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AN INVERSION FORMULA, MATRIX FUNCTIONS, COMBINATORIAL IDENTITIES AND GRAPHS

ANTONÍN VRBA, Praha

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INTRODUCTION

In the present paper an elementary proof is given of the combinatorial inversion formula (2.1) which can also be deduced from the Möbius inversion formula (cf. [1], [2]). The proof makes use of the properties of common matrix functions. Conversely, this formula is applied to obtain some expressions of these matrix functions in terms of each other; especially, the permanent is expressed in terms of the principal minors of the same matrix and vice versa. These formulae yield some combinatorial identities. Further, the close relationship between graphs and matrices makes it possible to express the number of hamiltonian circuits of a non-directed finite graph in terms of the principal minors of its incidence matrix.

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1. PRELIMINARIES

Let M be a set. Denote by $M/M_1 \dots M_k$ the partition of M into $M_1, \dots, M_k$, i.e. the (non-ordered) k-tuple of non-void mutually disjoint sets $M_1, \dots, M_k$ whose union is M. By $|M|$ denote the cardinality of M. Denote by $M//M_1 \dots M_k$ the partition of M into $M_1, \dots, M_k$ such that $|M_i| > 1$ for each $1 \leqq i \leqq k$. By $S(M)$ denote the family of all non-void subsets of M. Denote by $s(|M|, k)$ the number of partitions of M into k parts. This number is usually called the Stirling number of the second kind (v. [3]).

Let n be a positive integer. Denote $N = \{1, 2, \dots, n\}$.

Let $A = (a_{ik})$ be an $n \times n$ matrix. As usual, denote by $\det A$ the determinant of A and by per A the permanent $\sum \prod_{i=1}^{n} a_{ip_i}$ of A (summation is extended over all permutations $\{p_1, \ldots, p_n\}$ of N). Further, consider the matrix functions

$$\operatorname{cyd} A = (-1)^{n-1} \sum \prod_{i=1}^{n} a_{ip_i}$$

and

$$\operatorname{cyp} A = \sum \prod_{i=1}^{n} a_{ip_i},$$

where summations extend over all cyclic permutations $\{p_1, \ldots, p_n\}$ of N. Let $V \in S(N)$. Denote by $A(V)$ the principal submatrix obtained from A by deleting the rows and columns with indices from $N - V$. Thus, under this notation, $A = A(N)$, $a_{ii} = A(\{i\})$. Observe that there are the following connections between the above matrix functions:

$$\det A = \sum_{k=1}^{n} \sum_{N/M_1 \ldots M_k} \operatorname{cyd} A(M_1) \ldots \operatorname{cyd} A(M_k) \tag{1.1}$$

$$\operatorname{per} A = \sum_{k=1}^{n} \sum_{N/M_1 \ldots M_k} \operatorname{cyp} A(M_1) \ldots \operatorname{cyp} A(M_k) \tag{1.2}$$

They are based on the fact that each permutation is, roughly speaking, a composition of cycles.

Denote by I the $n \times n$ identity matrix and by J the $n \times n$ matrix each element of which is 1. The number of cyclic permutations of N being equal to $(n-1)!$, it holds $\operatorname{cyp} J = \operatorname{cyp} J - I = (n-1)!$ for $n > 1$. The number of permutations $\{p_1, \ldots, p_n\}$ of N such that $p_i \neq i$ for each $i \in N$ being $d_n = n! \sum_{k=0}^{n} (-1)^k/k!$, it holds $\operatorname{per}(J - I) = d_n$. Obviously, $\det(J - I) = (-1)^{n-1}(n-1)$.

2. AN INVERSION FORMULA

(2.1) *Let N be a finite set. Let c, d be two function defined on $S(N)$ such that*

$$d(M) = \sum_{k=1}^{|M|} \sum_{M/M_1 \ldots M_k} c(M_1) \ldots c(M_k)$$

for each $M \in S(N)$. Then

$$c(M) = \sum_{k=1}^{|M|} (-1)^{k-1} (k-1)! \sum_{M/M_1 \ldots M_k} d(M_1) \ldots d(M_k)$$

for each $M \in S(N)$.

Proof. First of all, prove that

$$c(M) = \sum_{k=1}^{|M|} q_k \sum_{M/M_1 \ldots M_k} d(M_1) \ldots d(M_k) \tag{$*$}$$

for each $M \in S(N)$, the coefficients q_k satisfying the recurrence

$$q_1 = 1$$

$$q_k = -\sum_{s=2}^{k} \sum_{\{1\ldots k\}/V_1\ldots V_s} q_{|V_1|} \ldots q_{|V_s|} \quad \text{for} \quad 1 < k \leqq |N|\,.$$

The case $|M| = 1$ being obvious, suppose that $1 < |M| \leqq |N|$ and that the last statement is true for each M' such that $|M'| < |M|$. It follows

$$c(M) = d(M) - \sum_{k=2}^{|M|} \sum_{M/M_1\ldots M_k} c(M_1) \ldots c(M_k) =$$

$$= d(M) - \sum_{k=2}^{|M|} \sum_{M/M_1\ldots M_k} \Big(\sum_{s=1}^{|M_1|} q_s \sum_{M_1/V_1\ldots V_s} d(V_1) \ldots d(V_s)\Big) \ldots$$

$$\ldots \Big(\sum_{s=1}^{|M_k|} q_s \sum_{M_k/V_1\ldots V_s} d(V_1) \ldots d(V_s)\Big)\,.$$

Further,

$$c(M) = d(M) - \sum_{k=2}^{|M|} \sum_{s=2}^{k} \sum_{\{1\ldots k\}/V_1\ldots V_s} q_{|V_1|} \ldots q_{|V_s|} \sum_{M/M_1\ldots M_k} d(M_1) \ldots d(M_k)\,,$$

which completes the first part of the proof. Thus the coefficients q_i in $(*)$ are independent of M.

To compute them, notice that according to (1.1), the relation $(*)$ is true for the functions $c(V) = \operatorname{cyd} A(V)$ and $d(V) = \det A(V)$ for each $n \times n$ matrix A. The substitution $A = J$ yields $q_{|M|} = (-1)^{|M|-1}(|M| - 1)!$ for each $M \in S(N)$.

(2.2) *Let N be the a finite set. Let d, p be two functions defined on $S(N)$ such that*

$$\sum_{k=1}^{|M|} (-1)^{k-1} (k-1)! \sum_{M/M_1\ldots M_k} d(M_1) \ldots d(M_k) =$$

$$= \sum_{k=1}^{|M|} (-1)^{|M|-k} (k-1)! \sum_{M/M_1\ldots M_k} p(M_1) \ldots p(M_k)$$

for each $M \in S(N)$. Then

$$p(M) = \sum_{k=1}^{|M|} (-1)^{|M|-k} k! \sum_{M/M_1\ldots M_k} d(M_1) \ldots d(M_k)$$

for each $M \in S(N)$.

Proof. First of all, prove that

$$(**) \qquad p(M) = \sum_{k=1}^{|M|} (-1)^{|M|-k} r_k \sum_{M/M_1\ldots M_k} d(M_1) \ldots d(M_k)$$

for each $M \in S(N)$, the coefficients r_k satisfying the recurrence

$$r_1 = 1$$

$$r_k = (k-1)! + \sum_{s=2}^{k} (-1)^s (s-1)! \sum_{\{1 \ldots k\}/V_1 \ldots V_s} r_{|V_1|} \ldots r_{|V_s|} \quad \text{for} \quad 1 < k \leqq |N|.$$

The case $|M| = 1$ being obvious, suppose that $1 < |M| \leqq |N|$ and that the last statement is true for each M' such that $|M'| < |M|$. It follows

$$p(M) = \sum_{k=1}^{|M|} (-1)^{|M|-k} (k-1)! \sum_{M/M_1 \ldots M_k} d(M_1) \ldots d(M_k) +$$

$$+ \sum_{k=2}^{|M|} (-1)^k (k-1)! \sum_{M/M_1 \ldots M_k} p(M_1) \ldots p(M_k) =$$

$$= \sum_{k=1}^{|M|} (-1)^{|M|-k} (k-1)! \sum_{M/M_1 \ldots M_k} d(M_1) \ldots d(M_k) +$$

$$+ \sum_{k=2}^{|M|} (-1)^k (k-1)! \sum_{M/M_1 \ldots M_k} \left(\sum_{s=1}^{|M_1|} (-1)^{|M_1|-s} r_s \sum_{M_1/V_1 \ldots V_s} d(V_1) \ldots d(V_s) \right) \ldots$$

$$\ldots \left(\sum_{s=1}^{|M_k|} (-1)^{|M_k|-s} r_s \sum_{M_k/V_1 \ldots V_s} d(V_1) \ldots d(V_s) \right) =$$

$$= \sum_{k=1}^{|M|} (-1)^{|M|-k} \left((k-1)! + \sum_{s=2}^{k} (-1)^s (s-1)! \sum_{\{1 \ldots k\}/V_1 \ldots V_s} r_{|V_1|} \ldots \right.$$

$$\left. \ldots r_{|V_s|} \right) \sum_{M/M_1 \ldots M_k} d(M_1) \ldots d(M_k)$$

which completes the first part of the proof. Thus the coefficients r_i in $(**)$ are independent of M.

To compute them, notice that according to (1.1) and (1.2), the relation $(*)$ is true for the functions $c(V) = \text{cyd}\, A(V)$, $d(V) = \det A(V)$ as well as for the functions $c(V) = \text{cyp}\, A(V)$, $d(V) = \text{per}\, A(V)$ for each $n \times n$ matrix A. Further, according to (2.1), the relation $(**)$ is true for the functions $d(V) = \det A(V)$, $p(V) = \text{per}\, A(V)$. The substitution $A = J$ yields $r_{|M|} = |M|!$ for each $M \in S(N)$.

3. MATRIX FUNCTIONS

Besides of and owing to (1.1) and (1.2), there are the following connections between the functions of an arbitrary $n \times n$ matrix A. They are an easy consequence of the results of the preceding section.

$$\text{(3.1)} \qquad \text{cyd}\, A = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{N/M_1 \ldots M_k} \det A(M_1) \ldots \det A(M_k)$$

$$(3.2) \qquad \operatorname{cyp} A = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{N/M_1 \ldots M_k} \operatorname{per} A(M_1) \ldots \operatorname{per} A(M_k)$$

$$(3.3) \qquad \det A = \sum_{k=1}^{n} (-1)^{n-k} k! \sum_{N/M_1 \ldots M_k} \operatorname{per} A(M_1) \ldots \operatorname{per} A(M_k)$$

$$(3.4) \qquad \operatorname{per} A = \sum_{k=1}^{n} (-1)^{n-k} k! \sum_{N/M_1 \ldots M_k} \det A(M_1) \ldots \det A(M_k) .$$

4. COMBINATORIAL IDENTITIES

The substitution of the matrices I, J and $J - I$ into (1.1), (1.2), (3.1)–(3.4) yields the following combinatorial identities. Many of them can be, of course, rewritten and proved in a more natural way.

$$\sum_{k=1}^{n} (-1)^k \sum_{N/M_1 \ldots M_k} (|M_1| - 1)! \ldots (|M_k| - 1)! = 0 \quad (n > 1)$$

$$\sum_{k=1}^{n} \sum_{N/M_1 \ldots M_k} (|M_1| - 1)! \ldots (|M_k| - 1)! = n!$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{N//M_1 \ldots M_k} (|M_1| - 1)! \ldots (|M_k| - 1)! = n - 1$$

$$\sum_{k=1}^{n} \sum_{N//M_1 \ldots M_k} (|M_1| - 1)! \ldots (|M_k| - 1)! = d_n$$

$$\sum_{k=1}^{n} (-1)^k (k-1)! \, s(n, k) = 0 \quad (n > 1)$$

$$\sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{N/M_1 \ldots M_k} |M_1|! \ldots |M_k|! = (n-1)!$$

$$\sum_{k=1}^{n} (k-1)! \sum_{N/M_1 \ldots M_k} (|M_1| - 1) \ldots (|M_k| - 1) = (n-1)! \quad (n > 1)$$

$$\sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{N/M_1 \ldots M_k} d_{|M_1|} \ldots d_{|M_k|} = (n-1)! \quad (n > 1)$$

$$\sum_{k=1}^{n} (-1)^{n-k} k! \, s(n, k) = 1$$

$$\sum_{k=1}^{n} (-1)^k k! \sum_{N/M_1 \ldots M_k} |M_1|! \ldots |M_k|! = 0 \quad (n > 1)$$

$$\sum_{k=1}^{n} k! \sum_{N/M_1 \ldots M_k} (|M_1| - 1) \ldots (|M_k| - 1) = d_n$$

$$\sum_{k=1}^{n} (-1)^{k-1} k! \sum_{N/M_1 \ldots M_k} d_{|M_1|} \ldots d_{|M_k|} = n - 1 .$$