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ON POWERS OF NON-NEGATIVE MATRICES

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1. INTRODUCTION

Denote by $p(A)$ the number of positive elements of a matrix A. Let A be square non-negative. Then, obviously, the behaviour of the sequence $\{p(A^r)\}$ is fully determined by the combinatorial structure of the positive elements of A. In the paper [1], Z. ŠIDÁK has noticed that this sequence is not necessarily non-decreasing even when A is primitive. Further, the following theorem was deduced there:

Let A be an irreducible non-negative matrix containing at most one zero element in its main diagonal. Then $p(B) \leqq p(AB)$ for each non-negative matrix B of the same size as A and, consequently, the sequence $\{p(A^r)\}$ is non-decreasing.

It is the purpose of this note to strengthen the quoted results.

2. PRELIMINARIES

Let $A = (a_{ik})$, $B = (b_{ik})$ be matrices of the same size. Write $A \subseteqq B$ if for each pair of indices $b_{ik} = 0$ implies $a_{ik} = 0$. Let A be square non-negative. If $A^r \subseteqq A^{r+1}$ for each positive integer r then the sequence of matrices $\{A^r\}$ is said to be non-decreasing, the sequence of integers $\{p(A^r)\}$ being obviously non-decreasing.

Let $A = (a_{ij})$ be an $n \times n$ matrix. For each permutation $\{p_1, p_2, \ldots, p_n\}$ of $N = \{1, 2, \ldots, n\}$ the product $\prod_{i=1}^{n} a_{ip_i}$ is called a diagonal product of A. The well known Frobenius-König theorem states that all diagonal products of A are zero if and only if A contains an $p \times q$ zero submatrix such that $p + q > n$ (v. [2]).

Given an $n \times n$ matrix $A = (a_{ij})$, denote by $G(A)$ the directed graph consisting of vertices $\{1, 2, \ldots, n\}$ and edges $\{i, k\}$ for each $a_{ik} \neq 0$. This graph is frequently used to describe combinatorial properties of A. A sequence $\{v, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-1}, w\}$ of edges of $G(A)$ is called a connection from v to w of the length l. Denote $A^r = (a_{ik}^{(r)})$. Notice that if A is non-negative then there exists a connection from v to w of the length l in $G(A)$ if and only if $a_{vw}^{(l)} > 0$.