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AN APPLICATION OF HALLS' THEOREMS TO MATRICES

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INTRODUCTION

It is the purpose of this paper to simplify proofs and to extend results of K. Čulík's paper [1] in which matrices which are singular (non-singular) together with all the matrices of the same combinatorial structure of zero elements are characterized. The well-known theorem of P. Hall concerning the existence of a system of distinct representatives of a system of sets and its quantitative refinement of M. Hall, Jr. are exploited.

PRELIMINARIES

Let $A = (a_{ik})$ be a matrix the elements of which belong to a given integral domain I of the characteristic h. Denote by $P(A)$ the class of all the matrices $B = (b_{ik})$ over I of the same size as A such that, for each pair of indices, $a_{ik} = 0$ if and only if $b_{ik} = 0$.

Let A be square. Then it is said to be absolutely singular if each matrix from $P(A)$ is singular. If $P(A)$ consists entirely of non-singular matrices then A is said to be absolutely non-singular. A is said to be pseudo-triangular if it arises from a triangular matrix with non-zero elements in the main diagonal by permutation of its rows and columns.

Let n be a positive integer. Denote $N = \{1, 2, \ldots, n\}$.

Let A be an $n \times n$ matrix. Then for each permutation $\{p_1, p_2, \ldots, p_n\}$ of N the product $\prod_{i=1}^{n} a_{ip_i}$ is called a diagonal product of A. Let $r, c \in N$. Denote by A_{rc} the submatrix obtained from A by deleting the r-th row and the c-th column. Let $\emptyset \neq R \subseteq N$, $\emptyset \neq C \subseteq N$. Denote by A_{RC} the submatrix of A obtained from A by deleting the rows and columns with indices from $N - R$ and $N - C$ respectively. Thus $A = A_{NN}$, $A_{rc} = A_{N-\{r\},N-\{c\}}$.

Let $S = \{S_1, S_2, \ldots, S_n\}$ be a system of sets. An n-tuple $\{s_1, s_2, \ldots, s_n\}$ such that $s_i \in S_i$ for each $i \in N$ is usually called a system of distinct representatives of S. Denote the cardinality of a set Z by $|Z|$ and $t = \min_{i \in N} |S_i|$.

A system S possesses a system of distinct representatives if and only if $|\bigcup_{i\in K} S_i| \geqq |K|$ *for each* $K \subseteqq N$. (P. Hall 1935.)

Assume a system S possessing a system of distinct representatives. If t is infinite then there exist at least t systems of distinct representatives of S. If t is finite and $t > n$ *then there exist at least* $t!/(t-n)!$ *systems of distinct representatives of S. For* $t \leqq n$ *there exist at least t! systems of distinct representatives of S.* (M. Hall, Jr. 1948.)

Proofs of these well-known theorems are available e.g. in [2] or [3].

Given an $n \times n$ matrix A, denote by $S(A)$ the system $\{S_1(A), S_2(A), \ldots, S_n(A)$ where $S_i(A) = \{k \in N \mid a_{ik} \neq 0\}$. Evidently, systems of distinct representatives of $S(A)$ are in one-to-one correspondence with non-zero diagonal products of A Notice that $A_{RC} = 0$ if and only if $C \subseteqq N - \bigcup_{i\in R} S_i(A)$.

If $h = 2$ the concepts of absolute singularity (absolute non-singularity) and singularity (non-singularity) merge and so this case is not of considerable interest. Moreover, the considerations in what follows are not valid for $h = 2$. Thus assume henceforward $h \neq 2$.

COMBINATORIAL CHARACTERIZATIONS

The following properties of an $n \times n$ *matrix A are equivalent*:

1. *A is absolutely singular.*
2. *Each diagonal product of A is zero.*
3. *A contains a zero* $p \times q$ *submatrix such that* $p + q > n$.

Proof. $1 \to 2$. The case $n = 1$ being obvious, suppose that $n > 1$ and that the implication is true for $(n-1) \times (n-1)$ matrices. Let some diagonal product of A, say $\prod_{i=1}^{n} a_{ip_i}$, be non-zero. Then, according to the induction hypothesis, there exists $B \in P(A)$ such that $\det B_{np_n} \neq 0$. It is easy to see from the expansion of det B by the n-th row that non-zero elements of this row could have been chosen such that $\det B \neq 0$.

In the case $h = 0$ the following simpler proof is valid: If $\prod_{i=1}^{n} a_{ip_i} \neq 0$ then put $b_{ip_i} = 1$ for each $i \in N$ and $b_{ik} = 0$ or $b_{ik} = 2$ otherwise in such a way that $B = (b_{ik}) \in P(A)$. Evidently, det B is odd.

$2 \leftrightarrow 3$. (This equivalence is due to G. Frobenius or D. König.) Each diagonal product of A is zero if and only if the system $S(A)$ does not possess a system of distinct representatives. According to the theorem of P. Hall, this takes place if and only if $|\bigcup_{i\in K} S_i(A)| < |K|$ for some $K \subseteqq N$. Further, this is equivalent to the existence of

$\emptyset \neq K \subseteq N$ such that $N - \bigcup_{i \in K} S_i(A) \neq \emptyset$ and $|K| + |N - \bigcup_{i \in K} S_i(A)| > n$, the submatrix $A_{K, N - \bigcup_{i \in K} S_i(A)}$ being zero.

2 → 1. If each diagonal product of A is zero then so is each diagonal product of each $B \in P(A)$, hence det $B = 0$.

The following properties of a square matrix A are equivalent:

1. *A is absolutely non-singular.*
2. *Exactly one diagonal product of A is non-zero.*
3. *A is pseudo-triangular.*

Proof. Denote by n the order of A.

1 → 2. The case $n = 1$ being obvious, suppose that $n > 1$ and that the implication is true for $(n - 1) \times (n - 1)$ matrices. If A is absolutely non-singular then there is a row of A containing exactly one non-zero element, say a_{ik}. (Otherwise it is easy to construct a matrix $B \in P(A)$ such that all its row sums are zero.) Then det $A =$ $= \pm a_{ik}$ det A_{ik}, hence A_{ik} is absolutely non-singular and, by the induction hypothesis, exactly one diagonal product of A_{ik} is non-zero. It follows that A has exactly one diagonal product as well.

2 → 3. The case $n = 1$ being obvious, suppose that $n > 1$ and that the implication is true for $(n - 1) \times (n - 1)$ matrices. Assertion 2 is equivalent to the fact that $S(A)$ possesses exactly one system of distinct representatives. According to the theorem of M. Hall, there exists $i \in N$ such that $S_i(A)$ consists of exactly one element, say k, i.e., a_{ik} is the only non-zero element of the i-th row of A. Exactly one diagonal product of A_{ik} being non-zero, A_{ik} is pseudo-triangular by the induction hypothesis. Accordingly, A is pseudo-triangular as well.

3 → 1. Obvious.

ALGEBRAIC CHARACTERIZATIONS

Let $r < n$ be positive integers. Then the following properties of an $n \times n$ matrix are equivalent:

1. *A is absolutely non-singular.*
2. *For each $B \in P(A)$ there exist $\emptyset \neq R \subset N$, $\emptyset \neq C \subset N$, $|R| = |C|$ such that* det B_{RC} det $B_{N-R,N-C} \neq 0$ *and either* det B_{RQ} det $B_{N-R,N-Q} = 0$ *for each $Q \subset N$, $|Q| = |R|$, $Q \neq C$, or* det B_{QC} det $B_{N-Q,N-C} = 0$ *for each $Q \subset N$, $|Q| = |C|$, $Q \neq R$.*
3. *A arises by permutations of rows and columns from an A' such that for each matrix from $P(A')$ the product of its $r \times r$ minor by the complementary minor is non-zero if and only if these minors are principal.*