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ON THE HEAT POTENTIAL OF THE DOUBLE DISTRIBUTION

JIŘÍ VESELÝ, Praha

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1. INTRODUCTORY REMARK

In the present article some properties of the heat potential of the double distribution are investigated. Similar questions for the Newtonian potential were studied in [5], [2]. Methods, notation and results from those papers and from [4] are used. Proofs are often very similar to proofs in the mentioned papers and because of it they are not given in detail. If necessary, some differences are described. The acquaintance with those articles is not expected but it is useful for understanding the process of the proofs.

Let R^k be the Euclidean k-space where $k = m$ or $k = m + 1$ and $m \geqq 3$. To distinguish the notation in R^m and R^{m+1} we shall use the symbol *. For example if $a \in R^k$, $\varrho > 0$, we shall put

$$\Omega(a, \varrho) = \{b \in R^k;\ |b - a| < \varrho\} \tag{1.1}$$

where $|\ldots|$ denotes the Euclidean norm. We shall denote by Γ the boundary $\partial\Omega(0, 1)$ of $\Omega(0, 1)$. Symbols $\Omega(0, 1)$ and Γ stand for ball and sphere in R^{m+1} and $\Omega^*(0, 1)$ and Γ^* have the same meaning in R^m.

To describe the point $z = [z_1, \ldots, z_{m+1}] \in R^{m+1}$ we shall use this convention: we shall write $z = [x, t]$ where $x = [z_1, \ldots, z_m] \in R^m$ and $z_{m+1} = t \in R^1$. Very often we shall use the symbol $\hat{z}$ for $[z_1, \ldots, z_m]$ so that we can write $z = [\hat{z}, z_{m+1}]$. For the differential operator $\nabla = [\partial_1, \ldots, \partial_{m+1}]$ we shall write similarly $\hat{\nabla} = [\partial_1, \ldots, \partial_m]$ and the same convention we shall apply for the Laplace operator Δ.

Let G be a fixed Borel set in R^m. We shall study the heat potential of the double distribution with the continuous density defined on $\partial(G \times R^1)$. No other restrictions about the boundary ∂G are introduced. The heat potential is usually introduced by the following standard process: let G be a bounded region in R^m with the "sufficiantly smooth" boundary $\partial G = B$. We shall denote by H_k the k-dimensional Hausdorff measure and by $n(y)$ the exterior normal to G at the point $y \in \partial G$. For each bounded continuous function f on $B \times R^1$ and $[x, t] \in R^{m+1} - (B \times R^1)$ we put

$$(1.2) \qquad W^G f(x, t) = \int_{-\infty}^{t} \left(\int_B f(y, u)\,(t - u)^{-m/2} \exp\left(- \frac{|x - y|^2}{4(t - u)} \right) . \right.$$
$$\left. . \frac{(y - x) \circ n(y)}{2(t - u)} \,\mathrm{d}H_{m-1}(y) \right) \mathrm{d}u \,.$$

Function $W^G f$ is called the heat potential of the double distribution with the density f. Sometimes we take $t \in \langle T_1, T_2 \rangle \subset R^1$ only.

We shall define $\mathscr{G}$ on R^{m+1} by

$$(1.3) \qquad \mathscr{G}(z) = z_{m+1}^{-m/2} \exp\left(- \frac{|\hat{z}|}{4 z_{m+1}} \right) \quad \text{for} \quad z_{m+1} > 0\,,$$
$$\mathscr{G}(z) = 0 \qquad \text{for} \quad z_{m+1} \leqq 0\,.$$

$\mathscr{G}$ is infinitely differentiable on $R^{m+1} - \{0\}$. Consequently the same is valid about $\mathscr{G}(z - w)$ on $R^{m+1} - \{w\}$ and $\mathscr{G}(z - w)$ is caloric on this set, i.e.

$$(1.4) \qquad \hat{\Delta}_z \mathscr{G}(z - w) - \frac{\partial}{\partial z_{m+1}} \mathscr{G}(z - w) = 0\,.$$

Similarly for $w \in R^{m+1} - \{z\}$ the following equation is valid

$$(1.5) \qquad \hat{\Delta}_w \mathscr{G}(z - w) + \frac{\partial}{\partial w_{m+1}} \mathscr{G}(z - w) = 0\,.$$

Changing the notation we have

$$(1.6) \qquad W^G f(x, t) = - \int_{-\infty}^{\infty} \left(\int_B f(y, u) \,.\, n(y) \circ \hat{\nabla} \mathscr{G}(x - y,\ t - u) \,\mathrm{d}H_{m-1}(y) \right) \mathrm{d}u$$

where f is a function continuous and bounded on $C = B \times R^1$ (we write again f instead of the restriction f/C). If we put for $T \in R^1$

$$(1.7) \qquad R_T = R^m \times (T, +\infty)\,,$$

the value $W^G f(z)$ for $z = [x, t] \in R^{m+1} - C$ depends on $f/(C - R_t)$ only. We can find for such a z the signed measure ν_z with the support spt $\nu_z \subset C - R_t$ and write (1.6) in the form

$$W^G f(z) = \int_{R^{m+1}} f \,\mathrm{d}\nu_z\,.$$

Function $W^G f$ is caloric on $R^{m+1} - C$ and we shall try in the next items to define $W^G f$ for a broad class of functions f under the minimal assumptions about G (or B). Then we shall study the behaviour of $W^G f$ for the fixed function f.

2. REPRESENTATION OF $W^G f(z)$ BY MEANS OF A MEASURE

Let G be a fixed Borel set with a compact nonvoid boundary B in R^m, the Euclidean m-space with $m \geqq 3$. Defining E, C by

$$E = G \times R^1\,, \quad C = B \times R^1 \tag{2.1}$$

we shall denote by $\mathscr{B}$ or $\mathscr{C}$ the space of all bounded Baire functions or bounded continuous functions on C, respectively. According to the convention from the introductory remark we shall denote by $\mathscr{B}^*$ and $\mathscr{C}^*$ similarly defined spaces of functions on B. The space $\mathscr{D}$ (or $\mathscr{D}^*$) is the space of all infinitely differentiable functions ψ with a compact support spt ψ in R^{m+1} (or R^m). We shall use the symbol $\|\ldots\|$ for the norm which is defined in all those spaces in the usual way by means of supremum. Given $z \in R^{m+1}$ we put

$$\mathscr{D}(z) = \{\psi \in \mathscr{D};\ z \notin \operatorname{spt} \psi\}\,. \tag{2.2}$$

In what follows a measure is a Borel signed measure which is finite on any compact subset of R^{m+1}. For a measure ν and a Borel set $A \subset R^{m+1}$ we denote by $|\nu|\,(A)$ the variation of ν on A.

For all $z \in R^{m+1}$, $\psi \in \mathscr{D}(z)$ we put

$$W^G\,\psi(z) = -\int_E (\hat{\nabla}_w \mathscr{G}(z - w) \circ \hat{\nabla}\psi(w) + \mathscr{G}(z - w)\,\partial_{m+1}\psi(w))\,\mathrm{d}w\,. \tag{2.3}$$

This integral is finite for all $\psi \in \mathscr{D}(z)$ (cf. estimates (10) and (11) in [4]). In the special case when the boundary B is a smooth hypersurface in R^m we can obtain (1.6) from (2.3) in the following way: we replace $\mathscr{G}$ by $\bar{\mathscr{G}} \in \mathscr{D}$ so that $\bar{\mathscr{G}}(w) = \mathscr{G}(z - w)$ in some neighborhood of spt ψ. We write $z = [x, t]$, $w = [y, u]$. Applying Fubini's theorem we obtain

$$\begin{aligned} W^G\,\psi(z) = &-\int_{-\infty}^{\infty}\left(\int_G \hat{\nabla}\bar{\mathscr{G}}(y, u) \circ \hat{\nabla}\psi(y, u)\,\mathrm{d}y\right)\mathrm{d}u\, - \\ &-\int_{-\infty}^{\infty}\left(\int_G \partial_u\psi(y, u)\,\bar{\mathscr{G}}(y, u)\,\mathrm{d}y\right)\mathrm{d}u\,. \end{aligned} \tag{2.4}$$

Now we change order of integration in the second integral, use (1.5) and integrating by parts. After a simple calculation we obtain

$$W^G\,\psi(z) = -\int_{-\infty}^{\infty}\left(\int_G \hat{\nabla}\bar{\mathscr{G}}(y, u) \circ \hat{\nabla}\psi(y, u) + \psi(y, u)\sum_{j=1}^{m}\partial^2_{y_j}\bar{\mathscr{G}}(y, u)\,\mathrm{d}y\right)\mathrm{d}u\,.$$

Making use of the Gauss's integral theorem we get easily (1.6).

If spt $\psi \cap C = \emptyset$ then there is a $\tilde{G}$ with the smooth boundary $\tilde{B}$ such that spt $\psi \cap \cap E \subset \tilde{G} \times R^1$. Preceding consideration gives us $W^G \psi(z) = W^{\tilde{G}} \psi(z) = 0$ so that $W^G \psi(z)$ depends just only on the values of ψ in a neighborhood of boundary C.

If $z \notin R^{m+1} - C$, we use this fact to extend $W^G \psi(z)$ from $\mathscr{D}(z)$ to $\mathscr{D}$ defining

$$W^G \psi(z) \overset{\text{def.}}{=\!=} W^G \tilde{\psi}(z)\,, \tag{2.5}$$

where $\tilde{\psi}$ is a function in $\mathscr{D}(z)$ coinciding with given $\psi \in \mathscr{D}$ in some neighborhood of C. In what follows we shall write $W\psi(z)$ instead $W^G \psi(z)$ because we shall work with fixed G only. $W\psi(z)$ may be considered as a distribution over $\mathscr{D}$. Some basic properties of $W\psi(z)$ are described in the following simple

2.1 Lemma. *For any $z \in R^{m+1} - C$ the support of the distribution $W\psi(z)$ over $\mathscr{D}$ is contained in $C - R_{z_{m+1}}$. For fixed $\psi \in \mathscr{D}$ the function $W\psi(z)$ is caloric function on $R^{m+1} - C$.*

We shall start with the following question: What necessary and sufficient restrictions are to be imposed on G to secure the existence of a measure ν_z with $|\nu_z|(C) < < +\infty$ with the property

$$W\psi(z) = \int_{R^{m+1}} \psi \, d\nu_z \tag{2.6}$$

for every $\psi \in \mathscr{D}(z)$? In such a case we shall say that $W\psi(z)$ is a measure.

We shall use some of the ideas developed in [4], [5]. Let S be an open segment or half-line (without end-points) in R^k and $A \subset R^k$ be a Borel set. Then $b \in S$ will be called a hit of S on A provided

$$H_1(\Omega(b, \varrho) \cap S \cap A) > 0\,, \quad H_1((\Omega(b, \varrho) \cap S) - A) > 0 \tag{2.7}$$

for any $\varrho > 0$. For $a \in R^k$, $\Theta \in R^k - \{0\}$ we denote by $n_r(a, \Theta)$ the number (possibly 0 or ∞) of all hits of $S_r(a, \Theta) = \{b \in R^k;\, b = a + \varrho\Theta,\, 0 < \varrho < r\}$ on A. Then $n_r(a, \Theta)$ is a Baire function of the variable Θ on Γ (cf. [5]) and we can define

$$v_r^A(a) = \int_\Gamma n_r(a, \Theta) \, dH_{m-1}(\Theta)\,. \tag{2.8}$$

In case that $r = +\infty$ we shall omit r in all introduced notations. The function v^A defined by (2.8) on R^k will be called cyclic variation of A. We shall use it for $A = = G \subset R^m$ and $A = E \subset R^{m+1}$.

Some simple properties of cylic variation are obvious. For example if $A \subset R^k$, $k \geqq 2$ is a bounded Borel set and $a \in A$ is an interior point of A, then

$$v^A(a) \geqq H_{k-1}(\Gamma) = \frac{2\pi^{k/2}}{\Gamma(k/2)}\,, \tag{2.9}$$

where Γ denotes now gamma-function.

2.2 Lemma. *Let* $z \in R^{m+1}$, $\psi \in \mathscr{D}(z)$. *We define the function* $S\psi$ *on* $(0, +\infty) \times \times (0, +\infty) \times \Gamma^*$ *by*

$$S\psi(\varrho, \gamma, \Theta^*) = \psi\left(\hat{z} + \varrho\Theta^*, z_{m+1} - \frac{\varrho^2}{4\gamma}\right), \tag{2.10}$$

where $\varrho, \gamma \in (0, +\infty)$, $\Theta^* \in \Gamma^*$. *Defining*

$$\tilde{G}(\Theta^*) = \{\varrho > 0;\ \hat{z} + \varrho\Theta^* \in G\} \tag{2.11}$$

we can write $W\psi(z)$ *in the form*

$$W\psi(z) = 2^{m-1} \int_{\Gamma^*} \mathrm{d}H_{m-1}(\Theta^*) \int_0^\infty e^{-\gamma} \gamma^{m/2-1}\,\mathrm{d}\gamma \int_{\tilde{G}(\Theta^*)} \partial_\varrho S\psi(\varrho, \gamma, \Theta^*)\,\mathrm{d}\varrho\,. \tag{2.12}$$

Proof. Applying Fubini's theorem in (2.3) we obtain

$$W\psi(z) = -\int_{-\infty}^{\infty} \left(\int_G \ldots \mathrm{d}\hat{w}\right) \mathrm{d}w_{m+1}\,.$$

Let us write $z = [x, t]$, $w = [y, u]$ and use (1.3). We get easily

$$W\psi(x, t) = -\int_{-\infty}^{t} \left(\int_G \ldots \mathrm{d}y\right) \mathrm{d}u\,.$$

Now we put $y = x + \Theta^* r$, $r > 0$, $\Theta^* \in \Gamma^*$ and after a simple calculation we use substitution

$$r = \varrho\,, \quad u = t - \frac{\varrho^2}{4\gamma}\,. \tag{2.13}$$

Using (2.10) we obtain (2.12) without difficulties. This proof is similar to the proof of Lemma 1.6 in [4] and we described only differences.

Comparing $W\psi(z)$ in (2.12) and $\tilde{W}\psi(z)$ in Lemma 1.6 of [4] we are able to state:

2.3 Lemma. *Let* $z \in R^{m+1}$, *fix* $R > 0$, $\varepsilon > 0$ *and put*

$$\bar{\mathscr{D}} = \{\psi \in \mathscr{D};\ \|\psi\| \leqq 1,\ \operatorname{spt}\psi \subset (\Omega(\hat{z}, R) - \{\hat{z}\}) \times (z_{m+1} - \varepsilon, z_{m+1})\}\,, \tag{2.14}$$
$$r(\gamma) = \min(R, 2\gamma\varepsilon^{1/2})\,, \quad \gamma > 0\,.$$

Then

$$\sup\{W\psi(z);\ \psi \in \bar{\mathscr{D}}\} = 2^{m-1} \int_0^\infty e^{-\gamma} \gamma^{m/2-1}\, v^G_{r(\gamma)}(\hat{z})\,\mathrm{d}\gamma\,. \tag{2.15}$$

Proof of this lemma and the proof of Proposition 1.8 in [4] are similar, only the definition of mapping Φ must be changed according to (2.13).

2.4 Lemma. *For $z \in R^{m+1}$ let us denote $\mathscr{D}^1(z) = \{\psi \in \mathscr{D}(z); \|\psi\| \leqq 1\}$. Then*

$$(2.16) \qquad \sup \{W\psi(z); \psi \in \mathscr{D}^1(z)\} = 2^{m-1} \Gamma \left(\frac{m}{2}\right) v^G(\hat{z}) .$$

Proof. We put $R = \varepsilon = +\infty$ in (2.14) so we get $r(\gamma) = +\infty$ and it is enough to consider the connection between $\tilde{\mathscr{D}}$ and $\mathscr{D}^1(z)$. Let $\alpha > 0$. It is easy to see that for a $\psi \in \mathscr{D}^1(z)$ there is a function $\tilde{\psi} \in \tilde{\mathscr{D}}$ such that $|W\psi(z) - W\tilde{\psi}(z)| < \alpha$. But (2.16) now follows because $\tilde{\mathscr{D}} \subset \mathscr{D}^1(z)$.

2.5 Remark. Instead of using results from [4] above it would be possible to start with (2.12) and to prove directly (see [5], the part following 1.7) for all $\gamma \in (0, +\infty)$

$$\sup \left\{ \int\int_{\tilde{G}(\Theta^*)} \partial_\varrho S\psi(\varrho, \gamma, \Theta^*) \, d\varrho; \; \psi \in \mathscr{D}^1(z) \right\} = n(\hat{z}, \Theta^*) .$$

(For the notation see 2.2.) This process would be rather long and complicated.

2.6 Lemma. *Let $z \in R^{m+1}$. Then*

$$(2.17) \qquad v^G(\hat{z}) < +\infty$$

is a necessary and sufficient condition to secure for every sequence $\{\psi_k\}$ of functions $\psi_k \in \mathscr{D}(z)$ converging uniformly to $\psi \in \mathscr{D}(z)$

$$(2.18) \qquad \lim_{k \to \infty} W\psi_k(z) = W\psi(z) .$$

Proof. Supposing (2.17) we get from (2.16) the estimate

$$(2.19) \qquad |W\psi_k(z) - W\psi(z)| \leqq 2^{m-1} \Gamma \left(\frac{m}{2}\right) v^G(z) \, \|\psi_k - \psi\| .$$

In case that $v^G(\hat{z}) = +\infty$ we can find a sequence $\{\psi_k\}$ of functions from $\mathscr{D}^1(z)$ for which $\lim_{k \to \infty} W\psi_k(z) = +\infty$ is valid. From a such sequence we shall construct easily the sequence converging uniformly to 0 for which (2.17) is not valid.

2.7 Remark. From (2.1) we get for $\Theta \in \Gamma$

$$(2.20) \qquad n(z, \Theta) = n(z, [\hat{\Theta}, 0]) = n(\hat{z}, \hat{\Theta}) ,$$

where first two values are equal to the number of all hits on $E \subset R^{m+1}$, the third one the number of all hits on $G \subset R^m$. In case $\hat{\Theta} = 0$ the last term in (2.20) is not defined.

For $(T_1, T_2) \subset R^1$ we put $\tilde{E} = G \times (T_1, T_2)$. Let $z = [x, t]$ be a point in R^{m+1} with $t \in (T_1, T_2)$. Applying Fubini's theorem and integrating we find easily some estimates: there exist $K_1, K_2 > 0$ such that the inequality

$$K_1\, v^G(\hat{z}) \leqq v^E(z) \leqq K_2\, v^G(\hat{z})$$

holds. Especially for E we obtain the equality

$$v^E(z) = \pi\, v^G(\hat{z})\,, \tag{2.21}$$

which holds for all $z \in R^{m+1}$.

The following theorem could be derived also for some subspaces of $\mathscr{D}(z)$, for example the subspace of those ψ for which $\operatorname{spt}\psi \subset R_{T_1} - R_{T_2}$ with $T_1 < T_2$. We shall formulate simple version only:

2.8 Theorem. *Let $z \in R^{m+1}$. Then the linear functional $W\psi(z)$ over $\mathscr{D}(z)$ is a measure ν_z with $|\nu_z|\,(R^{m+1}) < +\infty$ if and only if (2.17) or*

$$v^E(z) < +\infty \tag{2.22}$$

holds. Equality (2.6) together with any of the following conditions

$$\nu_z(\{z\}) = 0\,, \tag{2.23}$$

or

$$|\nu_z|\,(C) = 2^{m-1}\Gamma\left(\frac{m}{2}\right) v^G(\hat{z}) \tag{2.24}$$

determine this measure ν_z uniquely.

Proof of this theorem follows from the integral representation of the linear functional and from (2.16).

For the mentioned more complicated situations we would find that the first part of the theorem (with the condition (2.22)) is valid again.

2.9 Notation. Assuming (2.17) or (2.22) we define for all $f \in \mathscr{B}$ the value of the heat potential of the double distribution $Wf(z)$ with the density f by

$$Wf(z) \overset{\text{def.}}{=\!=} \int_{R^{m+1}} f\, \mathrm{d}\nu_z \tag{2.25}$$

where ν_z is the measure from 2.8.

Another formula for $Wf(z)$ is given in the following

2.10 Lemma. *Let $z \in R^{m+1}$ and suppose that (2.22) holds. Let us define the function s on $(0, +\infty) \times \Gamma^*$ by $s(\hat{z}; \varrho, \Theta^*) = \sigma$, $\sigma \in \{-1, 1\}$ provided there is a $\delta > 0$ such that for H_1-almost every $u \in (0, \delta)$*

$$\hat{z} + (\varrho + \sigma u)\,\Theta^* \in R^m - G\,, \quad \hat{z} + (\varrho - \sigma u)\,\Theta^* \in G$$

is valid. For all remaining we put $s(\hat{z}; \varrho, \Theta^*) = 0$.

Now we define for $f \in \mathscr{B}$

$$\sum_f(\hat{z}; \gamma, \Theta^*) = \sum_{\varrho} f\left(\hat{z} + \varrho\Theta^*, z_{m+1} - \frac{\varrho^2}{4\gamma}\right) . s(\hat{z}; \varrho, \Theta^*) \tag{2.26}$$

provided $n(\hat{z}, \Theta^*) < +\infty$. If $n(\hat{z}, \Theta^*) = +\infty$ we define the value of the expression from (2.26) by 0. Then $\sum_f$ as the function of Θ^* is H_1-integrable over Γ^* for all $\gamma > 0$, the function

$$V_f(z, \gamma) = \int_{\Gamma^*} \sum_f(z; \gamma, \Theta^*)\, \mathrm{d}H_{m-1}(\Theta^*) \tag{2.27}$$

is a bounded Baire function of γ on $(0, +\infty)$, the integral in the following formula is finite and

$$Wf(z) = 2^{m-1} \int_0^\infty e^{-\gamma} \gamma^{m/2-1} V_f(z, \gamma)\, \mathrm{d}\gamma . \tag{2.28}$$

2.11 Remark. If a function F is defined on a set which contains C we put $WF(z) = Wf(z)$ where $f = F/C$ provided $f \in \mathscr{B}$. We shall usually denote F and its restriction f by the same symbol.

Proof of Lemma 2.10. For those $\Theta^* \in \Gamma^*$ for which $n(\hat{z}, \Theta^*) < +\infty$ and for any $\psi \in \mathscr{D}(z)$ we prove similarly to the proof of 2.1 in [4]

$$\int_{\tilde{G}(\Theta^*)} \partial_\varrho S\psi(\varrho, \gamma, \Theta^*)\, \mathrm{d}\varrho = \sum_\psi(z; \gamma, \Theta^*)$$

(for the notation see 2.2). The function $\partial_\varrho S\psi(\varrho, \gamma, \Theta^*)$ is continuous on $(0, +\infty) \times (0, +\infty) \times \Gamma^*$ and $H_{m-1}(\{\Theta^* \in \Gamma^*; n(\hat{z}, \Theta^*) = +\infty\}) = 0$ so that the function $\sum_\psi$ is H_{m-1}-integrable on Γ^* and

$$V\psi(z, \gamma) = \int_{\Gamma^*} \mathrm{d}H_{m-1}(\Theta^*) \int_{\tilde{G}(\Theta^*)} \partial_\varrho S\psi(\varrho, \gamma, \Theta^*)\, \mathrm{d}\varrho .$$

Comparing (2.28) and (2.12) we get (2.28) for any $\psi \in \mathscr{D}(z)$. Then we use 2.8 and show that v_z is characterized by (2.28) and that the formula holds for any $f \in \mathscr{B}$.

Following again [4] we can obtain as a consequence of preceding lemma:

2.12 Lemma. *Let $z \in R^{m+1}$ and suppose that (2.22) holds. For any $f \in \mathscr{B}$ we have*

$$|Wf(z)| \leqq 2^{m-1} \Gamma\left(\frac{m}{2}\right) v^G(\hat{z}) . \|f\| . \tag{2.29}$$

If $\{f_k\}$ is a pointwise convergent sequence of functions from $\mathscr{B}$ which are uniformly bounded and converge to f, then

$$\lim_{k \to \infty} Wf_k(z) = Wf(z) . \tag{2.30}$$

3. NON-TANGENTIAL LIMITS OF THE HEAT POTENTIAL OF THE DOUBLE DISTRIBUTION

We shall use some another notions and notations. Given a Borel set $A \subset R^k$ and $a \in R^k$ we put

$$(3.1) \qquad d_A(a) = \lim_{\varrho \to 0+} \frac{H_k(\Omega(a, \varrho) \cap A)}{H_k(\Omega(a, \varrho))}$$

provided the limit in (3.1) exists. This number $d_A(a)$ is called the k-dimensional density of A at a.

Assuming again $A \subset R^k$, $a \in R^k$ we put for $\Theta \in \Gamma$, $\Gamma \subset R^k$ $n(a) = \Theta$ provided the symmetric difference of A and the half-space

$$\{b \in R^k;\ (b - a) \circ \Theta < 0\}$$

has k-dimensional density 0 at a. If such a $\Theta \in \Gamma$ exists we call it the exterior normal to A in a in the sense of Federer. It is well-known that if the "classical" exterior normal exists it is equal to the Federer's one. At points where the exterior normal in the sense of Federer does not exist we put $n(a) = 0$ $(\in R^k)$.

For $a \in R^k$ and $\Theta \in R^k$, $\Theta \neq 0$ we put

$$(3.2) \qquad \begin{aligned} S(a, \Theta) &= \{b \in R^k;\ b = a + \Theta\varrho,\ \varrho > 0\}\,, \\ S(a) &= \{S(a, \Theta);\ \Theta \in \Gamma\}\,. \end{aligned}$$

For $A \subset R^k$ and $a \in R^k$ we denote

$$(3.3) \qquad \operatorname{contg}(a, A) = \{S(a, \Theta);\ \Theta\}\ ;$$

on the right side we take all those $\Theta \in \Gamma$ for which $a_n \in A$, $a_n \neq a$ exist fulfilling the conditions

$$\lim_{n \to \infty} a_n = a\,, \quad \lim_{n \to \infty} \frac{a_n - a}{|a_n - a|} = \Theta\,.$$

We call the set from (3.3) the contingent of A at a. It is obvious that $\operatorname{contg}(a, A) \subset S(a)$ and $\operatorname{contg}(a, A) = \emptyset$ in case a is not a point of accumulation of A.

The above introduced notation will be used in R^m for example for sets G, B and in R^{m+1} for sets E, C. From (2.1) some simple relations between normals and densities of G and E follows.

We shall assume again that G is a fixed Borel set with the compact boundary in R^m. Now we shall study $Wf(z)$ from the preceding text as a function of z for a fixed $f \in \mathscr{B}$. It would be possible to do it on the maximal set where $Wf(z)$ would be defined by (2.28). Such a "natural domain" would depend on f. Since we want to find results holding for all $f \in \mathscr{B}$ we define Wf for any fixed $f \in \mathscr{B}$ by (2.28) on the set

$$(3.4) \qquad M = \{z \in R^{m+1};\ v^E(z) < +\infty\}\,.$$

Denoting

$$M^* = \{x \in R^m;\ v^G(x) < +\infty\} \tag{3.5}$$

we obtain from 2.7 easily $M = M^* \times R^1$.

$P(G)$ will denote the perimeter of G defined by

$$P(G) = \sup \int_G \operatorname{div} p(x)\, dx$$

where $p = [p_1, \ldots, p_m]$ ranges over all vector-valued functions with m components $p_i \in \mathscr{D}^*$, $i = 1, \ldots, m$ for which

$$\left(\sum_{j=1}^{m} p_j^2(x)\right)^{1/2} \leqq 1$$

holds everywhere in R^m.

3.1 Remark. The function v^G is a lower semicontinuous function on R^m. If

$$P(G) < +\infty\,, \tag{3.6}$$

then denoting by $d(x)$ the distance of x from B we can get for $x \notin B$ the estimate

$$v^G(x) \leqq d^{1-m}(x)\, P(G)\,. \tag{3.7}$$

Now it is easily seen that (3.6) implies $R^m - B \subset M^*$. On the other hand if M^* contains such $(m + 1)$-tuple of points which are not situated on a single hyperplane, then (3.6) holds. These facts were proved in [5] (cf. 2.9 Lemma and 2.10 Proposition).

From this remark we obtain immediately

3.2 Lemma. *For the set M (on which we define Wf) from (3.4) exactly one of the following conditions takes place:*

(a) $\quad M^\circ = (M^* \times R^1)^\circ = \emptyset$ *and* $P(G) = +\infty$,

(b) $\quad R^{m+1} - C \subset M$ *and* $P(G) < +\infty$.

In the following we shall suppose (b), i.e. $P(G) < +\infty$. We define $n(x)$ as the exterior normal in the Federer's sense to G at x and put

$$B^r = \{x \in R^m;\ n(x) \neq 0\}\,. \tag{3.8}$$

This set is called the reduced boundary of G. It is wellknown (we suppose (3.6)) that

$$B^r \subset B\,, \quad H_{m-1}(B^r) < +\infty\,. \tag{3.9}$$

Also

$$(3.10) \qquad \int_G \operatorname{div} p(x)\,\mathrm{d}x = \int_B p(y) \circ n(y)\,\mathrm{d}H_{m-1}(y)$$

holds for any $p = [p_1, \dots, p_m]$ with $p_i \in \mathscr{D}^*$, $i = 1, \dots, m$ (compare 2.11 and references in [5]).

The following remarks are again consequences of relation (2.1) between G and E.

3.3 Remark. The equality

$$(3.11) \qquad d_G(\hat{z}) = d_E(z)$$

holds at any point $z \in R^{m+1}$ for which one of the expressions in (3.11) is defined. This can be proved by standard calculus methods directly from the definition of k-dimensional density.

3.4 Remark. Defining the normal $n(z)$ (the exterior normal in the sense of Federer – we shall omit it in the following text) for set E in R^{m+1} we put similarly to (3.8)

$$(3.12) \qquad C^r = \{z \in R^{m+1};\ n(z) \neq 0\}\,.$$

From the preceding remark we obtain relations

$$(3.13) \qquad n(z) = [\widehat{n(z)}, 0]\,, \quad \widehat{n(z)} = n(\hat{z})$$

which hold for any $z \in R^{m+1}$. In particular, we get $C^r = B^r \times R^1$.

3.5 Lemma. *For any $z \in R^{m+1}$ we have*

$$(3.14) \qquad \int_C |n(\hat{w}) \circ \hat{\nabla}_w \mathscr{G}(z - w)|\,\mathrm{d}H_m(w) = 2^{m-1}\,\Gamma\left(\frac{m}{2}\right) v^G(\hat{z})\,.$$

In case $v^G(\hat{z}) < +\infty$ we can characterize the measure v_z from 2.8 by the relation

$$(3.15) \qquad v_z(A) = -\int_A n(\hat{w}) \circ \hat{\nabla}_w \mathscr{G}(z - w)\,\mathrm{d}H_m(w)\,,$$

which holds for any Borel set $A \subset R^{m+1}$.

Proof can be done according to the following instruction: For chosen z we take an arbitrary $\psi \in \mathscr{D}(z)$. Let us write $z = [x, t]$, $w = [y, u]$ and put

$$p_u(y) = -\psi(y, u)\,\hat{\nabla}_y \mathscr{G}(x - y,\ t - u) \quad \text{for} \quad [y, u] \subset R^{m+1} - \{[x, t]\}\,,$$
$$p_t(x) = 0\,.$$

Now we apply to p_u for fixed $u \in R^1$ formula (3.10) and integrate obtained equality over R^1 so that we have

$$\int_{-\infty}^{\infty} \left(\int_G \operatorname{div} p_u(y) \, \mathrm{d}y \right) \mathrm{d}u = \tag{3.16}$$

$$= - \int_{-\infty}^{\infty} \left(\int_B \psi(y, u) \, n(y) \circ \hat{\nabla}_y \mathscr{G}(x - y, \, t - u) \, \mathrm{d}H_{m-1}(y) \right) \mathrm{d}u \, .$$

The integral on the left-hand side of (3.16) can be transformed similarly to the process following (2.4); we do the same operations but in the oposite order. Coming back to the original notation we obtain easily

$$W \psi(z) = - \int_C \psi(w) \, n(\hat{w}) \circ \hat{\nabla}_w \mathscr{G}(z - w) \, \mathrm{d}H_m(w) \tag{3.17}$$

for any function $\psi \in \mathscr{D}(z)$. Using some of the ideas developed in the preceding section we shall finish the proof.

3.6 Remark. By means of 3.4 we conclude from 3.5 for any $f \in \mathscr{B}$ and any $z \in M$

$$W f(z) = - \int_C f(w) \, n(w) \circ \hat{\nabla}_w \mathscr{G}(z - w) \, \mathrm{d}H_m(w) \, . \tag{3.18}$$

Compare this formula with that from the introductory remark.

3.7 Remark. The estimate (3.7) also holds if $d(x)$ denotes the distance of x from the reduced boundary B^r and if $d(x) > 0$. This can be proved in the same way as 2.9 in [5] is proved, only the boundary B must be replaced by the reduced boundary B^r. As a consequence we have

$$R^{m+1} - \bar{C}^r \subset M \, . \tag{3.19}$$

Function Wf is continuous and caloric on $R^{m+1} - \bar{C}^r$. Supposing (3.6) we are also able to tell more about the support of v_z in (2.6) provided the measure v_z exists. From our assumptions about G we are not able to conclude more than $\bar{C}^r \subset C$ and evidently those sets need not be equal.

Our assumptions about G allow to define the value of Newtonian potential of the double distribution with the density $f^* \in \mathscr{B}^*$ at $x \in M^*$ by the following formula

$$W^* f^*(x) = \int_B f^*(y) \, . \, \frac{n(y) \circ (y - x)}{|y - x|^m} \, \mathrm{d}H_{m-1}(x) \, . \tag{3.20}$$

Properties of the function W^*f^* were studied in [5] and [2]. The next lemma will allow us to derive easily some properties of the heat potential.

3.8 Lemma. *Suppose that $f \in \mathscr{B}$ is such a function for which there is a function $f^* \in \mathscr{B}^*$ with*

$$f(\zeta) = f^*(\hat{\zeta}) \tag{3.21}$$

for any $\zeta \in C$. Then we have for any $z \in M$ (see (3.4))

$$Wf(z) = 2^{m-1}\,\Gamma\left(\frac{m}{2}\right) W^* f^*(\hat{z}) . \tag{3.22}$$

Proof. We express $Wf(z)$ in the form (1.2) ($n(y)$ must be interpreted in the sense of Federer). We change the order of integration, then we use (3.21) and the substitution $l = |x - y|^2/4(t - u)$. Now we integrate "inside" and so we obtain coming back to the original notation (3.22).

Now we shall start the study of the boundary behaviour of Wf.

3.9 Theorem. *Given a set $S \subset R^{m+1} - C$ and $\zeta = [\eta, \xi] \in \bar{S} \cap C$ then*

$$\limsup_{z \to \zeta,\, z \in S} Wf(z) < +\infty \tag{3.23}$$

for any function $f \in \mathscr{C}$ or any $f \in \mathscr{B}$ respectively if and only if

$$\limsup_{z \to \zeta,\, z \in S} v^E(z) < +\infty . \tag{3.24}$$

Proof. Denoting $S^* = \{\hat{z};\, z \in S\}$ and $\tilde{S} = S^* \times R^1$ we use the connection between v^G and v^E. It can be easily observed that $S^* \subset R^m - B$, $\eta \in \bar{S}^* \cap B$ and that (3.24) is equivalent with

$$\limsup_{x \to \eta,\, x \in S^*} v^G(x) < +\infty . \tag{3.25}$$

Inequality (3.23) holds especially for all $f \in \mathscr{C}$ which fulfil the assumptions of 3.7 and which form the subspace of $\mathscr{C}$ isomorphic to $\mathscr{C}^*$. Hence we get

$$\limsup_{x \to \eta,\, x \in S^*} W^* f^*(x) < +\infty \tag{3.26}$$

for any $f^* \in \mathscr{C}^*$. Applying 1.1 from [2] we get immediately (3.25) and consecquently also (3.24). As a consequence of (3.24) and (2.29) we obtain (3.23) for any $f \in \mathscr{B}$.

We shall investigate the existence of limits of Wf at points of the boundary C. We shall use this notation: for $\alpha \in \langle 0, 1 \rangle$ we put

$$\begin{aligned} G_\alpha &= \{x \in R^m;\ d_G(x) = \alpha\} , \\ E_\alpha &= \{z \in R^{m+1};\ d_E(z) = \alpha\} . \end{aligned} \tag{3.27}$$

According to 3.3 we have $G_\alpha \times R^1 = E_\alpha$ and it is obvious that the following inclusions hold:

$$R^{m+1} - \bar{E} \subset E_0\,, \quad E_{1/2} \subset C\,, \quad E_1 \subset \bar{E}\,. \tag{3.28}$$

3.10 Theorem. *Let $S \subset R^{m+1} - C$, $\zeta = [\eta, \xi] \in \bar{S} \cap C$. Suppose there is a $\delta > 0$ such that $S \cap \Omega(\zeta, \delta) \subset E_\alpha$ for $\alpha = 0$ or $\alpha = 1$, respectively. Assume in addition that $f \in \mathscr{B}$ is a function continuous at ζ and that (3.24) holds. Then the following limit exists and we have*

$$\lim_{\substack{z\to\zeta\\ z\in S}} Wf(z) = Wf(\zeta) + 2^m\pi^{m/2}(\alpha - d_E(\zeta))\, f(\zeta)\,. \tag{3.29}$$

Proof will be again shortly described. We put $f_1(z) = f(\hat{z}, \xi)$ for any $z \in C$ and $f_2 = f - f_1$. Both defined functions belong to $\mathscr{B}$, they are continuous at ζ and $f_1(\zeta) = f(\zeta)$, $f_2(\zeta) = 0$. For any $z \in M$ we have

$$Wf(z) = Wf_1(z) + Wf_2(z)\,.$$

So it is sufficient only to prove

$$\lim_{\substack{z\to\zeta\\ z\in S}} Wf_1(z) = Wf_1(\zeta) + 2^m\pi^{m/2}(\alpha - d_E(\zeta))\, f_1(\zeta)\,, \tag{3.30}$$

$$\lim_{\substack{z\to\zeta\\ z\in S}} Wf_2(z) = Wf_2(\zeta)\,. \tag{3.31}$$

The function f_1 is constructed in such a way that we can find $f_1^* \in \mathscr{B}^*$ with the property (3.21). For the Newtonian potential we have (assuming practically the same) according to 1.1 from [2]

$$W^* f^*(x) \to W^* f^*(\eta) + H_{m-1}(\Gamma^*)\,(\alpha - d_G(\eta))\, f^*(\eta)\,.$$

Now we use this for $f^* = 2^{m-1}\Gamma(\tfrac{1}{2}m)\, f_1^*$, change $d_G(\eta)$ and $d_E(\zeta)$ and express $H_{m-1}(\Gamma^*)$ with the help of (2.9). We obtain the formula for f_1^* from which we get according to 3.8 the desired equality (3.30).

Now we use the idea of proof of 2.4 in [4]. From (3.24) arises the existence of such a $\delta > 0$ for which we have

$$\sup\,\{v^E(z);\, z \in S \cap \Omega(\zeta, \delta)\} < +\infty\,. \tag{3.32}$$

For an arbitrary $\varepsilon > 0$ we can write f_2 as a sum of functions f_ε, $g_\varepsilon \in \mathscr{B}$ where $f_\varepsilon(z) = 0$ for all z from some neighborhood of ζ and $\|g_\varepsilon\| < \varepsilon$. Using (3.32) we can obtain the uniform (with respect to z) estimate of $|Wg_\varepsilon(z)|$, which converges to 0 if $\varepsilon \to 0_+$. At the same time we have for any f_ε

$$\lim_{\substack{z\to\zeta\\ z\in S}} Wf_\varepsilon(z) = Wf_\varepsilon(\zeta)$$

and those functions converge to Wf uniformly on C (as $\varepsilon \to 0_+$). From those facts we conclude (3.31) and then the proof of 3.10.

3.11 Remark. Besides the relation between v^G and v^E which is the consequence of (2.21) we use the following relation between contingents of the sets B and C (or B^r and C^r). If $S(\eta, \Theta^*) \notin \operatorname{contg}(\eta, B^r)$ we have for example for any ζ, Θ, for which $\hat{\zeta} = \eta$, $\hat{\Theta} = \Theta^*$,

$$S(\zeta, \Theta) \notin \operatorname{contg}(\zeta, C^r)\,.$$

Propositions 1.3 and 1.4 from [2] and the preceding remark give us as a consequence the following

3.12 Theorem. *Assume* $S \subset R^{m+1} - C$, $\zeta = [\eta, \xi] \in \bar{S} \cap C$ *and*

$$\operatorname{contg}(\zeta, S) \cap \operatorname{contg}(\zeta, C^r) = \emptyset\,. \tag{3.33}$$

If

$$v^G(\eta) + \sup_{\varrho > 0} \frac{H_{m-1}(\Omega(\eta, \varrho) \cap B^r)}{\varrho^{m-1}} < +\infty \tag{3.34}$$

then (3.24) *is valid. If* $\Theta_i \in \Gamma$ $(i = 1, \ldots, m)$ *are such vectors that* $\hat{\Theta}_i$ *are linearly independent and there is a* $\delta > 0$ *with the property*

$$\sup \{v^E(z);\ z \in \bigcup_{i=1}^{m} S_\delta(\zeta, \Theta_i)\} < +\infty \tag{3.35}$$

(see (2.7) for the notation), *then this condition secures the finiteness of the second term in* (3.34) *and the validity of* (3.24).

Theorem 3.12 together with Theorem 3.10 give us a satisfactory solution of the problem of the existence and the computation of limit

$$\lim_{\substack{z \to \zeta \\ z \in S}} Wf(z) \tag{3.36}$$

at a point $\zeta \in C$ provided $S = S(\zeta, \Theta) \in S(\zeta) - \operatorname{contg}(\zeta, C)$. Those limits are called non-tangential limits (or angular limits) of the heat potential of the double distribution.

3.13 Remark. Using the notation that we introduced in the last theorems we can show that the validity of (3.24) secures the following condition

$$v^E(\zeta) + \sup_{\varrho > 0} \frac{H_m(\Omega(\zeta, \varrho) \cap C^r)}{\varrho^m} < +\infty\,. \tag{3.37}$$

The form of this condition is quite natural and we mention it only because E need not have the finite perimeter and the compact boundary.

3.14 Remark. Similar questions for the heat potential for $m = 2$ were studied in [6].

4. CONTINUOUS EXTENSION OF THE HEAT POTENTIAL OF THE DOUBLE DISTRIBUTION

We note that $G \subset R^m$ is again a fixed Borel set with compact boundary and the finite perimeter. The notation from the preceding parts is used. Till now we studied the problem of the existence of the limit at a single point with respect to the special set. The following result is a simple consequence of the properties of the cyclic variation v^E, Remark 3.8 and Theorem 3.9.

4.1 Lemma. *Wf is bounded on $S = R^{m+1} - \bar{C}^r$ for any $f \in \mathscr{C}$ (or $f \in \mathscr{B}$) if and only if*

$$\sup \{v^E(\zeta); \zeta \in \bar{C}^r\} < +\infty \tag{4.1}$$

(or

$$\sup \{v^G(\eta); \eta \in \bar{B}^r\} < +\infty , \tag{4.2}$$

which is equivalent with (4.1)).

According to the fact that we shall study some questions connected with the problem of the existence of the continuous extension of Wf we shall assume in the following parts that (4.1) is valid. In that case v^E is bounded on R^{m+1} and at any point $z \in R^{m+1}$ the density $d_E(z)$ is defined (cf. Lemma 2.7 in [5]). We recall that $C^r \subset \subset E_{1/2} \subset C$. The behaviour of Wf on R^{m+1} is totally determined by the function $f \mid \bar{C}^r$. The following lemma which is again a consequence of 3.7 in [5] states more about the set $C - \bar{C}^r$:

4.2 Lemma. *Under the assumptions we made before we have*

$$R^{m+1} - \bar{C}^r \subset E_0 \cup E_1 . \tag{4.3}$$

Proof. In the mentioned lemma the following result is proved (see eventually references mentioned in [5]): Denoting

$$\tilde{B} = B \cap \{\eta; |d_G(\eta) - \tfrac{1}{2}| < \tfrac{1}{2}\} , \tag{4.4}$$

we have for every $\eta \in \tilde{B}$, $\varrho > 0$

$$H_{m-1}(B^r \cap \Omega(\eta, \varrho)) > 0 \tag{4.5}$$

and B^r is dense in $\tilde{B}$.

4.3 Lemma. *Supposing that G has all properties mentioned above we can extend Wf for any $f \in \mathscr{C}$ continuously from the set E_α onto $\bar{E}_\alpha$, $\alpha = 0$ or $\alpha = 1$. For any point $\zeta \in \bar{E}_\alpha \cap C$ the formula (3.29) is valid provided we put $S = E_\alpha$.*

Proof. This lemma is a consequence of Theorem 3.10; compare also 2.15 in [5].

For any $\gamma \in R^1$ and all $\zeta \in C$ we put

$$\overline{W}_\gamma f(\zeta) = Wf(\zeta) + 2^m \pi^{m/2}(\gamma - d_E(\zeta)) f(\zeta) . \tag{4.6}$$

We denote

$$V = \sup \{v^G(\eta);\ \eta \in B\} . \tag{4.7}$$

The operator $\overline{W}_\gamma$ from (4.6) is well-defined on $\mathscr{B}$ or $\mathscr{C}$; we note we suppose $V < +\infty$. If we extend Wf for $f \in \mathscr{C}$ continuously from E_α onto $\bar{E}_\alpha$, $\alpha = 0$ or $\alpha = 1$, then we can restrict these extensions on $\bar{E}_\alpha \cap C$, $\alpha = 0$ or $\alpha = 1$ and these restrictions are again continuous functions. These restrictions are described by (4.6) and we can easily obtain the following

4.4 Theorem. *Let $\gamma \in R^1$, $C \subset \bar{E}_\alpha$, $\alpha = 0$ or $\alpha = 1$. The operator*

$$\overline{W}_\gamma : f \mapsto \overline{W}_\gamma f \tag{4.8}$$

which is defined on the space $\mathscr{C}$ by (4.6) is a bounded linear operator and maps $\mathscr{C}$ into $\mathscr{C}$. We have the following estimate for the norm

$$\|\overline{W}_\gamma\| \leqq 2^{m-1} \left[V\Gamma\left(\frac{m}{2}\right) + 2(1 + |\gamma|)\, \pi^{m/2} \right] . \tag{4.9}$$

Proof. This theorem is a consequence of 4.3 and (2.29); compare also with 2.8 in [4].

4.5 Remark. We supposed $V < +\infty$ for the whole this paragraph. This condition is quite natural in some questions. For example $\overline{W}_\gamma$ from preceding theorem maps $\mathscr{C}$ into $\mathscr{C}$ if and only if $V < +\infty$.

In the rest of this paragraph we mention some facts connected with the first boundary value problem for the heat equation. We shall suppose that G is open set. Assuming it we have $C \subset E_\alpha$ for $\alpha = 1$.

The set of all functions from $\mathscr{C}$ for which $f(\zeta) = 0$ holds for any $\zeta \in C - R_T$ (see (1.7) for the notation) forms the closed linear subspace $\mathscr{C}_T$ of the space $\mathscr{C}$.

4.8 Lemma. *Let $\gamma \in R^1$. Then for the operator $\overline{W}_\gamma$ the inclusion*

$$\overline{W}_\gamma(\mathscr{C}_T) \subset \mathscr{C}_T \tag{4.10}$$

holds if and only if V from (4.7) is finite.