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ON THE RELATION BETWEEN YOUNG'S AND KURZWEIL'S CONCEPT OF STIELTJES INTEGRAL

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The considerations in this paper are limited to the closed interval $[a, b]$, $-\infty < < a < b < +\infty$ and to finite real functions defined on this interval. For a real function $g : [a, b] \to R$ we denote by $\operatorname{var}_a^b g$ the obvious (total) variation of g on $[a, b]$. The set of all real functions $g : [a, b] \to R$ with $\operatorname{var}_a^b g < +\infty$ is denoted by $BV(a, b)$.

1. THE RIEMANN AND YOUNG INTEGRALS

Let $\mathscr{D}$ be the set of all sequences $D = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ of points in the interval $[a, b]$ such that

$$(1,1) \qquad a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b .$$

We consider finite sequences (subdivisions of $[a, b]$) $B = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$. For a given $D = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \in \mathscr{D}$ we denote by $\mathscr{B}^*(D)$, $\mathscr{B}(D)$ the sets of all subdivisions $B = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$ such that, respectively,

$$(1,2) \qquad \text{a)} \quad \alpha_{j-1} \leqq \tau_j \leqq \alpha_j , \qquad \text{b)} \quad \alpha_{j-1} < \tau_j < \alpha_j$$

for all $j = 1, 2, \ldots, k$.

On $\mathscr{D}$ we define the binary relation $\succ$ in the following manner: for $D, D' \in \mathscr{D}$ we have $D' \succ D$ if D' is a refinement of D, i.e. if any point α_j from D appears also in D'.

If we define $|D| = \max_{j=1,\ldots,k} |\alpha_j - \alpha_{j-1}|$ for $D \in \mathscr{D}$ then another binary relation $\gg$ may be defined on $\mathscr{D}$ by $D' \gg D$ if $|D'| \leqq |D|$.

It can be easily shown that $(\mathscr{D}, \succ)$ and $(\mathscr{D}, \gg)$ are directed sets.

Let now be given finite functions $f, g : [a, b] \to R$; for every $B = \{\alpha_0, \tau_1, \alpha_1, \ldots \ldots, \tau_k, \alpha_k\}$ satisfying (1,1) and (1,2) a) we put

$$(1,3) \qquad R(B) = \sum_{j=1}^{k} f(\tau_j) \left(g(\alpha_j) - g(\alpha_{j-1})\right) .$$

Definition 1,1. *The function $f:[a, b] \to R$ is Riemann-Stieltjes integrable (Riemann-Stieltjes norm integrable) on the interval $[a, b]$ with respect to $g:[a, b] \to$ $\to R$ if there is a real number I such that to every $\varepsilon > 0$ there exists $\bar{D} \in \mathscr{D}$ so that*

$$|R(B) - I| < \varepsilon$$

for all $B \in \mathscr{B}^*(D)$ if $D \succ \bar{D}(D \gg \bar{D})$. The number I will be denoted by $R \int_a^b f\, dg$ ($NR \int_a^b f\, dg$) and is called the Riemann-Stieltjes (Riemann-Stieltjes norm) integral of f with respect to g on $[a, b]$.

Supposing that for the function $g:[a, b] \to R$ the limits $\lim_{s \to t+} g(s) = g(t+)$, $\lim_{s \to t-} g(s) = g(t-)$ exist for all $t \in [a, b]$ (for the endpoints of $[a, b]$ the corresponding onesided limits) then we put for $f:[a, b] \to R$ and $B = \{\alpha_0, \tau_1, \ldots, \tau_k, \alpha_k\}$ satisfying (1,1), (1,2) b)

$$(1,4) \qquad Y(B) = \sum_{j=1}^{k} [f(\alpha_{j-1})(g(\alpha_{j-1}+) - g(\alpha_{j-1}) + f(\tau_j)(g(\alpha_j-) - g(\alpha_{j-1}+)) + $$
$$+ f(\alpha_j)(g(\alpha_j) - g(\alpha_j-))] =$$
$$= \sum_{j=1}^{k} [f(\alpha_{j-1})\, \Delta^+ g(\alpha_{j-1}) + f(\tau_j)(g(\alpha_j-) - g(\alpha_{j-1}+)) + f(\alpha_j)\, \Delta^- g(\alpha_j)] =$$
$$= \sum_{j=0}^{k} f(\alpha_j)\, \Delta g(\alpha_j) + \sum_{j=1}^{k} f(\tau_j)(g(\alpha_j-) - g\, \alpha_{j-1}+))$$

where $\Delta^+ g(\alpha_j) = g(\alpha_j+) - g(\alpha_j)$, $\Delta^- g(\alpha_j) = g(\alpha_j) - g(\alpha_j-)$, $j = 1, 2, \ldots, k-1$, $\Delta^+ g(b) = \Delta^- g(a) = 0$ and $\Delta g(\alpha_j) = \Delta^+ g(\alpha_j) + \Delta^- g(\alpha_j)$, $j = 0, 1, 2, \ldots, k$.

Definition 1,2. *If for $g:[a, b] \to R$ the limits $g(t+)$, $g(t-)$ exist for all $t \in [a, b]$ then the function $f:[a, b] \to R$ is said to be Young (Young norm) integrable on the interval $[a, b]$ with respect to g if there is a number I such that to every $\varepsilon > 0$ there exists $\tilde{D} \in \mathscr{D}$ so that*

$$|Y(B) - I| < \varepsilon$$

for all $B \in \mathscr{B}(D)$ if $D \succ \tilde{D}(D \gg \tilde{D})$. The number I will be denoted by $Y \int_a^b f\, dg$ ($NY \int_a^b f\, dg$) and is called the Young integral (Young norm integral) of f with respect to g on $[a, b]$.

Remark 1,1. From Def. 1,1 and Def. 1,2 it is clear that if $NR \int_a^b f\, dg$, $NY \int_a^b f\, dg$ exist then also $R \int_a^b f\, dg$, $Y \int_a^b f\, dg$ exist respectively, because evidently $D \succ D'$ implies $D \gg D'$. The concept of the Stieltjes type integral from Def. 1,2 is in detail described and studied in the book [2] (cf. II.19.3 in [2]).

In the sequel we suppose that $g \in BV(a, b)$. Hence $Y(B)$ from (1,4) is defined, because $g(t-)$, $g(t+)$ exist for any $t \in [a, b]$.

For the Riemann-Stieltjes integral the following result is known (cf. II.10.10 in [2] or [1])

Theorem 1,1. *If* $f : [a, b] \to R$, $g \in BV(a, b)$ *and* $R \int_a^b f \, dg$ *exists, then* f *is bounded on a finite number of closed intervals which are complementary to a finite number of open intervals on which the function* g *is constant.*

In [2] (Theorem 19.3.1 in [2]) the same statement is asserted, $R \int_a^b f \, dg$ being replaced by $Y \int_a^b f \, dg$. Unfortunately, this statement does not hold in general. This fact can be demonstrated in the following way: Let $g \in BV(a, b)$, $g(a) = g(b) = g(t+) = g(t-)$ for all $t \in (a, b)$ (i.e. g is different from a constant on a countable set of points in (a, b)). Further let $f : [a, b] \to R$ be an arbitrary finite function. For any $D \in \mathscr{D}$ and $B = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \in \mathscr{B}(D)$ we have

$$Y(B) = \sum_{j=0}^{k} f(\alpha_j) \, \Delta g(\alpha_j) + \sum_{j=1}^{k} f(\tau_j) \left(g(\alpha_j -) - g(\alpha_{j-1} +) \right) = 0$$

because $g(\alpha_j +) = g(\alpha_j -)$ and $\Delta g(\alpha_j) = 0$. This yields the following.

Proposition 1,1. *Let* $g \in BV(a, b)$, $g(a) = g(b) = g(t+) = g(t-)$ *for all* $t \in (a, b)$. *Then* $Y \int_a^b f \, dg$ *exists and equals zero for every finite function* $f : [a, b] \to R$.

Example 1,1. Let us define $g(1/(k + 1)) = 2^{-k}$, $k = 1, 2, \ldots$, $g(t) = 0$ for $t[0, 1] - \{1/(k + 1)\}_{k=1}^{\infty}$. We put $f(1/(k + 1)) = 2^k$, $f(0) = f(1) = 0$ and we suppose that f is linear in $[\frac{1}{2}, 1]$, $[1/(k + 2), 1/(k + 1)]$, $k = 1, 2, \ldots$ The Young integral $Y \int_0^1 f \, dg$ exists by Proposition 1,1 and equals zero by the same Proposition. Any finite number of closed intervals which are complementary to a finite number of open intervals on which g is constant contains necessarily an interval of the form $[0, \alpha]$, $\alpha > 0$ on which g is not constant and the function f defined above is not bounded. Hence we obtain that Theorem 19.3.1 from Chapter II. in [2] is false.

For the Young integral the following Theorem (an analogue to Theorem 1,1) holds:

Theorem 1,2. *If* $f : [a, b] \to R$, $g \in BV(a, b)$ *and* $Y \int_a^b f \, dg$ *exists, then* f *is bounded on a finite number of closed intervals which are complementary to a finite number of open intervals* $J_i = (a_i, b_i)$, $a_i < b_i$, $i = 1, 2, \ldots, l$ *such that* $g(a_i +) = g(b_i -) = g(t+) = g(t-)$ *for all* $t \in J_i$, $i = 1, 2, \ldots, l$.

Proof. By definition for every $\varepsilon > 0$ there exists a $\tilde{D} \in \mathscr{D}$ such that $|Y(B) - Y \int_a^b f \, dg| < \varepsilon$ for all $B \in \mathscr{B}(D)$ if $D \succ \tilde{D}$. We choose a fixed $D = \{\alpha_0, \alpha_1 \ldots \ldots, \alpha_k\} \in \mathscr{D}$, $D \succ \tilde{D}$. We have evidently

$$|Y(B)| = \left| \sum_{j=0}^{k} f(\alpha_j) \, \Delta g(\alpha_j) + \sum_{j=1}^{k} f(\tau_j) \, g(\alpha_j -) - g(\alpha_{j-1} +)) \right| < \left| Y \int_a^b f \, dg \right| + \varepsilon$$

for all $B \in \mathscr{B}(D)$, i.e. for all $\tau_j \in (\alpha_{j-1}, \alpha_j)$, $j = 1, 2, \ldots, k$. Hence there is a constant $K > 0$ ($K = |\sum_{j=0}^{k} f(\alpha_j)\,\Delta g(\alpha_j) + |Y \int_a^b f\,dg| + \varepsilon$) such that

$$(1,5) \qquad |\sum_{j=1}^{k} f(\tau_j)\,(g(\alpha_j -) - g(\alpha_{j-1}+))| \leqq K$$

for all $\tau_j \in (\alpha_{j-1}, \alpha_j)$, $j = 1, 2, \ldots, k$.

Let us suppose that f is unbounded in some (α_{j-1}, α_j). If $g(\alpha_j -) - g(\alpha_{k-1}+) \neq 0$ then $f(\tau_j)\,(g(\alpha_j -) - g(\alpha_{j-1}+))$ would be arbitrarily large for a suitable choice of $\tau_j \in (\alpha_{j-1}, \alpha_j)$, but this contradicts (1,5). Therefore we have necessarily $g(\alpha_j -) = g(\alpha_{j-1}+) = c$, where c is a constant. Let now $a \in (\alpha_{j-1}, \alpha_j)$ be given; by the assumption f is not bounded either in (α_{j-1}, a) or in (a, α_j). If we add the point a to D then we obtain $D' = \{\alpha_0, \alpha_1, \ldots, \alpha_{j-1}, a, \alpha_j, \alpha_{j+1}, \ldots, \alpha_k\} \in \mathscr{D}$ where evidently $D' \succ D \succ \tilde{D}$ and the same argument as above gives either $g(a-) = c$ or $g(a+) = c$. In this way we obtain that if f is not bounded in some (α_{j-1}, α_j) then $g(\alpha_j -) = g(\alpha_{j-1}+) = c$ and for any $a \in (\alpha_{j-1}, \alpha_j)$ we have either $g(a+) = c$ or $g(a-) = c$. Since we suppose $g \in BV(a, b)$, the limits $g(t+)$ and $g(t-)$ exist for any $t \in (\alpha_{j-1}, \alpha_j)$ and it is a matter of routine to show that $g(a+) = g(a-) = c$ for all $a \in (\alpha_{j-1}, \alpha_j)$. This proves the Theorem, since the number of intervals (α_{j-1}, α_j) is finite.

Remark 1,2. Evidently in Theorem 1,2 the assumption $g \in BV(a, b)$ can be replaced by the requirement that the limits $g(t+)$ and $g(t-)$ exist for all $t \in [a, b]$ (with the corresponding onesided limits at the endpoints of $[a, b]$).

Corollary 1,1. *Let $g \in BV(a, b)$ be given and let $J_i = (a_i, b_i)$, $i = 1, 2, \ldots, l$ be a finite system of open intervals in $[a, b]$ such that $g(a_i+) = g(b_i)- = g(t+) = g(t-)$ holds for all $t \in J_i$. If for $f : [a, b] \to R$ the integral $Y \int_a^b f\,dg$ exists and if $\tilde{f} : [a, b] \to R$ is such a function that $f(t) = \tilde{f}(t)$ for all $t \in [a, b] - \bigcup_{i-1}^{l} J_i$ then $Y \int_a^b \tilde{f}\,dg$ exists and $Y \int_a^b \tilde{f}\,dg = Y \int_a^b f\,dg$. The same statement holds also for the Young norm integral.*

The proof follows easily from the definition of the Young integral and from the fact that the term from $Y(B)$ (cf. (1,4)) which corresponds to some $[\alpha_{j-1}, \alpha_j] \subset J_i$ equals zero for any function f.

The Young integral is an extension of the Riemann-Stieltjes integral; the following theorem holds:

Theorem 1,3. (cf. II.19.3.3 in [2]). *If $f : [a, b] \to R$, $g \in BV(a, b)$ and $R \int_a^b f\,dg$ exists then $Y \int_a^b f\,dg$ exists and the two integrals are equal. (The same holds for the norm integrals.)*

In the opposite direction we have the following

Theorem 1,4. (cf. II.19.3.4 in [2]). *If* $f:[a, b] \to R$, $g \in BV(a, b)$ *g is continuous in* $[a, b]$ *and* $Y \int_a^b f \, dg$ *exists then* $R \int_a^b f \, dg$ *exists and both integrals are equal. The same statement is valid for the norm integrals.*

For continuous $g \in BV(a, b)$ we can state the following Theorem which is a reversion of the statement given in Remark 1,1.

Theorem 1,5. *Let* $f:[a, b] \to R$, $g \in BV(a, b)$, *g continuous and let* $Y \int_a^b f \, dg$ *exist. Then* $NY \int_a^b f \, dg$ *exists and* $Y \int_a^b f \, dg = NY \int_a^b f \, dg$.

Proof. Let $\varepsilon > 0$ be given. By definition there is a $\tilde{D} = \{a_0, a_1, \ldots, a_k\} \in \mathscr{D}$ such that $|Y(B') - Y \int_a^b f \, dg| < \varepsilon$ for all $B' \in \mathscr{B}(D')$, $D' \succ D$. Regarding Theorem 1,2 and Corollary 1,1 we can suppose without any loss of generality that the function f is bounded, i.e. $|f(t)| \leqq M$ for all $t \in [a, b]$. If this is not satisfied, then we define the function $\tilde{f}$ by Corollary 1,1 so that $\tilde{f}$ is bounded and we work with the integral $Y \int_a^b \tilde{f} \, dg$ instead of $Y \int_a^b f \, dg$.

From the continuity of g at all points a_i, $i = 1, \ldots, k$ we obtain the existence of a $\delta > 0$ such that $|g(t) - g(a_i)| < \varepsilon/2Mk$ provided $|t - a_i| < \delta$, $i = 1, \ldots, k$.

Let $D = \{\alpha_0, \alpha_1, \ldots, \alpha_l\} \in \mathscr{D}$ be an arbitrary subdivision such that $|D| < \delta$ and let us construct a subdivision D' which is a common refinement of D and $\tilde{D}$; evidently $D' \succ \tilde{D}$. For a given $B \in \mathscr{B}(D)$ and $B' \in \mathscr{B}(D')$ we give an estimate of $|Y(B) - Y(B')|$.

If it occurs that $\alpha_{j-1} < a_{h+1} < \ldots < a_{h+m_j} < \alpha_j$ then

$$s_j = f(\tau_j)\,(g(\alpha_j) - g(\alpha_{j-1})) =$$
$$= f(\tau_j)\,(g(\alpha_j) - g(a_{h+m_j})) + (g(a_{h+m_j}) - g(a_{h+m_j-1})) + \ldots + (g(a_{h+1}) - g(\alpha_{j-1}))$$

is the term of $Y(B)$ corresponding to $\alpha_{j-1} < \tau_j < \alpha_j$ and the terms of $Y(B')$ are of the form

$$s'_j = f(\tau'_{q+m_j})\,(g(\alpha_j) - g(a_{h+m_j})) + f(\tau'_{q+m_j-1})\,(g(a_{h+m_j}) - g(a_{h+m_j-1})) + \ldots$$
$$\ldots + f(\tau'_q)\,(g(a_{h+1}) - g(\alpha_{j-1}))\,.$$

The difference $s_j - s'_j$ consists of $m + 1$ terms of the form

$$(f(\tau_j) - f(\tau'_{q+\varkappa}))\,(g(u) - g(v))$$

where $|u - v| < \delta$ (since $|D| < \delta$) and either u or v equals to some a_i. Hence

$$|f(\tau_j) - f(\tau'_{q+\varkappa}))\,(g(u) - g(v))| < 2M\,.\,(\varepsilon/2Mk) = \varepsilon/k$$

and

$$|s_j - s'_j| < \varepsilon(m_j + 1)/k = \varepsilon m_j/k + \varepsilon/k\,.$$

If the interval (α_{j-1}, α_j) does not contain points from $\tilde{D}$ then the corresponding terms from $Y(B)$ and $Y(B')$ are equal. Hence we have

$$|Y(B) - Y(B')| < \varepsilon \sum (m_j + 1)/k$$

where the sum on the right hand side is taken over all j for which (α_{j-1}, α_j) contains points from $\tilde{D}$. The number of such intervals is at most $k - 1$ and $\sum m_j \leqq k$; this yields

$$|Y(B) - Y(B')| < \varepsilon(1 + ((k - 1)/k) < 2\varepsilon\,.$$

In this way we obtain

$$\left|Y(B) - Y\int_a^b f\,\mathrm{d}g\right| \leqq |Y(B) - Y(B')| + \left|Y(B') - Y\int_a^b f\,\mathrm{d}g\right| < 3\varepsilon$$

for all $\dot{B} \in \mathscr{B}(D)$, $|D| < \varepsilon$, i.e. $NY\int_a^b f\,\mathrm{d}g$ exists and is equal to $Y\int_a^b f\,\mathrm{d}g$.

If $g, h \in BV(a, b)$, $f : [a, b] \to R$, $|f(t)| \leqq M$ for all $t \in [a, b]$ and if $B = \{\alpha_0, \tau_1, \alpha_1, \dots, \tau, \alpha_k\} \in \mathscr{B}(D)$ for some $D = \{\alpha_0, \dots, \alpha_k\} \in \mathscr{D}$ then we denote

$$Y_h(B) = \sum_{j=0}^{k} f(\alpha_j)\,\Delta h(\alpha_j) + \sum_{j=1}^{k} f(\tau_j)\,(h(\alpha_j-) - h(\alpha_{j-1}+))$$

and similarly $Y_g(B)$ denotes the Young sum for g (cf. (1,4)).

Evidently the inequality

$$(1,6) \qquad |Y_g(B) - Y_h(B)| \leqq M \operatorname{var}_a^b (g - h)$$

holds.

Similarly for $f, \tilde{f} : [a, b] \to R$ and $g \in BV(a, b)$ we have

$$(1,7) \qquad |Y^f(B) - Y^{\tilde{f}}(B)| \leqq \sup_{t \in [a,b]} |f(t) - \tilde{f}(t)| \operatorname{var}_a^b g$$

for any $B \in \mathscr{B}(D)$, $D \in \mathscr{D}$, where $Y^{\tilde{f}}(B) = \sum_{j=0}^{k} \tilde{f}(\alpha_j)\,\Delta g(\alpha_j) + \sum_{j=1}^{k} \tilde{f}(\tau_j)\,(g(\alpha_j-) - g(\alpha_{j-1}+))$ and similarly for $Y^f(B)$ (cf. (1,4)).

The inequality (1,6) immediately leads to the following

Proposition 1,2. (cf. II. 19.3.9 in [2]). *If $g_n, g \in BV(a, b)$, $n = 1, 2, \dots$ $\lim_{n\to\infty} \operatorname{var}_a^b (g_n - g) = 0$, $f : [a, b] \to R$, $|f(t)| \leqq M$ for all $t \in [a, b]$ and $Y\int_a^b f\,\mathrm{d}g_n$ exists for all $n = 1, 2, \dots$ then both $Y\int_a^b f\,\mathrm{d}g$ and $\lim_{n\to\infty} Y\int_a^b f\,\mathrm{d}g_n$ exist and are equal.*

Corollary 1,2. *If $g_b \in BV(a, b)$ is a pure break function and $f : [a, b] \to R$ is bounded ($|f(t)| \leqq M$ for $t \in [a, b]$) then $Y\int_a^b f\,\mathrm{d}g_b$ exists and we have $Y\int_a^b f\,\mathrm{d}g_b = \sum_{t\in[a,b]} f(t)\,\Delta g_b(t)$.*

Proof. To every pure break function $g_b \in BV(a, b)$ there exists a sequence $g_n \in BV(a, b)$, $n = 1, 2, \dots$ of break functions with a finite number of discontinuities such that $\lim_{n\to\infty} \operatorname{var}_a^b (g_n - g) = 0$. Therefore by Proposition 1,2 it is sufficient to prove that $Y\int_a^b f \, dg$ exists for any pure break function $g \in BV(a, b)$ with a finite number of discontinuities at the points $\{t_1, \ldots, t_\nu\} \subset [a, b]$; let us now prove it: we choose an arbitrary $\tilde{D} = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \in \mathscr{D}$ such that $\{t_1, \ldots, t_\nu\} \subset \tilde{D}$. For every $B = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \in \mathscr{B}(D)$, $D \succ \tilde{D}$ we have

$$Y(B) = \sum_{j=1}^{k} f(\alpha_j) \, \Delta g(\alpha_j) + \sum_{j=1}^{k} f(\tau_j) \, (g(\alpha_j -) - g(\alpha_{j-1}+)) = \sum_{i=1}^{\nu} f(t_i) \, \Delta g(t_i)$$

because $g(\alpha_j -) - g(\alpha_{j-1}+) = 0$ for all $j = 1, 2, \ldots, k$ and $\Delta g(\alpha_j) = 0$ if $\alpha_j \notin \{t_1, \ldots, t_\nu\}$. This implies the existence of $Y\int_a^b f \, dg$ and moreover we have obtained the equality

$$Y \int_a^b f \, dg = \sum_{i=1}^{\nu} f(t_i) \, \Delta g(t_i) \, .$$

From the inequality (1,7) we obtain

Proposition 1,3. (cf. II. 19.3.8 in [2]). *If $f_n : [a, b] \to R$, $\lim f_n = f$ uniformly in $[a, b]$, $g \in BV(a, b)$ and if $Y\int_a^b f_n \, dg$ exists for all $n = 1, 2, \ldots$ then $Y\int_a^b f \, dg$ as well as $\lim_{n\to\infty} Y\int_a^b f_n \, dg$ exist and are equal.*

Corollary 1,3. *If $f, g \in BV(a, b)$ then $Y\int_a^b f \, dg$ exists.*

Proof. It is known that every $f \in BV(a, b)$ is representable as the uniform limit of a sequence f_n of step-functions on $[a, b]$ (see for example 7.3.2.1 in [1]), i.e. every f_n is a pure break function with a finite number of points of discontinuity $\{t_1, t_2, \ldots, t_{\nu_n}\} \subset [a, b]$. We prove that $Y\int_a^b f_n \, dg$ exists for all $n = 1, 2, \ldots$ Let $\tilde{D} \in \mathscr{D}$ be an arbitrary subdivision of $[a, b]$ with $\{t_1, t_2, \ldots, t_{\nu_n}\} \subset D$; let be $D \succ \tilde{D}$, $B = \{\alpha_0, \tau_1, \ldots, \tau_k, \alpha_k\} \in \mathscr{B}(D)$ and let us suppose that $a < t_1 < \ldots < t_{\nu_n} < b$.

Hence using the fact that the function f_n is constant with values $f(a), f(t_i+)$, $i = 1, \ldots, \nu_n - 1$, $f(b)$ in the intervals $[a, t_1), (t_i, t_{i+1})$ $i = 1, \ldots, \nu_n - 1$, $(t_{\nu_n}, b]$ respectively, we obtain

$$Y(B) = \sum_{j=0}^{k} f_n(\alpha_j) \, \Delta g(\alpha_j) + \sum_{j=1}^{k} f_n(\tau_j) \, (g(\alpha_j -) - g(\alpha_{j-1}+)) =$$

$$= f(a) \, \Delta^+ g(a) + \sum_{i=1}^{\nu_n} f(t_i) \, \Delta g(t_i) + f(b) \, \Delta^- g(b) +$$

$$+ f(a+) \, (g(t_1 -) - g(a+)) + \sum_{i=1}^{\nu_n} f(t_i+) \, (g(t_{i+1}-) - g(t_i+)) +$$

$$+ f(b-) \, (g(b-) - g(t_{\nu_n}+)) = \sum_{i=1}^{\nu_n} f(t_i) \, \Delta g(t_i) + \sum_{i=1}^{\nu_n - 1} f(t_i+) \, (g(t_{i+1}-) - g(t_i+)) +$$

$$+ f(a) \, (g(t_1 -) - g(a)) + f(b) \, (g(b) - g(t_{\nu_n}+)) \, ,$$

i.e. the Young sum depends only on $t_1, \ldots, t_{v_n}$ and is independent of the choice of $D \succ \tilde{D}$ and $B \in \mathscr{B}(D)$. This implies that the integral $Y \int_a^b f_n \, dg$ exist and has the value $Y(B)$ evaluated above.

The analogous argument gives the same result if $a = t_1$ or $b = t_{v_n}$. The existence of $Y \int_a^b f \, dg$ follows now from Proposition 1,3.

2. THE KURZWEIL INTEGRAL

Let for any $\tau \in [a, b]$ a $\delta = \delta(\tau) > 0$ be given (i.e. $\delta : [a, b] \to (0, +\infty)$).

Put

(2,1) $$S = \{(\tau, t) \in R^2;\ a \leqq \tau \leqq b,\ \tau - \delta(\tau) \leqq t \leqq \tau + \delta(\tau)\}$$

and denote by $\mathscr{S} = \mathscr{S}(a, b)$ the system of all such sets $S \in R^2$. Any set $S \in \mathscr{S}$ can be evidently characterized by a function δ: $[a, b] \to (0, +\infty)$.

We consider finite sequences of numbers $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$ such that

(2,2) $$a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b\,,$$

(2,3) $$\alpha_{j-1} \leqq \tau_j \leqq \alpha_j\,, \quad j = 1, \ldots, k\,.$$

For a given set $S \in \mathscr{S}$, A is called a subdivision of $[a, b]$ subordinate to S if

(2,4) $$(\tau_j, t) \in S \quad \text{for} \quad t \in [\alpha_{j-1}, \alpha_j]\,, \quad j = 1, 2, \ldots, k\,.$$

The set of all subdivisions A of $[a, b]$ subordinate to $S \in \mathscr{S}$ let be denoted by $A(S)$ (cf. Definition 1,1,3 in [3]). In [3], Lemma 1,1,1 it is proved that $A(S) \neq \emptyset$ for any $S \in \mathscr{S}$.

Let $f : [a, b] \to R$, $g : [a, b] \to R$ be given. For every $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$ satisfying (2,2) and (2,3) we put

(2,5) $$K(A) = \sum_{j=1}^{k} f(\tau_j)\,(g(\alpha_j) - g(\alpha_{j-1}))\,.$$

Definition 2,1. *The function $f : [a, b] \to R$ is Stieltjes integrable on the interval $[a, b]$ with respect to $g : [a, b] \to R$ in the sense of Kurzweil if there is a number I such that to every $\varepsilon > 0$ there exists such a set $S \in \mathscr{S}$ that*

(2,6) $$|K(A) - I| < \varepsilon$$

if $A \in A(S)$. The number I will be denoted by $K \int_a^b f \, dg$ and called the Kurzweil integral of f with respect to g on $[a, b]$.

The following proposition is an obvious consequence of the completeness of R and of Def. 2,1:

Proposition 2,1. *Let $f, g : [a, b] \to R$. The integral $K \int_a^b f \, dg$ exists if and only if for any $\varepsilon > 0$ there is a set $S \in \mathscr{S}$ such that*

$$(2,7) \qquad |K(A_1) - K(A_2)| < \varepsilon$$

for all $A_1, A_2 \in A(S)$.

Remark 2,1. The above Def. 1. follows the definition given in [3] (see 1.2 in [3]). In [3] the notation $\int_a^b DU(\tau, t)$ with $U(\tau, t) = f(\tau) \, g(t)$ is used instead of our symbol $K \int_a^b f \, dg$. Some fundamental theorems (additivity etc). about the Kurzweil integral can be found in [3] (cf. 1,3 in [3]).

Remark 2,2. It is almost evident that if the Riemann-Stieltjes norm integral $NR \int_a^b f \, dg$ exists then also the Kurzweil integral $K \int_a^b f \, dg$ exists and both integrals are equal. To prove this fact it is sufficient to set $\delta(\tau) = |\bar{D}|$ for any $\varepsilon > 0$ where $\bar{D}$ is the subdivision from Def. 1,1.

Though it is not immediately apparent, the Kurzweil integral from Def. 2,1 is equivalent to the Perron-Stieltjes integral if we suppose $g \in BV(a, b)$.

Remark 2,3. For given finite $f : [a, b] \to R$, $g \in BV(a, b)$ we denote by $P \int_a^b f \, dg$ the Perron-Stieltjes integral of the point function f with respect to the additive function G of a interval in $[a, b]$ which is defined by the relation $G(I) = g(d) - g(c)$ for $I = [c, d] \subset [a, b]$ (cf. [4]).

The following theorem states the result promised above.

Theorem 2,1. *Let $f : [a, b] \to R$ be finite, $g \in BV(a, b)$. Then the integral $K \int_a^b f \, dg$ exists if and only if the integral $P \int_a^b f \, dg$ exists and both integrals have the same value.*

Proof. 1. Let $P \int_a^b f \, dg$ exist. From the definition (cf. [4]) we have: For any $\varepsilon > 0$ there is a major function U and a minor function V *) (U and V are additive functions of interval in $[a, b]$) of f with respect to G such that

$$(2,8) \qquad U([a, b]) - V([a, b]) < \varepsilon$$

Let $\delta_1 : [a, b] \to (0, +\infty)$, $\delta_2 : [a, b] \to (0, +\infty)$ be the function occuring in the definition of the minor function V and the major function U, respectively. Let us put $\delta(\tau) = \min (\delta_1(\tau), \delta_2(\tau))$ for any $\tau \in [a, b]$ and let $S \in \mathscr{S}$ be the set which corresponds to $\delta : [a, b] \to (0, +\infty)$ by (2,1). We suppose that an arbitrary $A = \{\alpha_0, \tau_1, \alpha_1, \ldots$

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*) An additive function of an interval V is said to be a minor function of f with respect to G on $[a, b]$ if to each point $\tau \in [a, b]$ there corresponds a number $\delta_1 = \delta_1(\tau) > 0$ such that $V([c, d]) \leq f(\tau) \, G([c, d]) = f(\tau) \, (g(d) - g(c))$ for every interval $[c, d]$ such that $\tau \in [c, d]$ and $|d - c| < \delta_1(\tau)$. The major function U is defined analogously.

$\ldots, \tau_k, \alpha_k\} \in A(S)$ is given. The properties of a subdivision from $A(S)$ as well as those of a major and minor function guarantee the inequality

$$V([\alpha_{j-1}, \alpha_j]) \leqq f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) \leqq U([\alpha_{j-1}, \alpha_j])$$

for any $j = 1, 2, \ldots, k$. Hence the additivity of U and V implies

$$V([a, b]) \leqq \sum_{j=1}^{k} f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) = K(A) \leqq U([a, b]).$$

From (2,8) we obtain in this way the inequality $|K(A_1) - K(A_2)| < \varepsilon$ for all $A_1, A_2 \in A(S)$ which means that by Prop. 2,1 the integral $K \int_a^b f \, dg$ exists. Considering that $P \int_a^b f \, dg = \inf_U U([a, b]) = \sup_V V([a, b])$ we have evidently also $K \int_a^b f \, dg = P \int_a^b f \, dg$.

2. Now we suppose that $K \int_a^b f \, dg$ exists. Let an arbitrary $\varepsilon > 0$ be given. According to Prop. 2,1 we choose a set $S \in \mathscr{S}$ (characterized by $\delta : [a, b] \to (0, +\infty)$) such that

$$|K(A_1) - K(A_2)| < \varepsilon \tag{2,9}$$

for all $A_1, A_2 \in A(S)$.

For a given τ, $a < \tau \leqq b$ let A_τ be a subdivision of $[a, \tau]$ subordinate to $S(A_\tau \in A(S, \tau)$, $A(S, \tau)$ is the set of all subdivisions of $[a, \tau]$ subordinated to S). Let us define

$$M(\tau) = \sup K(A_\tau), \quad m(\tau) = \inf K(A_\tau),$$

$M(a) = m(a) = 0$. We put $U([c, d]) = M(d) - M(c)$, $V([c, d]) = m(d) - m(c)$ for $[c, d] \subset [a, b]$. Hence by definition and by (2,9) we have

$$0 \leqq U([a, b]) - V([a, b]) = M(b) - m(b) \leqq \varepsilon. \tag{2,10}$$

U is a major function of f with respect to G: Let $\delta : [a, b] \to (0, +\infty)$ be the function which characterizes the set S. For fixed $\tau \in [a, b]$ let $[c, d] \subset [a, b]$, $\tau \in [c, d]$, $|d - c| < \delta(\tau)$. Then by definition

$$f(\tau)\, G([c, d]) + M(c) = f(\tau)(g(d) - g(c)) + M(c) \leqq M(d),$$

i.e.

$$f(\tau)\, G([c, d]) \leqq M(d) - M(c) = U([c, d]).$$

In a similar way it can be proved that V is a minor function of f with respect to G in $[a, b]$.

The existence of the Perron-Stieltjes integral $P \int_a^b f \, dg$ follows immediately from (2,10).

Definition 2,2. *Let $g : [a, b] \to R$ be given. A point $t \in [a, b]$ is called a point of variability of the function g if to every $\varepsilon > 0$ there is a $t' \in [a, b]$, $|t - t'| < \varepsilon$*

such that $g(t) \neq g(t')$. The set of all points of variability of g in $[a, b]$ is denoted by V_g while $C_g = [a, b] - V_g$.

It is easy to prove that the set V_g is closed in $[a, b]$.

Proposition 2,2. *Let $f_1, f_2, g : [a, b] \to R, f_1(t) = f_2(t)$ for $t \in V_g$ and let $K \int_a^b f_1 \, dg$ exist. Then $K \int_a^b f_2 \, dg$ exists and equals $K \int_a^b f_1 \, dg$.*

Proof. For every $\tau \in C_g = [a, b] - V_g$ there is by definition a $\tilde{\delta}(\tau) > 0$ such that for all $\tau' \in [a, b]$, $|\tau - \tau'| < \tilde{\delta}(\tau)$ we have $g(\tau) = g(\tau')$. Since $K \int_a^b f_1 \, dg$ exists, we can choose to every $\varepsilon > 0$ a set $S \in \mathcal{S}$ (characterized by a function $\delta : [a, b] \to \to (0, +\infty)$) such that

$$(2,11) \qquad \left| \sum_{j=1}^{k} f_1(\tau_j) \left(g(\alpha_j) - g(\alpha_{j-1}) \right) - K \int_a^b f_1 \, dg \right| < \varepsilon$$

for any $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \subset A(S)$. We define $\delta^*(\tau) = \delta(\tau)$ for $\tau \in V_g$ and $\delta^*(\tau) = \min(\delta(\tau), \tilde{\delta}(\tau)/2)$ for $\tau \in C_g$; evidently $\delta^*(\tau) \leqq \delta(\tau)$ for all $\tau \in [a, b]$ and $S^* \subset S$ if $S^* \in \mathcal{S}$ is the set in R^2 characterized by the function $\delta^* : [a, b] \to (0, +\infty)$. Let further $A \in A(S^*)$, then also $A \in A(S)$ and (2,11) holds for any $A = \{\alpha_0, \tau_1, \alpha_1, \ldots \ldots, \tau_k, \alpha_k\} \in A(S^*)$. If $\tau_j \in C_g$ then we have from (2,3) that $|t - \tau_j| \leqq \delta^*(\tau_j) \leqq \leqq \tilde{\delta}(\tau_j)/2 < \tilde{\delta}(\tau_j)$ for all $t \in [\alpha_{j-1}, \alpha_j]$ and therefore $g(\alpha_j) - g(\alpha_{j-1}) = 0$. Hence for all $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \in A(S)$ we have by assumption

$$\sum_{j=1}^{k} f_1(\tau_j) \left(g(\alpha_j) - g(\alpha_{j-1}) \right) = \sum_{j=1}^{k} f_2(\tau_j) \left(g(\alpha_j) - g(\alpha_{j-1}) \right)$$

and by (2,11) also

$$\left\| \sum_{j=1}^{k} f_2(\tau_j) \left(g(\alpha_j) - g(\alpha_{j-1}) \right) - K \int_a^b f_1 \, dg \right\| < \varepsilon$$

for any $A \in A(S^*)$. This completes the proof.

Proposition 2,3. *Let $g_l, g \in BV(a, b)$, $l = 1, 2, \ldots$ and $\lim_{l \to \infty} \operatorname{var}_a^b (g_l - g) = 0$. Further we assume that for $f : [a, b] \to R$ it is $|f(t)| \leqq M$ for all $t \in [a, b]$ and that $K \int_a^b f \, dg_l$ exists for all $l = 1, 2, \ldots$ Then also $K \int_a^b f \, dg$ and the limit $\lim_{l \to \infty} K \int_a^b f \, dg_l$ exist and the equality*

$$\lim_{l \to \infty} K \int_a^b f \, dg_l = K \int_a^b f \, dg$$

holds.

Proof. For every subdivision $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$ we have evidently

$$(2,12) \qquad |K(A) - K_l(A)| \leqq M \,.\, \operatorname{var}_a^b (g - g_l)$$

where $K_l(A)$ is the Kurzweil sum for f and g_l.

Let $\varepsilon > 0$ be given. We choose l_0 such that $\text{var}_a^b(g_l - g) < \varepsilon/4M$ for $l > l_{0_b}$. (If $M = 0$ then the proposition is evidently valid.) Since $K \int_a^b f \, dg_l$ exists for all l we can find for a given $l > l_0$ a set $S \in \mathcal{S}$ such that for any $A_1, A_2 \in A(S)$ we have $|K_l(A_1) - K_l(A_2)| < \varepsilon/2$ (cf. Prop. 3,1). Hence

$$|K(A_1) - K(A_2)| \leqq |K(A_1) - K_l(A_1)| + |K_l(A_1) - K_l(A_2)| + |K_l(A_2) - K(A_2)| \leqq$$
$$\leqq 2M \, \text{var}_a^b(g_l - g) + \varepsilon/2 < \varepsilon$$

for any $A_1, A_2 \in A(S)$ and $K \int_a^b f \, dg$ exists by Prop. 2,1. The other part of the proposition is a consequence of the inequality (2,12).

Corollary 2,1. *If $g_b \in BV(a, b)$ is a pure break function and $f : [a, b] \to R$ is bounded then $K \int_a^b f \, dg_b$ exists and we have $K \int_a^b f \, dg_b = \sum_{t \in [a,b]} f(t) \, \Delta g_b(t)$.*

Proof. Similarly as in the proof of Corollary 1,2 it is sufficient to prove that $K \int_a^b f \, dg$ exists for any pure break function $g \in BV(a, b)$ which is discontinuous at the points of a finite set $\{t_1, t_2, \ldots, t_\nu\} \subset [a, b]$ and that $K \int_a^b f \, dg = \sum_{i=1}^{\nu} f(t_i) \, \Delta g(t_i)$. Let us suppose that $a \leqq t_1 < t_2 < \ldots t_\nu < b$ and let us define

$$\delta(\tau) = \tfrac{1}{4}\varrho(\tau, \{a, t_1, \ldots, t_\nu, b\})$$

for $\tau \in (a, b)$, $\tau \neq t_i$, $i = 1, \ldots, \nu$, where ϱ is the Euclidean distance; further we define

$$\Delta_j = \max_{\tau \in (t_j, t_{j+1})} \delta(\tau), \quad j = 1, \ldots, \nu - 1$$

and $\Delta_0 = \max_{\tau \in (a, t_1)} \delta(\tau)$, $\Delta_\nu = \max_{\tau \in (t_\nu, b)} \delta(\tau)$ if $a < t_1$, $t_\nu < b$, respectively and we set $\delta(a) = \delta(t_j) = \delta(b) = \Delta$, $j = 1, \ldots, \nu$, where $\Delta = \min_j (\Delta_j)$. In this way we have defined a function $\delta : [a, b] \to (0, +\infty)$ which provides a set S defined by (2,1).

Let now $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \in A(S)$. By definition we have $[\alpha_{j-1}, \alpha_j] \subset [\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)]$ for any $j = 1, \ldots, k$ and the following assertions are valid:

1) if $\tau_j \in \{a, t_1, \ldots, t_\nu, b\}$ then $|\alpha_j - \alpha_{j-1}| \leqq 2\delta(\tau_j) = 2\Delta$ and $[\alpha_{j-1}, \alpha_j] \cap \{a, t_1, \ldots, t_\nu, b\} = \tau_j$,

2) if $\tau_j \notin \{a, t_1, \ldots, t_\nu, b\}$ then $|\alpha_j - \alpha_{j-1}| \leqq 2\delta(\tau_j) = \tfrac{1}{2}\varrho(\tau_j, \{a, t_1, \ldots, t_\nu, b\})$ and therefore $[\alpha_{j-1}, \alpha_j] \cap \{a, t_1, \ldots, t_\nu, b\} = \emptyset$.

Hence $\{a, t_1, \ldots, t_\nu, b\} \subset \{\tau_1, \ldots, \tau_k\}$ and

$$K(A) = \sum_{j=1}^{k} f(\tau_j) \, (g(\alpha_j) - g(\alpha_{j-1})) = f(a) \, (g(a+) - g(a)) +$$
$$+ \sum_{i=1}^{\nu} f(t_i) \, (g(t_i+) - g(t_i-)) + f(b) \, (g(b) - g(b-)) = \sum_{i=1}^{\nu} f(t_i) \, \Delta g(t_i)$$

for any $A \in A(S)$, i.e. $K \int_a^b f\, dg$ exists and equals $\sum_{i=1}^{v} f(t_i)\, \Delta g(t_i)$. This proves the corollary.

Proposition 2,4. *Let $T \subset (a, b)$ be given such that $[a, b] - T$ is dense in $[a, b]$ (i.e. $\overline{[a, b] - T} = [a, b]$) and let $g(t) = 0$ for $t \in [a, b] - T$. If $K \int_a^b f\, dg$ exists then necessarily $K \int_a^b f\, dg = 0$.*

Proof. For any $\delta : [a, b] \to (0, +\infty)$ we choose from the system of intervals $(\tau - \delta(\tau), \tau + \delta(\tau))$, $\tau \in [a, b]$ a finite system $(\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) = J_j$, $j = 1, \ldots, k$ such that $\tau_j < \tau_{j+1}$, $[a, b] \subset \bigcup_{j=1}^{k} J_j$ and $[a, b] - \bigcup_{\substack{j=1 \\ j \neq r}}^{k} J_j \neq \emptyset$ for any $r = 1, \ldots, k$. Hence $J_j \cap J_{j+1} \neq \emptyset$ is an interval for all $j = 1, \ldots, k - 1$ and the density of $[a, b] - T$ implies that there is an $\alpha_j \in (J_j \cap J_{j+1}) \cap ([a, b] - T)$ for $j = 1, \ldots, k - 1$. If we set $\alpha_0 = a$, $\alpha_k = b$, then we evidently obtain a subdivision $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \in A(S)$, where S is determined by δ (cf. (2,1)) and $g(\alpha_i) = 0$ for $i = 0, 1, \ldots, k$. Hence we have $K(A) = 0$ for this subdivision A and our proposition follows immediately from Def. 2,1.

Example 2,1 (due to I. Vrkoč). Let $g(1/(l + 1)) = 2^{-l}$, $l = 1, 2, \ldots$, $g(t) = 0$ for $t \in [0, 1] - \{1/(l + 1)\}_{l=1}^{\infty}$. Evidently $g \in BV(a, b)$. Let us put $f(1/(l + 1)) = 2^l$, $f(t) = 0$ for $t \in [0, 1] - \{1/(l + 1)\}_{l=1}^{\infty}$. We show that the integral $K \int_0^1 f\, dg$ does not exist. For an arbitrary $\delta : [0, 1] \to (0, +\infty)$ we set $\alpha_0 = \tau_0 = 0$. Since $1/(l + 1) \to 0$ for $l \to \infty$, in $(0, \delta(0))$ there exists a point of the form $1/(l_0 + 1)$. We set further $\alpha_1 = \tau_1 = 1/(l_0 + 1)$ and choose points $\alpha_2, \ldots, \alpha_k$ and $\tau_2, \ldots, \tau_k$ such that $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\} \in A(S)$ where S is the set given by δ (cf. (2,1)) and $g(\alpha_j) = 0$ for $j = 2, \ldots, k$.

This choice of $A \in A(S)$ yields

$$K(A) = \sum_{j=1}^{k} f(\tau_j)\,(g(\alpha_j) - g(\alpha_{j-1})) = f(\tau_1)\, g(\alpha_1) =$$

$$= f(1/(l_0 + 1))\, g(1/(l_0 + 1)) = f(1/(l_0 + 1))\, g(1/(l_0 + 1)) = 1$$

for any $\delta : [0, 1] \to (0, +\infty)$. Hence the integral $K \int_a^b f\, dg$ cannot exist. Indeed, if it existed, its value would be zero by Prop. 2,4 the set $T = \{1/(l + 1)\}_{l=1}^{\infty}$ having all properties required in Prop. 2,4. However, for any S we have constructed an $A \in A(S)$ such that $K(A) = 1$ and Definition 2,1 yields a contradiction with the existence of $K \int_a^b f\, dg$.

The set $T = \{1/(l + 1)\}_{l=1}^{\infty} = V_g$ is the set of all points of variability of g. The function g is evidently of bounded variation in $[0, 1]$ ($g \in BV(0, 1)$). By Prop. 2,2 the integral $K \int_a^b f\, dg$ does not exist for g given above and for any arbitrary function f satisfying $f(1/(l + 1)) = 2^{-1}$, $f : [0, 1] \in R$ (e.g. for the function from Example 1,1).

In this way functions $g \in BV(0, 1)$ are constructed such that the Young integral $Y \int_0^1 f\, dg$ exists but the Kurzweil integral $K \int_0^1 f\, dg$ does not.

3. COMPARISON OF $Y \int_a^b f\, dg$ AND $K \int_a^b f\, dg$ FOR $g \in BV(a, b)$

In this section we assume that $g \in BV(a, b)$, $f : [a, b] \to R$ and $Y \int_a^b f\, dg$ exists. The aim of our study is to find additional properties of f and g guaranteeing the existence of the integral $K \int_a^b f\, dg$.

For the function $g \in BV(a, b)$ let us denote by $N_S \subset (a, b)$ the set of all points $t \in (a, b)$ of discontinuity of the function g for which $g(t-) = g(t+)$, i.e.

$$N_S = \{t \in (a, b);\, g(t-) = g(t+),\ g(t) \neq g(t-)\}$$

and let us define $g_S(t) = g(t) - g(t-)$ for $t \in N_S$, $g_S(t) = 0$ for $t \in [a, b] - N_S$; we have evidently $g_S \in BV(a, b)$ because $\operatorname{var}_a^b g_S = 2 \sum_{t \in N_S} (g(t) - g(t-)) < \operatorname{var}_a^b g$. In Prop. 1,1 we have proved that $Y \int_a^b f\, dg_S$ exists for any function $f : [a, b] \to R$ and $Y \int_a^b f\, dg_S = 0$.

We denote further $g_R = g - g_S$; evidently $g_R \in BV(a, b)$ and if $g_R(t+) = g_R(t-)$ then $g_R(t) = g_R(t-)$, i.e. g_R is continuous at all points of continuity of g as well as for all $t \in N_S$.

Since $Y \int_a^b f\, dg_s$ exists by the assumption, the integral $Y \int_a^b f\, dg_R$ exists as well and equals $Y \int_a^b f\, dg - Y \int_a^b f\, dg_S = Y \int_a^b f\, dg$. Using the existence of $Y \int_a^b f\, dg_R$ we obtain from Theorem 1,2 that f is bounded on a finite number of closed intervals which are complementary to a finite number of open intervals on which the function g_R is constant. It is possible to assume that $|f(t)| \leqq M$ for all $t \in [a, b]$; in the opposite case we set $\tilde{f} = f$ on the set on which f is bounded and $\tilde{f} = 0$ otherwise. By Corollary 1,1 the existence of $Y \int_a^b f\, dg_R$ is equivalent to the existence of $Y \int_a^b \tilde{f}\, dg_R$ and we have $Y \int_a^b f\, dg_R = Y \int_a^b \tilde{f}\, dg_R$.

Now we uset the usual decomposition $g_R = g_c + g_{Rb}$ of $g_R \in BV(a, b)$ into the continuous part g_c and a pure break function g_{Rb}. Corollary 1,2 guarantees the existence of $Y \int_a^b f\, dg_{Rb}$ and so we obtain also the existence of $Y \int_a^b f\, dg_c$. Moreover, we have

$$Y \int_a^b f\, dg_{Rb} = \sum_{t \in [a,b]} f(t)\, \Delta g_{Rb}(t) = \sum_{t \in [a,b]} f(t)\, \Delta g(t)\,.$$

Since $g_c \in BV(a, b)$ is continuous the norm integral $NY \int_a^b f\, dg_c$ exists by Theorem 1,5 and by Theorem 1,4 also the Riemann-Stieltjes norm integral $NR \int_a^b f\, dg_c$ exists. From Remark 2,2 the existence of $K \int_a^b f\, dg_c$ and the equality $K \int_a^b f\, dg_c = Y \int_a^b f\, dg_c$ immediately follows. Further, Corollary 2,1 implies the existence of $K \int_a^b f\, dg_{Rb}$ since the function f is bounded, and also the equality $K \int_a^b f\, dg_{Rb} = Y \int_a^b f\, dg_{Rb}$.