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ON THE ZEROS OF GENERALIZED JACOBI'S ORTHOGONAL POLYNOMIALS

FRANTIŠEK PÚCHOVSKÝ, Žilina

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1. INTRODUCTION

1,1. We employ the following notation:

1. I is the closed interval $[-1, 1]$.

2. c_i $(i = 1, 2, \ldots)$ are positive constants independent of n as well as of $x \in I$ or of x in the interval in question.

3. $c_i(x)$ $(i = 1, 2, \ldots)$ are functions of the variable x such that

$$|c_i(x)| < c_1 .$$

The numbering of c_i a $c_i(x)$ is independent for every section.

1,2. In this paper the zeros of the orthonormal polynomials

$$Q_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}, \quad a_0^{(n)} > 0, \quad n = 0, 1, \ldots \tag{1,2a}$$

associated with the function

$$Q(x) = (1 - x)^\alpha (1 + x)^\beta e^{u(x)} = J(x) . e^{u(x)} \tag{1,2b}$$

on the interval I are investigated. Here $\alpha > -1$, $\beta > -1$ and $u(x)$ is a real function satisfying the following conditions:

1. $u'''(x)$ exists in the interval $[-1, 1]$.

2. If we put for brevity

$$\Delta_x f(t) = \frac{f(x) - f(t)}{x - t}, \quad v_1(t) = \Delta_x u''(t), \quad v_2(t) = \frac{\partial^2}{\partial x\, \partial t} \Delta_x u'(t), \quad v_3(t) = \Delta_x u'''(t),$$

then for $i = 1, 2, 3$

(1,2c) $$\min(\alpha, \beta) \geqq \tfrac{1}{2} \Rightarrow \int_I (t - t^2)^{3/2} |v_i(t)| \, dt = c_1(x)$$

and

(1,2d) $$\min(\alpha, \beta) < \tfrac{1}{2} \Rightarrow \Delta_x v_i(t) = c_2(x).$$

1,3. In my paper "On a class of generalized Jacobi's orthonormal polynomials"[1]) I have established the following differential equation for the above polynomials $Q_n(x)$:

(1,3a)

$$Q^{-1}(x) \frac{d}{dx} [(1 - x^2) Q_n'(x) Q(x) + (1 - x^2) b_n(x) Q_n'(x) + [\lambda_n^2 + a_n(x)] Q_n(x) = 0.$$

Herein

(1,3b) $$\lambda_n = \sqrt{(n(n + \alpha + \beta + 1))}$$

(We suppose n to be so large that λ_n is real.)

Further

(1,3c) $$a_n(x) = n\, c_3(x),$$

(1,3d) $$b_n(x) = n^{-1} c_4(x),$$

$b_n'(x)$ exists in the interval $[-1, 1]$ and

(1,3e) $$b_n'(x) = n^{-1} c_5(x).$$

1,4. We denote by $J_n(x)$ the orthonormal polynomial associated with the function $J(x)$ on the interval $[-1, 1]$. $J_n(x)$ are normalized Jacobi's polynomials.

1,5. The results of my investigations are contained in the second chapter. The theorems on the zeros of the polynomials $J_n(x)$ are a generalization of the known results of Szegö (See [7] p. 9 and [1] pp. 135–136).

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[1]) See Čas. pěst. mat. *97* (1972), 361–378.

2. THEOREMS ON THE ZEROS OF THE POLYNOMIALS $Q_n(x)$

2,1. *Let $\{x_{\nu,n}\}_{n=1}^{\infty}$ be the increasing sequence of the zeros of Bessel function $I_\nu(x)$ of the first kind and of order ν.*

Let $\{x_k^{(n)}\}_{k=1}^{n}$ be the increasing sequence of zeros of the polynomial $Q_n(x)$.

Let $k = 1, 2, \ldots$ be independent of n. Then for $n \to +\infty$

(2,1a) $$x_k^{(n)} = -1 + \frac{x_{\beta,k}^2}{2n^2}[1 + O(n^{-1})]$$

and

(2,1b) $$x_{m-k+1}^{(n)} = 1 - \frac{x_{\alpha,k}^2}{2n^2}[1 + O(n^{-1})] .$$

(*The constants in O depend on k.*)

The proof of this theorem is contained in Chapter 5.

2,2. *Let $Q_n(x) = J_n(x)$ where $J_n(x)$ is defined in Section 1,4. If we put*

(2,2a) $$j(\alpha, \beta) = j = \tfrac{1}{6}(\alpha^2 + 3\alpha\beta + 3\alpha + 3\beta + 2), \quad j_1 = j(\beta, \alpha),$$

then

(2,2b) $$x_k^{(n)} = -1 + \frac{x_{\beta,k}^2}{2n^2}\left[1 - \frac{\alpha+\beta+1}{n} - \frac{(\alpha+\beta+1)^2 + j_1}{n^2} - \frac{(\alpha+\beta+1)[2j_1 + (\alpha+\beta+1)^2]}{n^3}\right] - \frac{x_{\beta,k}^4}{24n^4}\left[1 - \frac{2(\alpha+\beta+1}{n}\right] + O(n^{-6})$$

and

(2,2c) $$x_{n-k+1}^{(n)} = 1 - \frac{x_{\alpha,k}^2}{2n^2}\left[1 - \frac{\alpha+\beta+1}{n} - \frac{(\alpha+\beta+1)^2 + j}{n^2} - \frac{(\alpha+\beta+1)[2j + (\alpha+\beta+1)^2]}{n^3}\right] + \frac{x_{\alpha,k}^4}{24n^4}\left[1 - \frac{2(\alpha+\beta+1)}{n}\right] + O(n^{-6}) .$$

The proof is in Chapter 6.

2,3. Theorem on the distance of the consecutive zeros of the function $Q_n(\sin z)$.

Notations.

(2,3a) $$|\alpha| \leqq \tfrac{1}{2} \Rightarrow \alpha_1 = 0 ; \quad |\alpha| > \tfrac{1}{2} \Rightarrow \alpha_1 = \tfrac{1}{2}\sqrt{(4\alpha^2 - 1)} ;$$

(2,3b) $$|\beta| \leqq \tfrac{1}{2} \Rightarrow \beta_1 = 0 ; \quad |\beta| > \tfrac{1}{2} \Rightarrow \beta_1 = -\tfrac{1}{2}\sqrt{(4\beta^2 - 1)} .$$

$\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$ are arbitrary real numbers independent of n;

(2,3c) $$a_n \in (\alpha_0, n), \quad b_n \in (-n, \beta_0)$$

are arbitrary numbers which may depend on n;

(2,3d) $$J_n = \left(-\frac{\pi}{4}, \frac{\pi}{2} - \frac{a_n}{n}\right), \quad J_n^{(1)} = \left(-\frac{\pi}{2} + \frac{b_n}{n}, \frac{\pi}{4}\right);$$

(2,3e) $$\lambda_n = \sqrt{(n(n + \alpha + \beta + 1))};$$

(2,3f) $$\varrho(x) = \lambda_n^2 + \frac{1 - 4\alpha^2}{4x^2}, \quad \varrho_1(x) = \lambda_n^2 + \frac{1 - 4\beta^2}{4x^2};$$

(2,3g) z_1 and z_2, $z_1 < z_2$ are arbitrary two consecutive zeros of the function

$$Q_n(\sin z).$$

Assertion.

(2,3h) $$[z_1, z_2] \subset J_n \Rightarrow z_2 - z_1 = \pi\varrho^{-1/2}\left(\frac{\pi}{2} - z_1\right) + \delta_1^{(n)}$$

and

(2,3i) $$[z_1, z_2] \subset J_n^{(1)} \Rightarrow z_2 - z_1 = \pi\varrho_1^{-1/2}\left(-\frac{\pi}{2} + z_1\right) + \delta_2^{(n)}.$$

Herein

(2,3j) $$|\delta_1^{(n)}| < cn^{-2}(na_n^{-3} + 1),$$

(2,3k) $$|\delta_2^{(n)}| < cn^{-2}(n|b_n|^{-3} + 1),$$

where c is a constant independent of n, a_n, b_n, z_1 and z_2, that is, c is the same number for any two consecutive zeros z_1, z_2 located in J_n and $J_n^{(1)}$ respectively.

For the proof see Chapter 7.

2,4. *Let $\delta \in (0, \pi/4)$ be a constant independent of n and*

(2,4a) $$J_\delta = \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right).$$

Then in terms of the notation of Section 2,3

(2,4b) $$[z_1, z_2] \subset J_\delta \Rightarrow z_2 - z_1 = \frac{\pi}{n} + \vartheta_n$$

where

(2,4c) $$|\vartheta_n| < cn^{-2},$$

c is a constant with the same properties as that in (2,3j) and (2,3k).

For the proof see Chapter 7.

2,5. *For the zeros of the function $J_n(\sin z)$ the following inequalities hold if we employ the notation introduced in Section 2,3*

(2,5a) $$|\delta_1^{(n)}| < cn^{-2}(na_n^{-3} + n^{-1}),$$

(2,5b) $$|\delta_2^{(n)}| < cn^{-2}(n|b_n|^{-3} + n^{-1}),$$

where $\delta_1^{(n)}$, $\delta_2^{(n)}$ are defined by (2,3h) *and* (2,3i) *respectively.*

For the proof see Chapter 7.

3. A TRANSFORMATION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION

3,1. We shall employ the following notations

(3,1a) $$z = \arcsin x,$$

(3,1b) $$y' = \frac{dy}{dz}, \quad y'' = \frac{d^2y}{dz^2},$$

(3,1c) $$\omega(z) = (1 + \alpha + \beta)\,\mathrm{tg}\, z + (\alpha - \beta)\sec z,$$

(3,1d) $$J(x) = (1 - x)^\alpha (1 + x)^\beta,$$

(3,1e) $$q(x) = \sqrt{(\cos z\, J(\sin z))} = \exp\left[-\frac{1}{2}\int_0^z \omega(t)\, dt\right],$$

(3,1f) $$\gamma(z) = \tfrac{1}{2}[\omega'(z) - \tfrac{1}{2}\omega^2(z)],$$

(3,1g) $$\alpha_n(z) = \lambda_n^2 + a_n(\sin z) + \gamma(z) - \tfrac{1}{2}[b_n'(\sin z) + u''(\sin z)]\cos^2 z -$$
$$- \tfrac{1}{4}[b_n(\sin z) + u'(\sin z)]\{[b_n(\sin z) + u'(\sin z)]\cos^2 z - 2\omega(z)\cos z - 2\sin z\}.$$

$$\left(\text{Here } b_n'(x) = \frac{db_n(x)}{dx},\ u^{(k)}(\sin z) = \frac{d^k[u(x)]}{dx^2} \quad (k = 1, 2).\right)$$

(3,1h) $$q_n(z) = Q_n(\sin z)\, q(z) \exp\left\{\frac{1}{2}\int_{\pi/2}^z [b_n(\sin t) + u'(\sin t)]\cos t\, dt\right\}.$$

3,2. *In the above notation the function $Q_n(\sin z)$ is a solution of the differential equation*

(3,2a) $$y'' + \{[u'(\sin z) + b_n(\sin z)]\cos z - \omega(z)\}\, y' + [\lambda_n^2 + a_n(z)]\, y = 0$$

and the function $q_n(z)$ satisfies the differential equation

(3,2b) $$y'' + \alpha_n(z)\, y = 0\,.$$

Proof follows from (1, 3a).

3,3. Remark. In the following all the assertions are derived for $x \in [0, 1]$, that is for $z \in [0, \pi/2]$. The same assertions hold for $z \in [-\pi/2, 0]$ if we replace α by β.

3,4. *For $\zeta \to 0+$*

(3,4a) $$q\left(\frac{\pi}{2} - \zeta\right) = 2^{(\beta-\alpha)/2}\,.\,\zeta^{\alpha+1/2}[1 + O(\zeta^2)]\,,$$

(3,4b) $$\omega\left(\frac{\pi}{2} - \zeta\right) = (1 + 2\alpha)\,\zeta^{-1} - \tfrac{1}{6}(\alpha + 3\beta + 2)\,\zeta + O(\zeta^3)\,,$$

(3,4c) $$\gamma\left(\frac{\pi}{2} - \zeta\right) = \tfrac{1}{4}(1 - 4\alpha^2)\,\zeta^{-2} + j + O(\zeta^2)\,,$$

where j is defined by (2,2a).

Proof. Trivial.

3,5. *For brevity, put*

(3,5a) $$\omega_n(\zeta) = \alpha_n\left(\frac{\pi}{2} - \zeta\right) - \lambda_n^2 + \frac{4\alpha^2 - 1}{4\zeta^2}\,.$$

Then

(3,5b) $$\zeta \in \left[0\, \frac{\pi}{2}\right] \Rightarrow |\omega_n(\zeta)| < c_1 n\,.$$

The proof follows from (3,5a), (1,3c), (1,3d), and (1,3e).

3,6. *Let $|\alpha| > \frac{1}{2}$. Denote by $\alpha^{(n)}$ the greatest real zero of the function $\alpha_n(z)$ defined by (3,1g). Then for $n \to +\infty$*

(3,6a) $$\alpha^{(n)} = \frac{\pi}{2} - \frac{\alpha_1}{n}\left[1 + O\left(\frac{1}{n}\right)\right],$$

where for brevity

(3,6b) $$\alpha_1 = \tfrac{1}{2}\sqrt{(4\alpha^2 - 1)}\,.$$

Remark. For almost all values of n there exists one and only one positive zero of $\alpha_n(z)$ (provided $|\alpha| > \frac{1}{2}$).

Proof. According to (3,5a) and (3,5b) it is

$$\frac{\pi}{2} - \alpha^{(n)} = \frac{\lambda_n^{-1}}{2}\left\{(4\alpha^2 - 1)\Big/\left[1 + \lambda_n^{-2}\omega_n\left(\frac{\pi}{2} - \alpha^{(n)}\right)\right]\right\}^{1/2} = \frac{\alpha_1}{n}\,[1 + O(n^{-1})]\,.$$

3,7. *Let $|\alpha| > \frac{1}{2}$ and let $\alpha_0 > \alpha_1$ be a constant independent of n, where α_1 is defined by* (3,6b). *Then for $z \in [0, \pi/2 - \alpha_0/n]$*

(3,7a) $$0 < \alpha_n^{-1}(z) < c_1 n^{-2}$$

for almost all values of n.

If $\alpha \leqq -\frac{1}{2}$, then (3,7a) *holds for every $\alpha_0 > 0$.*

Proof. Put

$$f(x) = \frac{1 - 4\alpha^2}{4x^2}\,.$$

Hence $f(\alpha_1) = -1$. Since $f(x)$ is an increasing function for $x > 0$, there exists in virtue of (3,5a) and (3,5b) a constant $c > 0$ independent of n such that for almost all values of n

$$\zeta \in \left(\frac{\alpha_0}{n}, \frac{\pi}{2}\right) \Rightarrow \alpha_n\left(\frac{\pi}{2} - \zeta\right) = \lambda_n^2 + f(\zeta) + \omega_n(\zeta) > \lambda_n^2 + n^2[f(\alpha_0) - f(\alpha_1)] +$$

$$+ n^2 f(\alpha_1) - cn = \lambda_n^2 - n^2 - cn + \frac{\alpha_0^2 - \alpha_1^2}{4\alpha_0^2\alpha_1^2}(4\alpha^2 - 1)\,n^2 > \frac{\alpha_0^2 - \alpha_1^2}{8\alpha_1^2\alpha_0^2}(4\alpha^2 - 1)\,n^2\,.$$

3,8. *For brevity, put*

(3,8a) $$\psi_n(x) = q_n\left(\frac{\pi}{2} - x\right).$$

Then for $x \to 0+$

(3,8b) $$\psi_n(x) = 2^{(\beta-\alpha)/2} x^{\alpha+1/2}\, Q_n(1)\,[1 + O(x^2)]\,,$$

where

(3,8c) $$Q_n(1) > 0\,.$$

Proof. For brevity, put

(1) $$\varepsilon_n(x) = \exp\left\{-\frac{1}{2}\int_{\pi/2-x}^{\pi/2} [b_n(\sin t) + u'(\sin t)\cos t\,dt\right\} =$$

$$= \exp\left\{-\frac{1}{2}\int_0^x [b_n(\cos t) + u'(\cos t)]\sin t\,dt\right\} = 1 + O(x^2) \quad \text{for} \quad x \to 0+\,.$$

Further

(2) $$Q_n(\cos x) = Q_n(1) + O(x^2).$$

Since

$$\psi_n(x) = Q_n(\cos x)\, q\left(\frac{\pi}{2} - x\right) \varepsilon_n(x),$$

(3.8b) follows from (1), (2) and (3,4a).

By a well known theorem

$$Q_n(x) \neq 0 \quad \text{for} \quad x \geqq 1$$

and in virtue of (1,1a) it is $Q_n(+\infty) = +\infty$. This shows that (3,8c) is true.

4. LEMMAS

4,1. In the following we employ the Bessel functions $I_\alpha(x)$ of the order α and of the first kind as well as the Bessel functions $Y_\alpha(x)$ of the order α and the second kind.

It is well known that

(4,1a) $$I_\alpha(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{\alpha+2\nu}}{\nu!\Gamma(\alpha + \nu + 1)}$$

and provided $\alpha \geqq 0$ is an integer,

(4,1b) $$Y_\alpha(x) = \frac{2}{\pi}\left[C + \lg \frac{x}{2}\right] I_\alpha(x) - \frac{1}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{\alpha+2\nu}}{\nu!\,(\nu + \alpha)!}\, \sigma_\nu - S_\alpha(x).$$

Herein C is the Euler constant and

$$\alpha > 0 \Rightarrow \sigma_0 = \sum_{k=1}^{\alpha} \frac{1}{k}, \quad \alpha = 0 \Rightarrow \sigma_0 = 1,$$

$$\nu > 0 \Rightarrow \sigma_\nu = \sum_{k=1}^{\nu} \frac{1}{k} + \sum_{k=1}^{\nu+\alpha} \frac{1}{k},$$

$$S_0(x) = 0, \quad \alpha > 0 \Rightarrow S_\alpha(x) = \frac{1}{\pi} \sum_{\nu=0}^{\alpha-1} \frac{(\alpha - \nu - 1)! \left(\frac{x}{2}\right)^{2\nu-\alpha}}{\nu!}.$$

4,2. Put

(4,2a) $$v(x) = \sqrt{(x)}\, I_\alpha(x)$$

and if α is not an integer,

(4,2b) $$w(x) = \sqrt{(x)}\, I_{-\alpha}(x)\,.$$

If α is an integer, then

(4,2c) $$w(x) = \sqrt{(x)}\, Y_\alpha(x)\,.$$

$v(x)$ and $w(x)$ are linearly independent solutions of the differential equation

(4,2d) $$y'' + \left(1 + \frac{1 - 4\alpha^2}{4x^2}\right) y = 0\,.$$

(See [I] pp. 29–30.)

It is easily seen that for any real number k the functions $v(kx)$ and $w(kx)$ are linearly independent solutions of the differential equation

(4,2e) $$y'' + \left[k^2 + \frac{1 - 4\alpha^2}{4x^2}\right] y = 0\,.$$

(See [I] p. 31.)

4,3. The following theorem will be used:

Let $p(x)$ and $q(x) < 0$ be real functions continuous on the interval (a, b) and let $\varphi(x)$ be a solution of the differential equation

(4,3a) $$y'' + p(x)\, y' + q(x)\, y = 0\,.$$

Then the function $\varphi(x)\,.\,\varphi'(x)$ has at most one zero in the closed interval $[a, b]$. Herein a or b are also zeros of $\varphi(x)\,\varphi'(x)$ if for $i = 0,1$

$$\lim_{x\to a+} \varphi^{(i)}(x) = 0 \quad \text{or} \quad \lim_{x\to b-} \varphi^{(i)}(x) = 0\,.$$

Proof. (See [2] pp. 164–165.)

4,4. *Let $\{x_{\alpha,n}\}_{n=1}^{\infty}$ and $\{x'_{\alpha,n}\}_{n=1}^{\infty}$ be the increasing sequences of all the positive zeros of the functions $v(x)$ and $v'(x)$ respectively.*

Let $\{\zeta_n\}_{n=0}^{\infty}$ and $\{\zeta'_n\}_{n=0}^{\infty}$ be the increasing sequences of all the positive zeros of the functions $\psi_n(x)$ and $\psi'_n(x)$ respectively.[2]

If $|\alpha| > \frac{1}{2}$, then

(4,4a) $$x_{\alpha,1} > x'_{\alpha,1} > \tfrac{1}{2}\sqrt{(4\alpha^2 - 1)} = \alpha_1\,.$$ [3]

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[2]) See (3,8a).

[3]) See (3,6b).

and

(4,4b) $$\tilde{\alpha}_n\left(\frac{\pi}{2} - \zeta_1\right) > \alpha_n\left(\frac{\pi}{2} - \zeta_1'\right) > 0\,.$$

Proof. Since

$$v(0) = \psi_n(0) = 0$$

and $y = v(x)$ is a solution of the equation (4,2d) our assertion is a consequence of theorem in Section 4,3.

4,5. *Let $v(x)$ and $w(x)$ be the functions defined by* (4,2a), (4,2b) *and* (4,2c) *respectively and let $\psi_n(x)$ be defined by* (3,8b).

For brevity, put

(4,5a) $$W(x, t) = v(x)\, w(t) - v(t)\, w(x)\,,$$

(4,5b) $$l^{-1} = v'(x)\, w(x) - v(x)\, w'(x)\,,$$

(4,5c) $$l_n = \sqrt{(\lambda_n^2 + \tau_n)}\,,$$

where $\lambda_n = \sqrt{(n(n + \alpha + \beta + 1))}$ *and*

(4,5d) $$\tau_n = O(n)$$

is a real number depending on n.

Further, put

(4,5e) $$\psi_n = \lim_{x \to 0+} \frac{v(l_n x)}{\psi_n(x)} = \frac{2^{-(\alpha+\beta)/2} l_n^{\alpha+1/2}}{\Gamma(\alpha + 1)\, Q_n(1)}\,,$$

(4,5f) $$\chi_n(x) = \psi_n\, \psi_n(x)$$

and

(4,5g) $$\beta_n(t) = \omega_n(t) - \tau_n$$

where $\omega_n(t)$ is defined by (3,5a).

Then for $x \in (0, 1)$

(4,5h) $$\chi_n(x) = v(l_n x) - \varrho_n(x)$$

where

(4,5i) $$\varrho_n(x) = l l_n^{-1} \int_0^x \beta_n(t)\, W(l_n x, l_n t)\, \chi_n(t)\, \mathrm{d}t\,.$$

Proof. 1. Denote by k_i ($i = 1, 2, \ldots$) positive constants independent of x and t in the interval $[0, 1]$. (They may depend on n.)

In virtue of (3,8b) and (3,5b) we may write for $t \in (0, 1)$

(1) $$|\chi_n(t)| < k_1 t^{\alpha+1/2}, \quad |\beta_n(t)| < k_2 .$$

By applying (4,1a) and (4,1b) we deduce that for $x \in (0, 1)$ and $t \in [0, 1)$ and $x > t$

(2) $$|W(l_n x, l_n t)| < k_3\, \delta(x, t) = k_3[(xt^{-1})^\alpha + (x^{-1}t)^\alpha]\, \sqrt{(xt)}\, \lg^{m_0} \left|\frac{ex}{t}\right| ,$$

where $m_0 = 1$ if $\alpha = 0$, and $m_0 = 0$ if $\alpha \neq 0$.

From (1) and (2) it follows for $x \in (0, 1)$

(3) $$|\varrho_n(x)| < k_4 \int_0^x t^{\alpha+1/2}\, \delta(x, t)\, dt < k_5 x^{\alpha+5/2} .$$

2. The function $\chi_n(x)$ defined by (4,5f) is a solution of the differential equation

$$y'' + \alpha_n\left(\frac{\pi}{2} - x\right) y = 0 .$$

Hence

(4) $$\chi_n''(x) + \left[l_n^2 + \frac{1 - 4\alpha^2}{4x^2}\right] \chi_n(x) = -\beta_n(x)\, \chi_n(x) .$$

By (4) we derive the equation

(5) $$\chi_n(x) = C_1\, v(l_n x) + C_2\, w(l_n x) - \varrho_n(x) ,$$

where C_1 and C_2 are constants.

Let α be non integer. Making use of (3,8b), (4,1a), (4,5e) and (3) we deduce by (5) that for $x \to 0+$

$$\frac{(l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha + 1)} + O(x^{\alpha+5/2}) = \frac{C_1(l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha + 1)} + O(x^{\alpha+5/2}) + \frac{C_2(l_n x)^{-\alpha+1/2}}{2^{-\alpha}\Gamma(1 - \alpha)}\, [1 + O(x^2)] .$$

Hence

(6) $$C_1 + \frac{2^{2\alpha}\Gamma(\alpha + 1)}{\Gamma(1 - \alpha)}\, l_n^{-2\alpha} x^{-2\alpha}[1 + O(x^2)]\, C_2 = 1 + O(x^2) .$$

From (6) it is easily seen that

(7) $$\alpha > 0 \Rightarrow C_2 = O(x^{2\alpha}) \Rightarrow C_2 = 0 , \quad C_1 = 1$$

and

$$\alpha < 0 \Rightarrow C_1 = 1 + O(x^{-2\alpha}) \Rightarrow C_1 = 1 , \quad C_2 = O(x^{2-2\alpha}) \Rightarrow C_2 = 0 .$$

If α is an integer, then by (3,8b), (4,1b) and (3) we deduce that for $x \to 0+$

$$\frac{(l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha+1)} + O(x^{\alpha+5/2}) = \frac{C_1(l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha+1)} + O(x^{\alpha+5/2}) +$$

$$+ \frac{1}{\pi}\left[\frac{(l_n x)^{\alpha+1/2}}{2^{\alpha-1}\Gamma(\alpha+1)} \lg x + 2^\alpha(\alpha-1)!\,(l_n x)^{-\alpha+1/2}\right][1 + O(x^2)]\, C_2 .$$

Hence we deduce $C_1 = 1$, $C_2 = 0$ by a similar argument as above.

4,6. *Let $a > 0$ be an arbitrary number independent of n and*

(4,6a) $$I_a = \left(0, \frac{a}{n}\right).$$

Further denote by $\gamma_n(x)$ a real function defined in the interval I_a such that

(4,6b) $$t \in I_a \Rightarrow |\gamma_n(t)| < \gamma_n .$$

Put

(4,6c) $$\sigma_n(x) = \int_0^x \gamma_n(t)\, W(l_n x, l_n t)\, \chi_n(t)\, dt ,$$

where $\chi_n(x)$ is defined by (4,5f).

Then

(4,6d) $$x \in I_a \Rightarrow |\sigma_n(x)| < c_1 n^{-1} \gamma_n .$$

From (4,5i) and (4,6d) we deduce that

(4,6e) $$x \in I_a \Rightarrow |\varrho_n(x)| < c_2 n^{-1} .$$

Proof. 1. For brevity, put

(1) $$l_n(x) = x^{-\alpha-1/2}\chi_n(x), \quad S_n = \sup_{x \in I_a} |l_n(x)| .$$

Making use of (4,6b) and (2) in Section 4,5, we obtain from (4,6c)

(2) $$x \in I_a \Rightarrow |\sigma_n(x)| < c_3 \gamma_n x S_n x^{\alpha+1/2} < c_4 \gamma_n n^{-1} S_n x^{\alpha+1/2} .$$

2. Put $\gamma_n(t) = \beta_n(t)$, where $\beta_n(t)$ is defined by (4,5g). In this case we may put $\gamma_n = c_5 n$ so that we obtain from (4,5i) and (2)

(3) $$x \in I_a \Rightarrow |\varrho_n(x)| < c_6 n^{-1} n n^{-1} x^{\alpha+1/2} S_n < c_7 n^{-1} x^{\alpha+1/2} S_n .$$

Since

(4) $$x \in I_a \Rightarrow |v(l_n x)| < c_8 (l_n x)^{\alpha+1/2}$$

and by (4,5h)

(5) $$l_n(x) = [v(l_n x) - \varrho_n(x)]\, x^{-\alpha-1/2}$$

we deduce by (2) and (5)

$$S_n < c_9 n^{\alpha+1/2} + c_{10} n^{-1} S_n \Rightarrow S_n < c_{11} n^{\alpha+1/2}\,.$$

Applying this result we obtain (4,6d) from (2) and (4,6e) from (3).

4,7. *Let $v(x)$ be defined by (4,2a) and let $x_{\alpha,k}$ $(k = 1, 2, \ldots)$ be the zeros of $v(x)$ introduced in Section 4,4. Let $A_n > 0$ satisfy the condition*

(4,7a) $$A_n = o(1) \quad \textit{for} \quad n \to +\infty\,.$$

If

(4,7b) $$x_{\alpha,0} = 0,\; x_{\alpha,k} + \eta n \in (x_{\alpha,k-1} + c_1,\, x_{\alpha,k+1} - c_1) \;\textit{and}\; |v(x_{\alpha,k} + n\eta)| < A_n\,,$$

then

(4,7c) $$|\eta| < c_2 n^{-1} A_n\,.$$

Proof. For brevity, put $x_{\alpha,k} = x_k$ and $x_k + n\eta = b$.

Let I_η be the interval (b, x_k) if $\eta < 0$ or (x_k, b) if $\eta > 0$. By (4,7a) and (4,7b) we deduce

(1) $$x \in I_\eta \Rightarrow |v(x)| < A_n\,.$$

Further

(2) $$v(b) = n\eta\, v'(x_k) + \tfrac{1}{2} n^2 \eta^2\, v''(\xi)\,,$$

where

(3) $$\xi \in I_\eta\,.$$

From the equation (4,2d) we obtain

(4) $$v''(\xi) = \left[\frac{4\alpha^2 - 1}{4\xi^2} - 1\right] v(\xi)\,.$$

Making use of (4), and (1) we deduce

(5) $$|v''(\xi)| < c_3 |v(\xi)| < c_4 A_n\,.$$

Since $v'(x_k) \neq 0$ it follows from (2), (5) and (4,7a) that

$$A_n > |v(b)| > n|\eta|\, |v'(x_k)| - c_5 |v''(\xi)| > n\eta |v'(x_k)| - c_6 A_n$$

for almost all values of n.

4,8. *Following the notation of Section* 4,6 *we put*

(4,8a) $$h_n(x) = v(l_n x) + \eta_n(x),$$

where l_n *is defined by* (4,5c)

(4,8b) $$x \in I_a \Rightarrow |\eta_n(x)| < A_n .$$

Here A_n *satisfies* (4,7a).

Let $\{\xi_n\}_{n=1}^{N}$ *be the increasing sequence of all the zeros of the function* $h_n(x)$ *contained in the interval* I_a. *Then the following assertions are true*:

a) *For every positive integer* k *there exists an integer* $r > 0$ *such that for* $n \to +\infty$

(4,8c) $$\xi_k = \frac{x_{\alpha,r}}{l_n}[1 + O(A_n)] .$$

b) *For every integer* $m > 0$ *there exists an integer* s *such that for* $n \to +\infty$

(4,8d) $$\xi_s = \frac{x_{\alpha,m}}{l_n}[1 + O(A_n)] .$$

Proof. 1. Let $\{x'_{\alpha,n}\}_{n=1}^{\infty}$ be the increasing sequence of all the positive zeros of the function $v'(x)$.

From (4,8b) and (4,7a) we deduce the following assertion A: For every integer $v > 0$ there is at least one zero of the function $h_n(x)$ in the interval $(x'_{\alpha,v}/l_n, x_{\alpha,v+1}/l_n)$.

2. Put

(1) $$\xi_k = \frac{x_{\alpha,r}}{l_n} + l_n^{-1} n\eta ,$$

where $x_{\alpha,r}$ is the zero of the function $v(l_n x)$ nearest to the number ξ_k. From the above proposition

(2) $$\xi_k < \frac{x'_{\alpha,k+1}}{l_n} < \frac{x_{\alpha,k+2}}{l_n} \in I_a .$$

From (2) it is obvious that $r \leq k + 2$.

If $a > x_{\alpha,k+2}$ it follows from (4,8b) that

(3) $$|\eta_n(\xi_k)| < A_n .$$

By (4,8a) and (1) we deduce that

(4) $$0 = h_n(\xi_k) = v(x_r + n\eta) + \eta_n(\xi_k) .$$

Hence we obtain as a consequence of (3) and (4,8b) that

(5) $$|v(x_r + n\eta)| < A_n .$$

The proposition of Section 4,7 yields

$$|\eta| < A_n n^{-1}.$$

This inequality shows that (4,8c) is true.

3. Let

$$\frac{x_{\alpha,m}}{l_n} = \xi_s - n l_n^{-1} \eta' , \tag{6}$$

where ξ_s is a zero of the function $h_n(x)$ nearest to the number $x_{\alpha,m}/l_n$. From the above assertion A we see that

$$a > x'_{\alpha,m+2} \Rightarrow \xi_s < \frac{x'_{\alpha,m+2}}{l_n} \in I_a . \tag{7}$$

Making use of (4,8a) we obtain

$$0 = h_n(\xi_s) = v(x_{\alpha,m} + n\eta') + \eta_n(\xi_s) .$$

Hence, in virtue of (7) and (4,8b)

$$|v(x_m + n\eta')| < A_n \tag{8}$$

Hence by the statement of Section 4,7

$$|\eta'| < n^{-1} A_n . \tag{9}$$

(7) and (9) establish (4,8d).

5. PROOF OF (2,1a) AND (2,1b)

5,1. *In the notation introduced in Section 4,4, for* $k = 1, 2, \ldots$ *independent of* n *it is*

$$\zeta_k = \frac{x_{\alpha,k}}{n} [1 + O(n^{-1})] \quad for \quad n \to +\infty . \tag{5,1a}$$

Proof. 1. The zeros of the function $\psi_n(x)$ coincide with the zeros of the function $\chi_n(x)$ defined by (4,5f). Let I_a be defined by (4,6a) and choose a sufficiently large.

In virtue of (4,5h) and (4,6e) the theorem of Section 4,8 yields for $k = 1, 2, \ldots$ and $m = 1, 2, \ldots$ provided that $\zeta_k \in I_a$ and $x_{\alpha,m}/n \in I_a$,

$$\zeta_k = \frac{x_{\alpha,r}}{n} [1 + O(n^{-1})] \tag{1}$$

and

$$\zeta_s = \frac{x_{\alpha,m}}{n} [1 + O(n^{-1})] . \tag{2}$$

Herein $x_{\alpha,r}/l_n$ is the zero of $v(l_n x)$ nearest to the number ζ_k and ζ_s is the zero of $\chi_n(x)$ nearest to the number $x_{\alpha,m}/l_n$.

2. Put in (1) $k = 1$ and in (2) $m = 1$. Then

(3) $$n\zeta_1 \geqq x_{\alpha,1} + O(n^{-1})$$

and

(4) $$n\zeta_1 \leqq x_{\alpha,1} + O(n^{-1}) .$$

From (3) and (4) we see that

(5) $$\zeta_1 = \frac{x_{\alpha,1}}{n} [1 + O(n^{-1})] .$$

Hereby (5,1a) is established for $k = 1$.

3. Let $\omega_n(x)$ be defined by (3,5a) and put

(6) $$s_n = \sup_{x \in [0,\pi/2]} |\omega_n(x)| .$$

In virtue of (3,5b) we may choose $k_1 > 1$ independent of n and σ_n such that

(7) $$k_1 n > \sigma_n > s_n .$$

Put

(8) $$\lambda = \sqrt{(\lambda_n^2 - \sigma_n)} .$$

(5) enables us to choose σ_n so that

(9) $$\frac{x_{\alpha,1}}{\lambda} > \zeta_1 .$$

Since the functions $v(\lambda x)$ and $\chi_n(x)$ are solutions of the differential equations

(10) $$y'' + \left[\lambda^2 + \frac{1 - 4\alpha^2}{4x^2}\right] y = 0$$

and

(11) $$y'' + \left[\lambda_n^2 + \frac{1 - 4\alpha^2}{4x^2} + \omega_n(x)\right] y = 0$$

respectively it follows by the well-known Sturm's comparison theorem in virtue of (9) that in the interval $[0, \zeta_k]$ there are at most $(k - 1)$ zeros of the function $v(\lambda x)$. Hence we obtain for the number k and r in (1)

(12) $$r \leqq k .$$

3. Further, put

$$k_2 n > \mu_n > s_n\,, \quad \mu = \sqrt{(\lambda_n^2 + \mu_n)}\,, \tag{13}$$

where k_2 does not depend on n and s_n is defined by (6). Choose μ_n so that

$$\zeta_1 > \frac{x_{\alpha,1}}{\mu}\,. \tag{14}$$

Then there are at least $(k - 1)$ zeros of $v(\mu x)$ in the interval $[0, \zeta_k]$. Hence by (1)

$$\zeta_k = \frac{x_{\alpha,t}}{n}\,[1 + O(n^{-1})]\,, \tag{15}$$

where

$$t \geqq k\,. \tag{16}$$

From (1) and (15) we deduce that

$$0 = x_{\alpha,r} - x_{\alpha,t} + O(n^{-1})\,.$$

Hence

$$x_{\alpha,r} = x_{\alpha,t} \Rightarrow r = t\,. \tag{17}$$

(12), (16) and (17) show that $r = k$.

5,2. The proof of (2,1b). By (5,1a) we deduce

$$x_{n-k}^{(n)} = \sin\left(\frac{\pi}{2} - \zeta_{k+1}\right) = 1 - \frac{x_{\alpha,k+1}}{2n^2}\,[1 + O(n^{-1})]$$

for $n \to +\infty$.

5,3. For the proof of (2,1a) see Remark 3,3.

6. PROOF OF (2,2b) AND (2,2c)

6,1. 1. Put $Q_n(x) = J_n(x)$. Then by (3,5a)

$$\omega_n(t) = \gamma(t) - \frac{1 - 4\alpha^2}{4t^2}\,. \tag{1}$$

Put in (4,5c) and (4,5g)

$$l_n = (\lambda_n^2 + j)^{1/2}\,, \tag{2}$$

$$\beta_n(t) = \omega_n(t) - j\,. \tag{3}$$

where j is defined by (2,1a).

Let I_a be defined by (4,6a) and a sufficiently large. It is easly to see from (3,4c) and (1) that

(4) $$t \in I_a \Rightarrow |\beta_n(t)| < c_1 n^{-2}.$$

Then by (4,5i) and (4,6d)

(5) $$x \in I_a \Rightarrow |\varrho_n(x)| < c_2 n^{-4}$$

for in this case $\gamma_n = c_3 n^{-2}$.

Denote by $\{\zeta_k\}_{k=1}^n$ the increasing sequence of all the zeros of $J_n(\sin z)$.

By the theorem of Section 4,8 and by (5) we deduce that for every $k = 1, 2, \ldots$ there exists an integer $r > 0$ such that

(6) $$\zeta_k = \frac{x_{\alpha,r}}{l_n} + O(n^{-5}).$$

By (5,1a) we have

(7) $$\zeta_k = \frac{x_{\alpha,k}}{n} + O(n^{-2}).$$

From (6) and (7) it follows that

$$0 = x_{\alpha,r} - x_{\alpha,k} + O(n^{-1}).$$

Hence

$$x_{\alpha,r} = x_{\alpha,k} \Rightarrow r = k$$

so that by (6)

(8) $$\zeta_k = \frac{x_{\alpha,k}}{l_n} + O(n^{-5}).$$

2. Let $Q_n(x) = J_n(x)$. Then

(9) $$x_{n-k+1}^{(n)} = \cos \zeta_k = 1 - \frac{\zeta_k^2}{2} + \frac{\zeta_k^4}{24} + O(n^{-6}).$$

From (2) it is obvious that

(10)
$$n^2 l_n^{-2} = 1 - \frac{\alpha+\beta+1}{n} - \frac{j}{n^2} - \left[\frac{\alpha+\beta+1}{n} + \frac{j}{n^2}\right]^2 - \frac{(\alpha+\beta+1)^3}{n^3} + O(n^{-4}) =$$
$$= 1 - \frac{\alpha+\beta+1}{n} - \frac{(\alpha+\beta+1)^2 + j}{n^2} - \frac{(\alpha+\beta+1)\,[2j + (\alpha+\beta+1)^2]}{n^3} +$$
$$+ O(n^{-4}).$$

Further

$$(11) \qquad n^4 l_n^{-4} = 1 - \frac{2(\alpha + \beta + 1)}{n} + O(n^{-2}).$$

From (8)–(11) we may deduce (2,2c).

As for (2,2b), see Remark 3,3.

7. PROOF OF THE INEQUALITIES IN SECTIONS 2,3; 2,4 AND 2,5

7,1. *In the notations introduced in Section* 2,3

$$(7,1a) \qquad z \in J_n \Rightarrow c_1 n^2 < \alpha_n(z) < c_2 n^2 .$$

Proof. (7,1a) is a consequence of (3,5a), (3,5b). See also (3,7a).

7,2. *Let z_1 and z_2 be defined by* (2,3g). *Then*

$$(7,2a) \qquad (z_1, z_2) \subset J_n \Rightarrow z_2 - z_1 < c_1 n^{-1} .$$

Proof. Employing Sturm's comparison theorem we obtain from the differential euqation $y'' + \alpha_n(z)\, y = 0$

$$(1) \qquad z_2 - z_1 < \pi \sup_{z \in J_n} \alpha_n^{-1/2}(z) .$$

Now, (7,2a) is a consequence of (1) and (7,1a).

7,3. *In the notation of Section* 2,3

$$(7,3a) \qquad [z_1', z_2'] \subset [z_1, z_2] \Rightarrow \left| \varrho\left(\frac{\pi}{2} - z_1'\right) - \varrho\left(\frac{\pi}{2} - z_2'\right) \right| < c_1 n^2 a_n^{-3} .$$

Here c_1 does not depend on z_i, z_i' $(i = 1, 2)$.

Proof. For brevity, put

$$\xi_i' = \frac{\pi}{2} - z_i' \quad (i = 1, 2) .$$

From (2,3d) it follows

$$\xi_i' > \frac{a_n}{n} .$$

Now, (7,2a) yields

$$|\varrho(\xi_1') - \varrho(\xi_2')| = |\alpha^2 - \tfrac{1}{4}| \frac{(\xi_1' - \xi_2')(\xi_1' + \xi_2')}{\xi_1'^2 \cdot \xi_2'^2} < c_2 n^{-1} \xi_2'^{-3} < c_3 n^2 a_n^{-3} .$$

7,4 *According to the notation introduced in the preceding chapter*

$$\delta_n = |\alpha_n^{-1/2}(z_1') - \alpha_n^{-1/2}(z_2')| < c_1 n^{-2}(n a_n^{-3} + 1)\,. \tag{7,4a}$$

Proof. Making use of (7,3a), (3,5a) and (3,5b), we obtain

$$|\alpha_n(z_1') - \alpha_n(z_2')| = |\varrho(\xi_2') - \varrho(\xi_1') + \omega_n(\xi_2') - \omega_n(\xi_1')| < c_2 n(n a_n^{-3} + 1)\,.$$

Further, it follows from (7,1a) and (7,2a) that

$$\delta_n = |\alpha_n(\xi_1') - \alpha_n(\xi_2')|\,[\alpha_n(\xi_1')\,\alpha_n(\xi_2')]^{-1/2}\,[\sqrt{\alpha_n(\xi_1')} + \sqrt{\alpha_n(\xi_2')}]^{-1} <$$
$$< c_3 n^{-2}(n a_n^{-3} + 1)\,.$$

7,5. The proof of (2,3i).

Put

$$s_1 = \sup_{z\in(z_1,z_2)} \alpha_n^{-1/2}(z)\,, \quad s_2 = \inf_{z\in(z_1,z_2)} \alpha_n^{-1/2}(z)\,.$$

Making use of Sturm's comparison theorem, we deduce by the differential equation (3,2b)

$$\pi s_2 < z_2 - z_1 < \pi s_1\,.$$

Hence

$$z_2 - z_1 = \pi s_2 + \vartheta(s_1 - s_2) \tag{1}$$

where $\vartheta \in (0, 1)$. Put

$$s_1 = \alpha_n^{-1/2}(z_1) + \vartheta_1^{(n)}\,, \quad s_2 = \alpha_n^{-1/2}(z_1) + \vartheta_2^{(n)}\,, \quad s_1 - s_2 = \vartheta_3^{(n)}\,. \tag{7,5a}$$

From (7,4a) it follows for $i = 1, 2, 3$

$$|\vartheta_i^{(n)}| < c_1 n^{-2}(n a_n^{-3} + 1)\,. \tag{2}$$

By (7,5a), (7,1a), (3,5a), (3,5b), (1) and (2) we deduce that

$$z_2 - z_1 = \pi\alpha_n^{-1/2}(z_1) + \vartheta_4^{(n)} = \pi\varrho^{-\frac{1}{2}}\left(\frac{\pi}{2} - k_1\right) + O(n^{-2}) + \vartheta_4^{(n)} \tag{7,5b}$$

where $\vartheta_4^{(n)}$ satisfies (2) for $i = 4$.

7.6 The proof of (2.5a). It follows from (3,5a) for the polynomials $J_n(x)$ that

$$\omega_n(\zeta) = \gamma\left(\frac{\pi}{2} - \zeta\right) + \frac{4\alpha^2 - 1}{4\zeta^2}\,. \tag{1}$$

Hence

$$\frac{\pi}{2} - \zeta \in J_n \Rightarrow |\omega_n(\zeta)| < c_1\,. \tag{2}$$