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ON A GRAPH THEORY PROBLEM OF M. KOMAN

DRAGOŠ M. CVETKOVIĆ, Beograd

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We consider only finite undirected graphs without loops or multiple edges.

Let G be a graph with vertices numbered by $1, \ldots, s$. A walk of length k in G is a sequence $i_1, \ldots, i_{k+1}$ of vertices with i_j and i_{j+1} adjacent for every $j = 1, \ldots, k$. The adjacency matrix A of the graph G is defined by $A = \|a_{ij}\|_1^s$, where a_{ij} is equal to the number of edges connecting the vertex i with the vertex j. It is known that the element at the place (i, j) of matrix A^k is equal to the number of walks of length k leading from the vertex i to the vertex j [1], p. 124.

Let $B = \{\beta_1, \ldots, \beta_q\}$ be a set of n-tuples $\beta_f = (\beta_{f1}, \ldots, \beta_{fn})$, $f = 1, \ldots, q$ of the numbers 0 and 1 not containing n-tuple $(0, \ldots, 0)$. NEPS (incomplete extended p-sum of graphs [2]) with the basis B of graphs $G_1, \ldots, G_n$ is the graph $G = g_B(G_1, \ldots$ $\ldots, G_n)$, whose set of vertices is equal to the Cartesian product of the sets of vertices of graphs $G_1, \ldots, G_n$ and in which two vertices $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ are adjacent if and only if there is an n-tuple $(\beta_{f1}, \ldots, \beta_{fn})$, in B, such that $p_j = q_j$ exactly when $\beta_{fj} = 0$ and p_j is adjacent to q_j in G_j exactly when $\beta_{fj} = 1$.

We shall now deduce a relation between the numbers of walks in $G_1, \ldots, G_n$ and the number of walks in $g_B(G_1, \ldots, G_n)$; this relation is a little more precise than the corresponding ones in [3] and [2].

Let the vertices in every graph $G_1, \ldots, G_n$ be ordered (numbered). We shall give the lexicographic order to the vertices of NEPS (representing the ordered n-tuples of vertices of graphs $G_1, \ldots, G_n$) and we shall form adjacency matrix $\mathscr{A}$ of NEPS according to this ordering.

If $A_1, \ldots, A_n$ are the adjacency matrices of graphs $G_1, \ldots, G_n$, the adjacency matrix of $g_B(G_1, \ldots, G_n)$ is given by

$$(1) \qquad \mathscr{A} = \sum_{f=1}^{q} A_1^{\beta_{f1}} \otimes \ldots \otimes A_n^{\beta_{fn}},$$

where $\otimes$ denotes Kronecker's multiplication of matrices [2].

Let $B_f = A_1^{\beta_{f1}} \otimes \ldots \otimes A_n^{\beta_{fn}}, f = 1, \ldots, q$. Then

$$\mathscr{A}^k = (B_1 + \ldots + B_q)^k = \sum_{s_1,\ldots,s_q} \frac{k!}{s_1! \ldots s_q!} B_1^{s_1} \ldots B_q^{s_q} = \tag{2}$$

$$= \sum_{s_1,\ldots,s_q} \frac{k!}{s_1! \ldots s_q!} A_1^{l_1} \otimes \ldots \otimes A_n^{l_n},$$

where the sum is taken over all ordered partitions (compositions) of the number k and where $l_i = \sum_{f=1}^{q} \beta_{fi} s_f$ $(i = 1, \ldots, n)$.

Let x and y be two vertices of the graph to which a square matrix Z of the order equal to the number of vertices corresponds. $(Z)_{x,y}$ denotes the element of Z from the row corresponding to x and the column corresponding to y.

According to (2) we have

$$(\mathscr{A}^k)_{(x_1,\ldots,x_n),(y_1,\ldots,y_n)} = \sum_{s_1,\ldots,s_q} \frac{k!}{s_1! \ldots s_q!} (A_1^{l_1})_{x_1,y_1} \ldots (A_n^{l_n})_{x_n,y_n}. \tag{3}$$

Let $N^k_{(x_1,\ldots,x_n),(y_1,\ldots,z_n)}$ be the number of walks of length k in NEPS leading from the vertex $(x_1, \ldots, x_n)$ to the vertex $(y_1, \ldots, y_n)$ and ${}^iN^k_{x_i,y_i}$, $i = 1, \ldots, n$ the numbers of walks of length k in G_i leading from x_i to y_i. Relation (3) can be written in the following way:

$$N^k_{(x_1,\ldots,x_n),(y_1,\ldots,y_n)} = \sum_{s_1,\ldots,s_q} \frac{k!}{s! \ldots s_q!} {}^1N^{l_1}_{x_1,y_1} \ldots {}^nN^{l_n}_{x_n,y_n}. \tag{4}$$

According to [4] we can deduce that the numbers ${}^iN^k_{x_i,y_i}$ are of the form

$${}^iN^k_{x_i,y_i} = \sum_{j_i=0}^{b_i} {}^i_{j_i}C_{x_i,y_i} {}^i\lambda^k_{j_i}, \tag{5}$$

where ${}^i_{j_i}C_{x_i,y_i}$, ${}^i\lambda_{j_i}$ are real numbers and b_i nonnegative integers.

Substituting (5) into (4) we get

$$N^k_{(x_1,\ldots,x_n),(y_1,\ldots,y_n)} = \sum_{s_1,\ldots,s_q} \frac{k!}{s_1! \ldots s_q!} \sum_{j_1=0}^{b_1} {}^1_{j_1}C_{x_1,y_1} {}^1\lambda^{l_1}_{j_1} \ldots \sum_{j_n=0}^{b_n} {}^n_{j_n}C_{x_n,y_n} {}^n\lambda^{l_n}_{j_n} = \tag{6}$$

$$= \sum_{j_1,\ldots,j_n} {}^1_{j_1}C_{x_1,y_1} \ldots {}^n_{j_n}C_{x_n,y_n} \sum_{s_1,\ldots,s_q} \frac{k!}{s_1! \ldots s_q!} {}^1\lambda_{j_1}^{\sum_{f=1}^{q} \beta_{f1} s_f} \ldots {}^n\lambda_{j_n}^{\sum_{f=1}^{q} \beta_{fn} s_f} =$$

$$= \sum_{j_1,\ldots,j_n} {}^1_{j_1}C_{x_1,y_1} \ldots {}^n_{j_n}C_{x_n,y_n} \sum_{s_1,\ldots,s_q} \frac{k!}{s_1! \ldots s_q!} \prod_{f=1}^{q} ({}^1\lambda^{\beta_{f1}}_{j_1} \ldots {}^n\lambda^{\beta_{fn}}_{j_n})^{s_f} =$$

$$= \sum_{j_1,\ldots,j_n} {}^1_{j_1}C_{x_1,y_1} \ldots {}^n_{j_n}C_{x_n,y_n} \Big(\sum_{f=1}^{q} {}^1\lambda^{\beta_{f1}}_{j_1} \ldots {}^n\lambda^{\beta_{fn}}_{j_n}\Big)^k,$$

where $\sum_{j_1,\ldots,j_n}$ denotes the sum over all n-tuples $j_1, \ldots, j_n$ for which $0 \leqq j_i \leqq b_i$, $i = 1, \ldots, n$ holds.

Adding up relation (6) for all pairs of vertices in a NEPS we get the result of [3], i.e. of [2].

The number of walks of length k joining two given vertices in the graph G (G to be defined below) is determined in [5]. Vertices of G are all n-tuples $(p_1, \ldots, p_n)$, where $1 \leqq p_i \leqq r_i$, $i = 1, \ldots, n$. Two vertices are adjacent if and only if they differ in exactly one coordinate (in [5] directed graphs are treated but the above formulation using undirected graphs is equivalent). We shall extend obtained results to the graph of a somewhat more general form and we shall use a method different from that in [5].

NEPS with the basis containing all possible n-tuples having exactly one 1 is called the sum of graphs. The above described graph G can be represented as the sum of graphs $K_{r_1}, \ldots, K_{r_n}$, where K_r denotes the complete graph with r vertices. We shall consider an arbitrary NEPS of the mentioned complete graphs and we shall determine the number of walks of length k joining two given vertices of NEPS.

Primarily, we shall find the expressions for the number of walks of length k in a complete graph. Due to the symmetry, we shall distinguish only two case: 1°(2°) the first and the last vertex of the walk are (are not) identical.

Let a complete graph with r vertices be given. The number of all walks of length k is, obviously, equal to $r(r-1)^k$. The set of eigenvalues of the adjacency matrix of the graph contains the number $r-1$ as well as $r-1$ numbers equal to -1. The number of all walks of length k starting and terminating at the same vertex is equal to the trace of the matrix A^k, i.e., to $(r-1)^k + (r-1)(-1)^k$. The number of all walks of length k joining two non-identical vertices is then $r(r-1)^k - [(r-1)^k + (r-1) \cdot (-1)^k] = (r-1)(r-1)^k - (r-1)(-1)^k$. The number of walks starting and terminating at the given vertex is equal to $(r-1)^k/r + (r-1)(-1)^k/r$ and the number of walks starting at the given vertex and terminating at the other given vertex is equal to $(r-1)^k/r - (-1)^k/r$. The number of walks starting at the vertex i and terminating at the vertex j can be, in general, expressed by

$$N_{i,j} = \frac{1}{r}\left[(r-1)^k + (r\delta_{ij} - 1)(-1)^k\right], \tag{7}$$

where δ_{ij} denotes Kronecker's δ-symbol.

Let $G_i = K_{r_i}$ $(r_i \geqq 2)$ and let p_i and q_i be two vertices of G_i, for every i. Then

$${}^iN^k_{p_i,q_i} = \frac{1}{r_i} \sum_{j_i=0}^{1} (r_i\delta_{p_iq_i} - 1)^{j_i} (r_i - 1 - r_ij_i)^k . \tag{8}$$

The number of walks of length k starting at $(p_1, \ldots, p_n)$ and terminating at