and I is the identity operator. It follows that

$$\begin{aligned} R(\varrho, T)\,\chi_{\Omega(\varepsilon)} &\geqq R(\varrho, U)\,U\chi_{\Omega(\varepsilon)} + R(\varrho, V)\,\chi_{\Omega(\varepsilon)} > \\ &> \alpha(\chi_{\Omega(\varepsilon)})\,\frac{1}{\varrho - r(U)}\,x_0 + \frac{1}{\varrho - r(V) + \varepsilon}\,\chi_{\Omega(\varepsilon)} \geqq \\ &\geqq \left[\frac{\alpha(\chi_{\Omega(\varepsilon)})\,\beta(\varepsilon)}{\varrho - r(U)} + \frac{1}{\varrho - r(V) + \varepsilon}\right]\chi_{\Omega(\varepsilon)} = \gamma(\varrho)\,\chi_{\Omega(\varepsilon)}\,. \end{aligned}$$

According to Theorem 6.2 in [4] we conclude that

$$r(R(\varrho, T)) \geqq \gamma(\varrho)\,.$$

Obviously,

$$r(R(\varrho, V)) = \frac{1}{\varrho - r(V)}$$

and

$$\gamma(\varrho) - \frac{1}{\varrho - r(V)} = \frac{1}{[\varrho - r(V)]\,[\varrho - r(U)]\,[\varrho - r(V) + \varepsilon]} \times$$
$$\times\,\{\alpha(\chi_{\Omega(\varepsilon)})\,\beta(\varepsilon)\,[\varrho - r(V)]^2 + \varepsilon\,\alpha(\chi_{\Omega(\varepsilon)})\,\beta(\varepsilon)\,[\varrho - r(V)] - \varepsilon[\varrho - r(U)]\}\,.$$

We see that

$$\gamma(\varrho) - \frac{1}{\varrho - r(V)} > 0$$

for ϱ sufficiently large. This means that

$$r(R(\varrho, T)) > r(R(\varrho, V))$$

and since

$$r(R(\varrho, T)) = \frac{1}{\varrho - r(T)}$$

we deduce that

$$\frac{1}{\varrho - r(T)} > \frac{1}{\varrho - r(V)}$$

and this implies the required result. Theorem 5 is proved.

Remark 1. If U in Theorem 5 is compact then $T = U + V$ is a Radon - Nikolskii operator [5]. Thus, each spectral point λ for which $|\lambda| = r(T)$ is a pole of the resolvent