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ON THE LINE GRAPH OF THE SQUARE AND THE SQUARE OF THE LINE GRAPH OF A CONNECTED GRAPH

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Let $G = (V, X)$ be a nontrivial connected graph with p points and q lines. The square of G is the graph (V, X') where $uv \in X'$ if and only if the distance between u and v in G is either 1 or 2. The line graph of G is the graph (X, Z) where $xy \in Z$ if and only if x and y are adjacent lines in G. The square of G and the line graph of G will be denoted by G^2 and $L(G)$, respectively. Consequently, the line graph of the square of G and the square of the line graph of G will be denoted by $L(G^2)$ and $(L(G))^2$, respectively. In the present paper we shall prove that if $p \geqq 3$, then $L(G^2)$ is hamiltonian, and that if $q \geqq 3$, then $(L(G))^2$ is hamiltonian. (For the terminology of graph theory, see HARARY [1]; for some results relative to the present paper, see [1], [2], and [3].)

Lemma 1. *Let G be a connected graph with $p \geqq 3$ points and such that it contains a point u of degree 1 and a point w of degree $p - 1$. If v is a point of G such that $u \neq v \neq w$, then there exists a spanning path in $L(G)$ joining the points uw and vw of $L(G)$.*

Proof. The case when $p = 3$ is obvious. Assume that $p = n \geqq 4$ and that for $p = n - 1$ the lemma is proved. The case when G is a star is simple. Assume that G is not a star. Then there is a point t of G such that t has degree at least 2 and $v \neq t \neq w$. By $v_1, \ldots, v_k$ we denote the points of G different from w and adjacent to t. Obviously, there is a spanning path S in $L(G - t)$ joining the points uw and vw. There is a point rs of $L(G - t)$ such that $(rs)(v_1w)$ is a line in S. It is evident that either $v_1 \in \{r, s\}$ or $w \in \{r, s\}$. If $v_1 \in \{r, s\}$, then by P we denote the path $(rs)(tv_1) \ldots \ldots (tv_k)(tw)(v_1w)$. If $w \in \{r, s\}$, then by P we denote the path $(rs)(tw)(tv_k) \ldots \ldots (tv_1)(v_1w)$. If in S we replace the line $(rs)(v_1w)$ by the path P, we obtain a spanning path in $L(G)$ joining the points uw and vw.

Theorem 1. *Let G be a connected graph with $p \geqq 3$ points. Then $L(G^2)$ is hamiltonian.*

Proof. The case when $p = 3$ is obvious. Assume that $p = n \geqq 4$ and that for $p = n - 1$ the theorem is proved. The case when $G = K_p$ is simple. Assume that $G \neq K_p$. Then there is a point w of G with degree not exceeding $p - 2$ and such that $G - w$ is connected. By d and d' we denote the distance in G and in $G - w$, respectively. By F we denote the graph with the points t of G such that $d(t, w) \leqq 2$, and with the lines $\bar{t}\bar{\bar{t}}$ such that either $w \in \{\bar{t}, \bar{\bar{t}}\}$ and $1 \leqq d(\bar{t}, \bar{\bar{t}}) \leqq 2$, or $\bar{t} \neq w \neq \bar{\bar{t}}$ and $d(\bar{t}, \bar{\bar{t}}) = 2 < d'(\bar{t}, \bar{\bar{t}})$. Notice that the graphs $(G - w)^2$ and F are line-disjoint and that x is a line in G^2 if and only if it is a line either in $(G - w)^2$ or in F. There are points u and v of G such that v is adjacent to w in G, u is adjacent to v in G and $d(u, w) = 2$. Obviously, u and v are points both in $(G - w)^2$ and in F, and u has degree 1 in F. By Lemma 1, there is a spanning path S_0 in $L(F)$ joining uw with vw. Similarly, there is a spanning path S_1 in $L(F)$ joining vw with uw. By the induction hypothesis, there exists a hamiltonian cycle H in $L((G - w)^2)$. Consider a point rs of $L((G - w)^2)$ such that $(rs)(uv)$ is a line in H. If $u \in \{r, s\}$, then by P we denote the path $(rs)\, S_0(uv)$; if $v \in \{r, s\}$, then by P we denote the path $(rs)\, S_1(uv)$. It is easy to see that if in H we replace the line $(rs)(uv)$ by P we obtain a hamiltonian cycle in $L(G^2)$.

Lemma 2. *Let T be any tree with $q \geqq 3$ lines. Then $(L(T))^2$ is hamiltonian.*

Proof. The case when $q = 3$ is obvious. Let $q = n \geqq 4$ and assume that for any q, $3 \leqq q < n$, the lemma is proved. The case when T is a path is simple. We shall assume that T is not a path. Then T contains distinct points $v_0, \ldots, v_k$ such that $1 \leqq k \leqq q - 2$, $v_0 \operatorname{adj} v_1, \ldots, v_{k-1} \operatorname{adj} v_k$, v_0 has degree at least 3, v_k has degree 1, and if $0 < j < k$, then v_j has degree 2. By T_0 we denote the tree which we obtain from T by deleting the points $v_1, \ldots, v_k$. By $u_1, \ldots, u_i$ we denote the points which are adjacent to v_0 in T_0; obviously, $i \geqq 2$. There is a hamiltonian cycle H in $(L(T_0))^2$. It is easy to verify that H contains such a line xy of $(L(T_0))^2$ that x is incident with one of the points $u_1, \ldots, u_i$, and y is incident with v_0. By P we denote the path in $(L(T))^2$ such that if $k = 1$, then $P = x(v_0v_1)\, y$, and if $k \geqq 2$, then $P = x(v_0v_1)(v_2v_3) \ldots (v_{g-3}v_{g-2}) \cdot (v_{g-1}v_g)(v_hv_{h-1}) \ldots (v_2v_1)\, y$, where g is the greatest odd integer not exceeding k and h is the greatest even integer not exceeding k. If in H we replace xy by P, we obtain a hamiltonian cycle in $(L(T))^2$.

Theorem 2. *Let G be a connected graph with $q \geqq 3$ lines. Then $(L(G))^2$ is hamiltonian.*

Proof. Consider a spanning tree T_1 of G. Color the lines of T_1 in blue. Subdivide each uncolored line of G (if any) into two new lines and color one of them in blue and the other of them in yellow (the choice is arbitrary). By T_2 we denote the graph consisting of the blue lines. Obviously T_2 is a tree with at least 3 lines. It is easy to see that $L(T_2)$ is isomorphic to a spanning subgraph of $L(G)$. This implies that $(L(T_2))^2$ is isomorphic to a spanning subgraph of $(L(G))^2$. By Lemma 2, $(L(T_2))^2$ is hamiltonian. Hence the theorem follows.