5. GRAPHS

Let G be a finite non-directed graph of n vertices. Having chosen a fixed ordering of its vertices, assign to G an $n \times n$ matrix $A_G = (a_{ik})$ such that $a_{ik} = 1$ if G contains an edge between the i-th and the k-th vertices, $a_{ik} = 0$ otherwise. This matrix is usually called the incidence matrix of G.

(5.1) *Let G be a finite non-directed graph of n vertices and A_G its incidence matrix. Then the number of hamiltonian circuits of G is equal to*

$$\frac{1}{2}\sum_{k=1}^{n}(-1)^{n-k}(k-1)! \sum_{N/M_1\ldots M_k} \det A_G(M_1)\ldots\det A_G(M_k)\,.$$

Proof. Let G' be a directed graph obtained from G by replacing each (non-directed) edge of G by a pair of oppositely directed edges. Evidently, cyp A_G coincides with the number of cycles of the length n in G'. Pairs of oppositely oriented cycles of G' are in one-to-one correspondence with hamiltonian circuits of G. The required expression is obtained by combining this with (3.1).

(5.2) *Let G be a finite non-directed graph of n vertices and A_G its incidence matrix. Let $i, j \in N$, $i \neq j$. Denote by A'_G the matrix obtained from A_G by deleting the i-th row and the j-th column. Then the number of hamiltonian paths between the i-th and the j-th vertices of G is equal to*

$$\frac{1}{2}\sum_{k=1}^{n}(-1)^{n+i+j-k}(k-1)! \sum \det A'_G(M_1)\det A_G(M_2)\ldots\det A_G(M_k)$$

where summation extends over all the partitions $M_1, \ldots, M_k$ of N such that $i, j \in M_1$.

Proof. Differentiate (3.1) with respect to a_{ij}. The obtained formula implies the required result similarly as (3.1) implies (5.1).

References

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